

On Hecke eigenvalues at primes of the form $[g(n)]$

by

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1. Introduction. The estimation of mean-values of arithmetic functions over sparse sequences and the detection of primes in arithmetically interesting and sparse sets of natural numbers are often very hard and of great interest to analytic number theorists. In [1], we investigated a problem that addresses both of these questions, namely the distribution of Fourier coefficients of cusp forms for the full modular group at Piatetski-Shapiro primes. These are primes of the form $[n^c]$, where $c > 1$ is fixed. We successfully handled the c 's in the range $1 < c < 8/7$. In this paper, we extend our result in [1] to primes of the form $[g(n)]$, where $g(x)$ is a general “nice” function that grows much faster than a linear function. However, our result will be weaker in the sense that it covers the result in [1] only for the range $1 < c < 30/29$.

We first introduce some notations and conditions. By F we denote a holomorphic cusp form of weight κ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and by $\lambda_F(n)$ the normalized n th Fourier coefficient of F , i.e. we assume that

$$F(z) = \sum_{n=1}^{\infty} \lambda_F(n) n^{(\kappa-1)/2} e(nz)$$

for $\Im z > 0$. We note that the Ramanujan–Petersson conjecture, proved by P. Deligne [2, 3], gives a bound for the modulus of λ_F . It states that for any fixed $\varepsilon > 0$,

$$(1.1) \quad \lambda_F(n) \ll d(n) \ll n^\varepsilon,$$

where $d(n)$ is the number of divisors of n . If we assume, in addition, that F is an eigenform of all the Hecke operators, then F can be normalized such that $\lambda_F(1) = 1$ and with this normalization the implied constant in the first “ \ll ” in (1.1) can be taken to be 1.

2010 *Mathematics Subject Classification*: 11F11, 11F30, 11F60, 11L03, 11L07, 11L20.

Key words and phrases: Hecke eigenvalues, Piatetski-Shapiro primes.

Further, we assume that $g : [1, \infty) \rightarrow [1, \infty)$ is a function satisfying the following conditions:

- (i) g is increasing.
- (ii) g is infinitely differentiable.
- (iii) g satisfies the inequalities

$$(1.2) \quad x \leq g(x) \leq x^{30/29-\varepsilon}.$$

We note that then the inverse function $f : \text{range}(g) \rightarrow [1, \infty)$ of g exists and has the following corresponding properties:

- (a) f is increasing.
- (b) f is infinitely differentiable.
- (c) f satisfies the inequalities

$$(1.3) \quad x^{29/30+\varepsilon} \leq f(x) \leq x.$$

Moreover, we shall also suppose that the derivatives of f satisfy the following conditions:

- (d) The k th derivative of f satisfies

$$(1.4) \quad f^{(k)}(x) \asymp f(x)/x^k \quad \text{for all } x \text{ in the image of } g \text{ and } k \in \mathbb{N},$$

where the implied constants depend on k alone.

- (e) The second and third derivatives of f satisfy

$$(1.5) \quad 2f''(x) + xf'''(x) \asymp f(x)/x^2 \quad \text{for all } x \text{ in the image of } g.$$

Furthermore, we denote the set of primes by \mathbb{P} . The main result of this paper is the following.

THEOREM 1.1. *Let $g : [1, \infty) \rightarrow [1, \infty)$ be a function satisfying the conditions (i)–(iii) above. Suppose that the inverse function of g satisfies the conditions (a)–(e) above. Let $\lambda_F(n)$ be the normalized n th Fourier coefficient of a holomorphic cusp form F for the full modular group $\text{SL}_2(\mathbb{Z})$. Then there exists a constant $C > 0$ depending on g and F such that*

$$(1.6) \quad \sum_{\substack{n \leq N \\ [g(n)] \in \mathbb{P}}} \lambda_F([g(n)]) \ll N \exp(-C\sqrt{\log N}),$$

where the implied \ll -constant depends on g and F .

For comparison, our main result in [1] was as follows.

THEOREM 1.2. *Let $1 < c < 8/7$ and $\lambda_F(n)$ be the normalized n th Fourier coefficient of a holomorphic cusp form F for the full modular group $\text{SL}_2(\mathbb{Z})$.*

Then there exists a constant $C > 0$ depending on F such that

$$(1.7) \quad \sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda_F([n^c]) \ll N \exp(-C\sqrt{\log N}),$$

where the implied \ll -constant depends on c and the cusp form F .

Some parts of [1] generalize directly in the present paper, while others cannot be carried over. We indicate the differences in the following description of our method for the proof of Theorem 1.1. First, since every cusp form can be written as a linear combination of finitely many Hecke eigenforms, it will suffice to prove Theorem 1.1 for (normalized) Hecke eigenvalues. The advantages of working with Hecke eigenvalues are that they are multiplicative and real. Now we make a similar standard reduction of the problem to exponential sums with Hecke eigenvalues and the von Mangoldt function as in [1]. Then, just as in [1], we decompose the von Mangoldt function using a Vaughan-type identity, which leads to type I and type II sums. The type II sums are then treated by simply using van der Corput's method for exponential sums. In contrast, in [1], we used sophisticated estimates for exponential sums with monomials, which are not applicable in the present, more general situation. For the type I sums, we need to estimate smooth exponential sums with Hecke eigenvalues. Since we work with general functions $g(x)$ in place of x^c , it is not possible to apply Jutila's method utilized in [1]. Instead, we estimate the said exponential sums using a Weyl shift and a bound for shifted convolutions of Hecke eigenvalues with a weakly oscillating weight, a result analogous to that of W. Duke, J. B. Friedlander and H. Iwaniec [4] for the divisor function.

Notations. The following notations and conventions are used throughout the paper.

- $e(z) = \exp(2\pi iz) = e^{2\pi iz}$.
- η and ε are small positive real numbers, where ε may not be the same at each occurrence.
- $c > 1$ is a fixed number and we set $\gamma = 1/c$.
- $\lambda(n)$ denotes the normalized n th Fourier coefficients of a Hecke eigenform for the full modular group. In the following, we shall suppress the subscript F , used in the introduction, since the cusp form is fixed throughout the paper.
- $A(n)$ is the van Mangoldt function.
- $d(n)$ is the divisor function.
- $k \sim K$ means $K_1 \leq k \leq K_2$ with $K/2 \leq K_1 \leq K_2 \leq 2K$.
- $f = O(g)$ or $f \ll g$ means $|f| \leq cg$ for some unspecified positive constant c .

- $f \asymp g$ means $f \ll g$ and $g \ll f$.
- $[x]$ denotes the largest integer not exceeding x , and $\psi(x) = x - [x] - 1/2$ denotes the saw-tooth function.

2. Preliminary lemmas. For the estimation of exponential sums with Hecke eigenvalues, we need the following bound for shifted convolutions of Hecke eigenvalues.

LEMMA 2.1. *Set*

$$D_g(a, b; h) := \sum_{am \mp bn = h} \lambda(m)\lambda(n)g(am, bn),$$

where $a, b \geq 1$, $(a, b) = 1$, $h \neq 0$ and g is a smooth function on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying

$$x^i y^j g^{(ij)}(x, y) \ll \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} P^{i+j}$$

with some $P, X, Y \geq 1$ for all $i, j \geq 0$, the implied constant depending on i, j alone. Then

$$D_g(a, b; h) \ll P^{5/4}(X + Y)^{1/4}(XY)^{1/4+\varepsilon},$$

where the implied constant depends on ε only.

Proof. In [4], an analogous result was proved for the divisor function $d(n)$ in place of $\lambda(n)$. The same arguments based on the delta-method and the Voronoï summation formula lead to the above result. ■

To reduce our problem to the estimation of exponential sums, we shall use the following approximation of the saw-tooth function $\psi(x)$ due to J. D. Vaaler.

LEMMA 2.2 (Vaaler [9], [6, Theorem A6]). *For $0 < |t| < 1$, let*

$$W(t) = \pi t(1 - |t|) \cot \pi t + |t|.$$

Fix a positive integer J . For $x \in \mathbb{R}$ define

$$\begin{aligned} \psi^*(x) &:= - \sum_{1 \leq |j| \leq J} (2\pi i j)^{-1} W\left(\frac{j}{J+1}\right) e(jx), \\ \delta(x) &:= \frac{1}{2J+2} \sum_{|j| \leq J} \left(1 - \frac{|j|}{J+1}\right) e(jx). \end{aligned}$$

Then δ is non-negative, and

$$|\psi^*(x) - \psi(x)| \leq \delta(x) \quad \text{for all real } x.$$

At several places of the paper, we shall use the following classical estimate for exponential sums due to van der Corput.

LEMMA 2.3 (van der Corput, [6, Theorem 2.2]). *Suppose that f is a real-valued function with two continuous derivatives on $[N, N_1]$. Suppose also that there are $\lambda > 0$ and $\alpha \geq 1$ such that*

$$\lambda \leq |f''(x)| \leq \alpha\lambda \quad \text{on } [N, N_1], \text{ where } N_1 \geq N + 1.$$

Then

$$\sum_{N < n \leq N_1} e(f(n)) \ll \alpha(N_1 - N)\lambda^{1/2} + \lambda^{-1/2}.$$

The following is the prime number theorem for Hecke eigenvalues which is used to bound the main term.

LEMMA 2.4 (a special case of [8, Theorem 5.12]). *There exists a positive constant C such that*

$$\sum_{n \leq N} \Lambda(n)\lambda(n) \ll N \exp(-C\sqrt{\log N}),$$

where the implied \ll -constant and the constant C depend on the cusp form.

To bound the error term, we shall see that it suffices to prove that

$$(2.1) \quad \sum_{n \sim N} \Lambda(n)r(n) = O(N^{1-\eta})$$

for some fixed $\eta > 0$, where r is a certain function involving $\lambda(n)$ and an exponential sum. The following lemma reduces the above sum containing the von Mangoldt function to so-called type I and type II sums.

LEMMA 2.5 (Heath-Brown). *Let $r(n)$ be a complex-valued function defined on the natural numbers. Suppose that u, v and z are real parameters satisfying the conditions*

$$3 \leq u < v < z < 2N, \quad z - 1/2 \in \mathbb{N}, \quad z \geq 4u^2, \quad N \geq 32z^2u, \quad v^3 \geq 64N.$$

Suppose further that $1 \leq Y \leq N$ and $XY = N$. Assume that a_m and b_n are complex numbers. Write

$$(2.2) \quad K := \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} a_m r(mn),$$

$$(2.3) \quad L := \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} a_m b_n r(mn).$$

Then the estimate (2.1) holds if we uniformly have

$$\begin{aligned} K &\ll N^{1-2\eta} && \text{for } Y \geq z \text{ and any complex } a_m \ll 1, \\ L &\ll N^{1-2\eta} && \text{for } u \leq Y \leq v \text{ and any complex } a_m, b_n \ll 1. \end{aligned}$$

Proof. This is a consequence of Lemma 3 in [7]. ■

To separate the variables m and n appearing in Lemma 2.5, we shall use the following lemmas. The first of them is the multiplicative property of Hecke eigenvalues, and the second is a variant of Perron’s formula.

LEMMA 2.6. *Hecke eigenvalues are multiplicative and they satisfy*

$$\lambda(mn) = \sum_{d|\gcd(m,n)} \mu(d)\lambda\left(\frac{m}{d}\right)\lambda\left(\frac{n}{d}\right).$$

Proof. This follows by applying the Möbius inversion formula to the product formula for the Hecke eigenvalues. See, for example, Proposition 14.9 of [8]. ■

LEMMA 2.7 ([5, Lemma 6]). *Let $0 < M \leq N < \nu N < \kappa M$ and let a_m be complex numbers with $|a_m| \leq 1$. Then*

$$(2.4) \quad \sum_{N < n < \nu N} a_n = \frac{1}{2\pi} \int_{-M}^M \left(\sum_{M < m < \kappa M} a_m m^{-it} \right) N^{it} (\nu^{it} - 1) t^{-1} dt + O(\log(2 + M)),$$

where the implied O -constant depends only on κ .

To bound a certain error term, we shall need the following.

LEMMA 2.8. *Assume that $1 \leq N < N + 1 \leq N_1 \leq 2N$. Define the function δ as in Lemma 2.2. If f satisfies*

$$f(x) \asymp f(N), \quad f'(x) \asymp f(N)/N, \quad f''(x) \asymp f(N)/N^2 \quad \text{for } N < x \leq N_1,$$

then

$$\sum_{N < n \leq N_1} \delta(-f(n)) \ll J^{-1}N + J^{1/2}f(N)^{1/2} + J^{-1/2}Nf(N)^{-1/2}.$$

Proof. We prove this along the lines of [6, Lemma 2.8, p. 48]. Clearly,

$$(2.5) \quad \sum_{N < n \leq N_1} \delta(-f(n)) \ll \frac{1}{J} \sum_{|j| \leq J} \left| \sum_{N < n \leq N_1} e(jf(n)) \right| \ll \frac{N}{J} + \frac{1}{J} \cdot \sum_{1 \leq j \leq J} \left| \sum_{N < n \leq N_1} e(jf(n)) \right|.$$

Using Lemma 2.3, we get, for $j \geq 1$,

$$\sum_{N < n \leq N_1} e(jf(n)) \ll j^{1/2}f(N)^{1/2} + j^{-1/2}Nf(N)^{-1/2}.$$

Putting everything together, it follows that

$$\sum_{N < n \leq N_1} \delta(-f(n)) \ll J^{-1}N + J^{1/2}f(N)^{1/2} + J^{-1/2}Nf(N)^{-1/2}. \quad \blacksquare$$

We shall also need the following ‘‘Weyl differencing’’ lemma.

LEMMA 2.9 ([8, Lemma 8.17]). *For any complex numbers z_n , we have*

$$\left| \sum_{a < n < b} z_n \right|^2 \leq \left(1 + \frac{b-a}{Q} \right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q} \right) \sum_{a < n, n+q < b} z_{n+q} \overline{z_n},$$

where Q is any positive integer.

3. Exponential sums with Hecke eigenvalues. In this section, we consider exponential sums of the form

$$(3.1) \quad S = \sum_{N < n \leq N'} \lambda(n) e(f(n)),$$

where $3 \leq N < N' \leq 2N$ and $f \in C^\infty([N/2, 3N])$ satisfies

$$(3.2) \quad |f^{(k)}(x)| \ll_k T/N^k \quad \text{for all } x \in [N/2, 3N] \text{ and } k \in \mathbb{N}_0$$

with some

$$(3.3) \quad T \geq N^{3/4}.$$

We shall prove the following lemma.

LEMMA 3.1. *With S defined in (3.1) and the conditions (3.2) and (3.3) satisfied, we have*

$$(3.4) \quad S \ll N^{2/3+\varepsilon} T^{5/18} + N^{5/6} T^{-5/18},$$

where the implied constant depends on ε only.

Proof. We first do a ‘‘Weyl differencing’’, where we introduce an extra smooth weight function $\Phi \in C^\infty(\mathbb{R})$, compactly supported in $[N/2, 5N/2]$ and satisfying

$$\Phi^{(k)}(x) \ll_k N^{-k} \quad \text{for all } x \in \mathbb{R}^+ \text{ and } k \in \mathbb{N}_0$$

and

$$\Phi(x) = 1 \quad \text{for } N \leq x \leq N'.$$

Let Q be any positive integer and set

$$z_n := \begin{cases} 1 & \text{if } N < n \leq N', \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$S = \sum_n z_n \lambda(n) e(f(n)) = \sum_n z_{n+q} \lambda(n+q) e(f(n+q))$$

for any $q \in \mathbb{Z}$. We sum this up over q with $0 \leq q < Q \leq N/2$, getting

$$QS = \sum_{N-Q < n < N'} \sum_{0 \leq q < Q} z_{n+q} \lambda(n+q) e(f(n+q)).$$

Hence, by Cauchy’s inequality,

$$Q^2|S|^2 \leq (N' - N + Q) \sum_n \left| \sum_{0 \leq q < Q} z_{n+q} \lambda(n+q) e(f(n+q)) \right|^2.$$

It follows that

$$Q^2|S|^2 \ll N \sum_{N < n \leq N' - Q} \left| \sum_{0 \leq q < Q} \lambda(n+q) e(f(n+q)) \right|^2 + Q^3 N^{1+\varepsilon}$$

and further

$$Q^2|S|^2 \ll N \sum_n \Phi(n)^2 \left| \sum_{0 \leq q < Q} \lambda(n+q) e(f(n+q)) \right|^2 + Q^3 N^{1+\varepsilon}.$$

Expanding the square on the right-hand side and setting

$$G_{q_1, q_2}(m_1, m_2) := \Phi(m_1 - q_1) \Phi(m_2 - q_2), \quad F_{q_1, q_2}(m) := f(m) - f(m + q_1 - q_2)$$

gives

$$(3.5) \quad \begin{aligned} & Q^2|S|^2 \\ & \ll N \sum_{0 \leq q_1 < Q} \sum_{0 \leq q_2 < Q} \sum_{\substack{m_1, m_2 \\ m_1 - m_2 = q_1 - q_2}} G_{q_1, q_2}(m_1, m_2) \lambda(m_1) \lambda(m_2) e(F_{q_1, q_2}(m_1)) \\ & \quad + Q^3 N^{1+\varepsilon}. \end{aligned}$$

Now we impose the condition

$$Q \geq N/T.$$

Then a simple computation shows that

$$\frac{d^{i+j}}{dx^i dy^j} G_{q_1, q_2}(x, y) e(F_{q_1, q_2}(x)) \ll_{i,j} \left(\frac{TQ}{N^2} \right)^i N^{-j}$$

for

$$0 \leq q_1, q_2 < Q, \quad N/2 + q_1 \leq x \leq 5N/2 + q_1, \quad N/2 + q_2 \leq y \leq 5N/2 + q_2.$$

Now if $q_1 \neq q_2$, we use Lemma 2.1 with

$$g(x, y) := G_{q_1, q_2}(x, y) e(F_{q_1, q_2}(x))$$

and

$$a = b = 1, \quad X := N, \quad Y := N, \quad P := TQ/N$$

to deduce that the inner double sum on the right-hand side of (3.5) is

$$(3.6) \quad \sum_{\substack{m_1, m_2 \\ m_1 - m_2 = q_1 - q_2}} G_{q_1, q_2}(m_1, m_2) \lambda(m_1) \lambda(m_2) e(F_{q_1, q_2}(m_1)) \ll (TQ)^{5/4} N^{-1/2+\varepsilon}.$$

If $q_1 = q_2$, then we have the trivial bound

$$(3.7) \quad \sum_{\substack{m_1, m_2 \\ m_1 - m_2 = q_1 - q_2}} G_{q_1, q_2}(m_1, m_2) \lambda(m_1) \lambda(m_2) e(F_{q_1, q_2}(m_1)) \\ = \sum_m \Phi(m - q_1)^2 \lambda(m_1)^2 \ll N^{1+\varepsilon}.$$

Combining everything in this section, we obtain

$$(3.8) \quad S \ll (TQ)^{5/8} N^{1/4+\varepsilon} + \frac{N}{Q^{1/2}} + N^{1/2} Q^{1/2}$$

under the condition

$$(3.9) \quad N/T \leq Q \leq N/2.$$

Now we choose

$$Q := \frac{N^{2/3}}{T^{5/9}}.$$

Then, by $N \geq 3$ and (3.3), the condition in (3.9) is satisfied, and we get (3.4). ■

4. Reduction to exponential sums. Using $\lambda(n) \ll n^\varepsilon$, partial summation, and the fact that every cusp form can be written as a linear combination of finitely many Hecke eigenforms, Theorem 1.1, our main result, can be easily deduced from the following result whose proof will be the object of the remainder of this paper.

THEOREM 4.1. *Let $g : [1, \infty) \rightarrow [1, \infty)$ be a function satisfying conditions (i)–(iii) in Section 1. Suppose that the inverse function of g , f , satisfies conditions (a)–(e) in Section 1. Let $\lambda(n)$ be the normalized n th Fourier coefficient of a Hecke eigenform for the full modular group. Let $\Lambda(n)$ denote the von Mangoldt function. Then there exists a positive constant C depending on the cusp form such that*

$$(4.1) \quad \sum_{n \leq N} \Lambda([g(n)]) \lambda([g(n)]) \ll N \exp(-C \sqrt{\log N}),$$

where the implied \ll -constant depends only on C and the cusp form.

In this section, we reduce the left-hand side of (4.1) to exponential sums. We recall that $f := g^{-1}$ denotes the function inverse to g . Let $m, n \in \mathbb{N}$. Then $[g(n)] = m$ is equivalent to

$$-f(m + 1) < -n \leq -f(m).$$

Therefore,

$$(4.2) \quad \sum_{n \leq N} \Lambda([g(n)])\lambda([g(n)]) \\ = \sum_{g(1) \leq m \leq g(N)} ([-f(m)] - [-f(m+1)])\Lambda(m)\lambda(m) + O(\log N).$$

Breaking into dyadic intervals and using the fact that g is increasing, it hence suffices to prove that

$$(4.3) \quad S := \sum_{n \sim g(N)} ([-f(n)] - [-f(n+1)])\Lambda(n)\lambda(n) \ll N \exp(-C\sqrt{\log N})$$

for any $N > 1$. We write

$$(4.4) \quad S = S_1 + S_2,$$

where

$$S_1 = \sum_{n \sim g(N)} (f(n+1) - f(n))\Lambda(n)\lambda(n), \\ S_2 = \sum_{n \sim g(N)} (\psi(-f(n+1)) - \psi(-f(n)))\Lambda(n)\lambda(n),$$

with $\psi(n)$ being the saw-tooth function of Lemma 2.2.

By (1.4) and the mean value theorem, we have the bounds

$$f(x+1) - f(x) \ll \frac{f(x)}{x} \quad \text{and} \quad \frac{d}{dx}(f(x+1) - f(x)) \ll \frac{f(x)}{x^2}$$

for all x in the image of g . Hence, using partial summation, $f \circ g(x) = x$ and $g(x) \ll x^{30/29-\varepsilon}$, we deduce from Lemma 2.4 that

$$S_1 \ll N \exp(-C\sqrt{\log N}),$$

where the implied constant depends only on C and the cusp form.

Our treatment of the sum S_2 begins as in [6]. By Lemma 2.2, for any $J > 0$ there exist functions ψ^* and δ , with δ non-negative, such that

$$\psi(x) = \psi^*(x) + O(\delta(x)),$$

where

$$\psi^*(x) = \sum_{1 \leq |j| \leq J} a(j)e(jx), \quad \delta(x) = \sum_{|j| \leq J} b(j)e(jx)$$

with

$$a(j) \ll j^{-1}, \quad b(j) \ll J^{-1}.$$

Consequently,

$$\begin{aligned} S_2 &= \sum_{n \sim g(N)} (\psi^*(-f(n+1)) - \psi^*(-f(n)))\Lambda(n)\lambda(n) \\ &\quad + O\left((\log N) \sum_{n \sim g(N)} (\delta(-f(n+1)) + \delta(-f(n)))\right) \\ &= S_3 + O(S_4), \end{aligned}$$

say. We fix a small $\eta > 0$ and set

$$(4.5) \quad J := \frac{g(N)}{N} \cdot N^\eta.$$

Then, using (1.4), Lemma 2.8 and $g(N) \ll N^{30/29-\varepsilon}$, we obtain

$$S_4 \ll N^{1-\eta/2}.$$

The remaining task is to prove that

$$S_3 \ll N^{1-\eta/2}$$

provided that η is sufficiently small. We write

$$S_3 = \sum_{1 \leq |j| \leq J} \sum_{n \sim g(N)} \Lambda(n)\lambda(n)a(j)\phi_j(n)e(-jf(n)),$$

where $\phi_j(x) = 1 - e(j(f(x) - f(x+1)))$. Using partial summation and the bounds $a(j) \ll j^{-1}$ and

$$\phi_j(x) \ll \frac{jf(x)}{x} \quad \text{and} \quad \frac{d}{dx}\phi_j(x) \ll \frac{jf(x)}{x^2},$$

we deduce that it suffices to prove that

$$\sum_{1 \leq |j| \leq J} \left| \sum_{n \sim g(N)} \Lambda(n)\lambda(n)e(-jf(n)) \right| \ll g(N)N^{-\eta/2}.$$

Replacing $g(N)$ by N and N by $f(N)$, taking the definition of J in (4.5) into account, dividing the summation interval $1 \leq |j| \leq J$ into $O(\log 2J)$ dyadic intervals, and using the facts that $e(-x) = \overline{e(x)}$ and the Hecke eigenvalues are real, we see that the above bound holds if

$$(4.6) \quad \sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n)\lambda(n)e(hf(n)) \right| \ll N^{1-\eta}$$

for any $N \geq 1$ and $1 \leq H \leq N^{1+\eta}f(N)^{-1}$. The following lemma reduces the term on the left-hand side of (4.6) to trilinear exponential sums.

LEMMA 4.2. *Suppose that u, v and z are real parameters satisfying the conditions*

$$(4.7) \quad \begin{aligned} 3 \leq u < v < z < 2N, \quad z - 1/2 \in \mathbb{N}, \quad z \geq 4u^2, \\ N \geq 32z^2u, \quad v^3 \geq 64N. \end{aligned}$$

Suppose further that $1 \leq Y \leq N$, $XY = N$ and $H \geq 1$. Assume that A_m , B_n and C_h are complex numbers. For $d \in \mathbb{N}$ set

$$(4.8) \quad K_d := \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} A_m C_h \lambda(n) e(hf(d^2 mn)),$$

$$(4.9) \quad L_d := \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} A_m B_n C_h e(hf(d^2 mn)).$$

Then the estimate (4.6) holds if we uniformly have

$$(4.10) \quad K_d \ll N^{1-3\eta} d^{-1} \quad \text{for } Y \geq z, d \leq 2Y \text{ and any complex } A_m, C_h \ll 1,$$

$$(4.11) \quad L_d \ll N^{1-3\eta} d^{-1} \quad \text{for } u \leq Y \leq v, d \leq 2Y \text{ and any complex } A_m, B_n, C_h \ll 1.$$

Proof. We first write

$$\sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n) \lambda(n) e(hf(n)) \right| = \sum_{h \sim H} c_h \sum_{n \sim N} \Lambda(n) \lambda(n) e(hf(n)),$$

where c_h are suitable complex numbers with $|c_h| = 1$. We further set

$$r(n) := \lambda(n) \sum_{h \sim H} c_h e(hf(n))$$

so that

$$\sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n) \lambda(n) e(hf(n)) \right| = \sum_{n \sim N} \Lambda(n) r(n).$$

Now, by Lemma 2.5, the bound (4.6) holds if

$$(4.12) \quad K \ll N^{1-2\eta} \quad \text{and} \quad L \ll N^{1-2\eta}$$

under the conditions of the same lemma. Here K and L are defined as in (2.2) and (2.3). We may rewrite these terms in the form

$$K = \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} \sum_{h \sim H} a_m c_h \lambda(mn) e(hf(mn)),$$

$$L = \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} \sum_{h \sim H} a_m b_n c_h \lambda(mn) e(hf(mn)).$$

Using the multiplicative property of Hecke eigenvalues, Lemma 2.6, we have

$$(4.13) \quad K = \sum_{d \leq 2Y} \mu(d) \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{\substack{n \sim Y/d \\ h \sim H}} \sum_{h \sim H} a_{dm} \lambda(m) c_h \lambda(n) e(hf(d^2 mn)),$$

$$(4.14) \quad L = \sum_{d \leq 2Y} \mu(d) \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{\substack{n \sim Y/d \\ h \sim H}} \sum_{h \sim H} a_{dm} \lambda(m) b_{dn} \lambda(n) c_h e(hf(d^2 mn)).$$

Now, (4.12) follows from (4.10), (4.11), (4.13), (4.14) and the bound $\lambda(n) \ll n^\epsilon$. ■

In the following sections, we shall estimate the terms K_d and L_d .

5. Estimation of L_d . Our task in this section is to estimate L_d , defined in (4.9).

LEMMA 5.1. *For every sufficiently small and fixed $\eta > 0$, we have*

$$(5.1) \quad L_d \ll N^{1-3\eta} d^{-1}$$

provided that $f(N) \geq N^{8/9+30\eta}$, $1 \leq H \leq N^{1+\eta} f(N)^{-1}$, $1 \leq d \leq 2Y$ and

$$(5.2) \quad \frac{N^{2+100\eta}}{f(N)^2} \leq Y \leq \frac{f(N)^6}{N^{5+100\eta}}.$$

Proof. From (4.9), we have

$$L_d = \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{\substack{n \sim Y/d \\ h \sim H}} \sum_{h \sim H} A_m B_n C_h e(hf(d^2 mn)),$$

with

$$A_m \ll m^\epsilon, \quad B_n \ll n^\epsilon, \quad C_h \ll h^\epsilon.$$

Using Cauchy’s inequality, we get

$$(5.3) \quad L_d^2 \ll N^\epsilon \frac{X}{d} H \sum_h \sum_m \left| \sum_n B_n e(hf(d^2 mn)) \right|^2.$$

Using the “Weyl differencing”, Lemma 2.9, we have

$$(5.4) \quad \left| \sum_n B_n e(hf(d^2 mn)) \right|^2 \leq \left(1 + \frac{Y/d}{Q} \right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q} \right) \sum_{\substack{n \sim Y/d \\ n+q \sim Y/d}} B_{n+q} \overline{B_n} e(h(f(d^2 m(n+q)) - f(d^2 mn))),$$

where Q is a parameter to be chosen later and satisfying the condition

$$(5.5) \quad Q \leq Y/d.$$

Inserting the above into (5.3), we have, since $XY = N$ and $Q < Y/d$,

$$(5.6) \quad L_d^2 \ll N^\varepsilon \left(\frac{H^2 N^2}{Q d^4} + \frac{H N}{Q d^2} \sum_h \sum_{0 < |q| < Q} \sum_{\substack{n \sim Y/d \\ n+q \sim Y/d}} \left| B_{n+q} \overline{B_n} \sum_{m \in I} e(h(f(d^2 m(n+q)) - f(d^2 mn))) \right| \right).$$

The first term on the right-hand side of (5.6) is the contribution from $q = 0$, and I denotes the interval defined by the conditions

$$m \sim X/d, \quad mn \sim N/d^2, \quad m(n+q) \sim N/d^2.$$

Note that

$$\begin{aligned} \frac{d^2}{dm^2} (f(d^2 m(n+q)) - f(d^2 mn)) &= d^4 (n+q)^2 f''(d^2 m(n+q)) - d^4 n^2 f''(d^2 mn) \\ &= d^4 q n_0 (2f''(d^2 n_0 m) + n_0 d^2 m f'''(d^2 n_0 m)), \end{aligned}$$

by the mean-value theorem applied to the function $\tilde{f}(x) = x^2 f''(d^2 xm)$, for some n_0 between n and $n+q$. Using (1.5), it follows that

$$h \frac{d^2}{dm^2} (f(d^2 m(n+q)) - f(d^2 mn)) \asymp h d^3 |q| Y \frac{f(N)}{N^2}.$$

Hence, Lemma 2.3 gives

$$\begin{aligned} &\sum_m e(h(f(d^2 n_1 m) - f(d^2 n_2 m))) \\ &\ll X h^{1/2} d^{1/2} |q|^{1/2} Y^{1/2} \frac{f(N)^{1/2}}{N} + \frac{N}{h^{1/2} d^{3/2} |q|^{1/2} Y^{1/2} f(N)^{1/2}}. \end{aligned}$$

Now inserting the above estimate into (5.6), summing over all the relevant variables and mindful of $XY = N$, we get

$$(5.7) \quad L_d^2 \ll N^{2\varepsilon} \left(\frac{H^2 N^2}{Q d^4} + H^{5/2} N Q^{1/2} Y^{1/2} f(N)^{1/2} d^{-5/2} + \frac{H^{3/2} N^2 Y^{1/2}}{Q^{1/2} f(N)^{1/2} d^{9/2}} \right).$$

To equalize the first two terms above, we set

$$(5.8) \quad Q = H^{-1/3} N^{2/3} f(N)^{-1/3} Y^{-1/3}.$$

If the lower bound for Y in (5.2) holds, this choice of Q is in accordance with (5.5). It follows from (5.7) that

$$\begin{aligned} L_d^2 &\ll \frac{N^{2\varepsilon}}{d^2} (H^{7/3} N^{4/3} f(N)^{1/3} Y^{1/3} + H^{5/3} N^{5/3} Y^{2/3} f(N)^{-1/3}) \\ &\ll \frac{N^{2\varepsilon}}{d^2} (N^{11/3+7/3\eta} f(N)^{-2} Y^{1/3} + N^{10/3+5/3\eta} f(N)^{-2} Y^{2/3}), \end{aligned}$$

since $H \leq N^{1+\eta} f(N)^{-1}$. Taking the square root, we have the desired estimate in (5.1) provided that

$$(5.9) \quad \frac{N^{1/2+100\eta}}{f(N)^{1/4}} \leq Y \leq \frac{f(N)^6}{N^{5+100\eta}},$$

where we use that $f(N) \leq N$.

If, instead of choosing Q as in (5.8), we simply set

$$(5.10) \quad Q = Y/d,$$

which certainly satisfies the requirement in (5.5), then from (5.7), repeating the above computations with this choice of Q , we arrive at the estimate

$$\begin{aligned} L_d^2 &\ll \frac{N^{2\varepsilon}}{d^2} (N^2 Y^{-1} H^2 + N Y H^{5/2} f(N)^{1/2} + N^2 H^{3/2} f(N)^{-1/2}) \\ &\ll \frac{N^{2\varepsilon}}{d^2} (N^{4+2\eta} f(N)^{-2} Y^{-1} + N^{7/2+5/2\eta} Y f(N)^{-2} + N^{7/2+3/2\eta} f(N)^{-2}). \end{aligned}$$

This gives the desired majorant in (5.1) if

$$(5.11) \quad \frac{N^{2+100\eta}}{f(N)^2} \leq Y \leq \frac{f(N)^2}{N^{3/2+100\eta}}.$$

We note that $f(N) \geq N^{8/9+50\eta}$ implies

$$\frac{N^{2+100\eta}}{f(N)^2} \leq \frac{N^{1/2+100\eta}}{f(N)^{1/4}} \leq \frac{f(N)^2}{N^{3/2+100\eta}} \leq \frac{f(N)^6}{N^{5+100\eta}}.$$

Now joining the two Y -ranges in (5.9) and (5.11), we get the lemma. ■

6. Estimation of K_d . For small Y , we cannot directly exploit the smooth exponential sum over n with Hecke eigenvalue $\lambda(n)$. In this case, we treat $\lambda(n)$ like an arbitrary coefficient and hence K_d like L_d , obtaining the following result.

LEMMA 6.1. *For every sufficiently small fixed $\eta > 0$, we have*

$$(6.1) \quad K_d \ll N^{1-3\eta} d^{-1}$$

provided that $f(N) \geq N^{8/9+30\eta}$, $1 \leq H \leq N^{1+\eta} f(N)^{-1}$, $1 \leq d \leq 2Y$ and

$$(6.2) \quad \frac{N^{6+100\eta}}{f(N)^6} \leq Y \leq \frac{f(N)^2}{N^{1+100\eta}}.$$

Proof. This can be proved in essentially the same way as Lemma 5.1, but with the roles of X and Y reversed. Similarly to Lemma 5.1, we deduce that $K_d \ll N^{1-3\eta} d^{-1}$ provided that

$$\frac{N^{2+100\eta}}{f(N)^2} \leq X \leq \frac{f(N)^6}{N^{5+100\eta}}.$$

These inequalities are equivalent to (6.2) since $XY = N$. ■

For large Y , we employ Lemma 3.1 to deduce the following.

LEMMA 6.2. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta}d^{-1}$$

provided that $f(N) \geq N^{3/4+10\eta}$, $1 \leq H \leq N^{1+\eta}f(N)^{-1}$, $1 \leq d \leq 2Y$ and

$$(6.3) \quad Y \geq N^{23/6+100\eta}f(N)^{-3}.$$

Proof. We note that for every $k \in \mathbb{N}$, we have

$$(6.4) \quad \frac{d^k}{dy^k}hf(d^2my) \asymp \frac{hf(N)}{y^k}$$

by (1.4). Thus, we may apply Lemma 3.1 with N replaced by Y/d and $T = hf(N)$ to the sum over n provided that $f(N) \geq N^{3/4}$. Summing up the resulting estimate trivially over h and m , we obtain

$$(6.5) \quad K_d \ll H \cdot \frac{X}{d} \cdot \left(\frac{Y^{2/3+\varepsilon}}{d^{2/3}} \cdot (Hf(N))^{5/18} + \frac{Y^{5/6}}{d^{5/6}} \cdot (Hf(N))^{-5/18} \right).$$

Therefore, the lemma follows upon noting that $H \leq N^{1+\eta}f(N)^{-1}$, $XY = N$ and $f(N) \geq N^{3/4+10\eta}$. ■

Combining Lemmas 6.1 and 6.2, we arrive at the following conclusion.

LEMMA 6.3. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta}d^{-1}$$

provided that $f(N) \geq N^{29/30+100\eta}$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$\frac{N^{6+100\eta}}{f(N)^6} \leq Y \leq 2N.$$

Proof. Clearly, the Y -ranges in Lemma 6.1 and 6.2 overlap if $f(N) \geq N^{29/30+100\eta}$. This yields the desired conclusion. ■

We point out that the condition (1.3) on f arises from Lemma 6.3.

7. Proof of Theorems 4.1 and 1.1. We recall that Theorem 4.1, and hence Theorem 1.1, our main result, holds if (4.6) is valid for any $N \geq 1$ and $1 \leq H \leq N^{1+\eta}f(N)^{-1}$. Here f satisfies conditions (a)–(e) in the introduction, and η is sufficiently small, which we assume in the following. Furthermore, in Lemma 4.2 we formulated some conditions on bilinear sums K_d and L_d under which (4.6) holds. In the following, we check that these conditions are satisfied.

We choose the parameters u , v and z in Lemma 4.2 as follows:

$$u := N^{2+100\eta}f(N)^{-2}, \quad v := 4N^{1/3}, \quad z := [f(N)N^{-1/2-100\eta}] + 1/2.$$

The parameters u , v and z so chosen indeed satisfy the conditions in (4.7) if $f(N) \geq N^{9/10+\varepsilon}$ and η is sufficiently small. Moreover, the conditions (4.10)

and (4.11) hold by Lemmas 5.1 and 6.3 since

$$4N^{1/3} \leq \frac{f(N)^6}{N^{5+100\eta}} \quad \text{and} \quad \frac{N^{6+100\eta}}{f(N)^6} \leq \frac{f(N)}{N^{1/2+100\eta}}$$

if $f(N) \geq N^{13/14+\varepsilon}$ and η is sufficiently small. This completes the proof. ■

Acknowledgments. S. B. thanks the restaurant “Kreuzgang” on the Marktplatz in Göttingen for enabling him to eat his beloved liver everyday. During this work, S. B. was supported by an ELCC Grant 25871 and L. Z. by an AcRF Tier 1 Grant at Nanyang Technological University.

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