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K-finite Whittaker functions are of finite order one

by

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Introduction. Essentially, this paper is a vast revision of Chapter 2 of the author's thesis [14], with a much improved estimate. Relevant to this discussion is the Introduction in McKee [15].

In Jacquet [9], the holomorphy of Whittaker functions attached to K-finite sections of principal series representations of Chevalley groups is proved. Later, Schiffmann [18] extended holomorphy to smooth sections of real rank one groups, using intertwining estimates. Following ideas from both papers, Shahidi [20] extended holomorphy to smooth sections of real groups. Interwoven with these Whittaker functions is the meromorphy and functional equation of certain Eisenstein series. In connection to this, the early work of Shahidi [19]–[22] lets one see Jacquet's functional equation of Whittaker functions from a more representation-theoretic viewpoint. This is Shahidi's theory of "local coefficients" in connection with Whittaker functionals. Involved here is intertwining (coming from the functional equation of Eisenstein series, cf. Langlands [13]) and multiplicity one (cf. Shalika [25]).

More recently, in the paper of Gelbart and Shahidi [6], the boundedness of automorphic L-functions (appearing in the Langlands–Shahidi method) in vertical strips is proved. (This is a result which is necessary for the application of a converse theorem, needed for the somewhat recent cases of functoriality, as described in the Introduction to McKee [15].) These L-functions appear in the Fourier coefficients of cuspidally induced Eisenstein series. Whittaker functions appear in the *generic* coefficients. The paper [6] actually proves the finite order of these L-functions using the theory of Eisenstein series. Since functional equations are used, all that is needed in [6] for the Whittaker functions is a finite order bound of the archimedean Whittaker functions in a half-plane. It is here that the question arose as to if

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these Whittaker functions are of finite order globally. It turns out this is not true for smooth sections; in McKee [15] an infinite order smooth Whittaker function on $\mathrm{SL}_2(\mathbb{R})$ is constructed.

For K-finite sections the result is true; in this paper we prove a finite order type bound (actually, order one) for the archimedean K-finite Whittaker functions of Chevalley groups. This is the result of Theorem 5.2 below. The proof follows that in Jacquet [9] closely, making effective estimates, and using convexity in $\mathbb{C}^n$. Here n is the split rank of the Chevalley group.

To be a bit more precise, the content of Theorem 5.2 is as follows. G is a real or complex Chevalley group. By this we mean G is a real or complex split reductive group. NAK is an Iwasawa decomposition where K is a maximal compact subgroup, A is split, and N is a full unipotent radical. Let M_0 be the centralizer of A in K so that M_0A is a maximal torus, and M_0AN is a minimal parabolic. Let V be the unipotent radical opposite N.

For our Whittaker function, denoted by $E_{\mathcal{D},\eta,\chi}(g,\lambda)$, we have $g \in G$ (it is fixed), $\mathcal{D}$ is a K-type, η is a unitary character of M_0, χ is a generic character on V, while λ is in $\mathfrak{a}_{\mathbb{C}}^*$ which is the complex dual of the real Lie algebra of A. See (1.1), (1.2), and (1.4) and the surrounding discussion in Section 1 for the definition (for appropriate λ) of the Whittaker function with the above data. We are interested in $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ as a function of λ.

Jacquet's Theorem 3.4 in [9] in this context says that $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ is holomorphic in all of $\mathfrak{a}_{\mathbb{C}}^*$. Our Theorem 5.2 gives the estimate (for fixed g)

$$E_{\mathcal{D},\eta,\chi}(g,\lambda) = O(e^{\|\lambda\|^{1+\epsilon}})$$

for any $\epsilon > 0$, with the constant depending on ϵ, as well as on the data with which the Whittaker function is constructed. Here $\|\lambda\|$ can be one of two norms we use on $\mathfrak{a}_{\mathbb{C}}^*$ (see Section 1.1).

The fundamental paper of Jacquet [9] is now 46 years old. From a representation-theoretic standpoint, Jacquet's functional equation of Whittaker functions has been eclipsed by Shahidi's theory of local coefficients quite some time ago. However, it is easy to get estimates from [9], and the proof in [20] is somewhat ineffective. It looks possible to obtain the result of this paper using the proof in [20]. This would require the computation of Shahidi's local coefficient $C(s)$ (from [22]), along with an effective computation of intertwining estimates. This can easily be done, for example in rank one, using the Bernstein polynomial method of intertwining operators, of which effective computations are made in the appendices of Cohn [3].

The following is an outline for this paper.

In Section 1, we discuss the general assumptions and structure theory of the group G. In Sections 1–5 we assume G is *real*. The complex case will be sketched in Section 6.2. Some structure theory is necessary to define the type of Whittaker function we study. For our estimates, we quote Shahidi [20] and reduce to the case that G is semisimple and simply connected. We also discuss our notations (for the first six sections) compared to Jacquet [9] and Shahidi [20]. (To remain similar to other references, Section 7 does not use the same notation as the first six sections.) In Section 1.1 we set up the coordinate system on $\mathfrak{a}_{\mathbb{C}}^*$ coming from the fundamental weights. However, we will also use the "natural" coordinate system, due to the fact that the Weyl chambers have simple Euclidean properties. We prove a small lemma relating the norms of the two coordinate systems. In Section 1.2, we discuss some simple finite order properties of functions on $\mathbb{C}^n$, and also record a trivial lemma, since it used many times below.

In Section 2, we discuss Jacquet's proof of holomorphy of these Whittaker functions. In a nutshell, we are following this proof, making it effective. What in particular we need to look at from Jacquet [9] is Lemma 3.2, Proposition 3.3, and Theorem 3.4, with a few things from Sections 1 and 4.

The beginning of Section 2 serves as an introduction to the following subsections. In Section 2.1, we discuss and record some basic representation-theoretic facts pertaining to $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SO}(2)$. In Section 2.2 we discuss how $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow G$, for each independent root. (Sections 2.1 and 2.2 are relevant for the functional equation of our higher rank Whittaker function.) In Section 2.3 we discuss Lemma 3.2 of [9], and record an effective version of this lemma. In Section 2.4, we discuss Proposition 3.3 of [9], actually sketching its proof using the effective version of Lemma 3.2 of [9].

In Section 3, we prove many results which are effective versions of results from [9]. In Section 3.1 we prove an effective estimate (Lemma 3.1) for the Whittaker function in $B_\epsilon(V)$. Here $B_\epsilon(V)$ is *most* of the positive Weyl chamber (which we denote by $B(V)$), depending on $\epsilon > 0$. In Section 3.2, we prove effective estimates (Lemma 3.2 and Corollary 3.3) which correspond to Proposition 3.3 of [9]. In Section 3.3, we prove effective estimates (Proposition 3.4 and Corollary 3.5) which correspond to Theorem 3.4 of [9]. Indeed, Corollary 3.5 is an estimate on M_ϵ. Here M_ϵ is *most* of $\mathbb{C}^n$. More specifically, the complement of M_ϵ is contained in a small region (depending on ϵ) surrounding the intersection of walls of all Weyl chambers.

In Section 4, we discuss the simple geometry of M_ϵ. We prove two simple lemmas (4.1 and 4.2). These give a precise description of not only M_ϵ, but also of how the convex hull of M_ϵ encloses the complement of M_ϵ.

In Section 5, we prove our main result, Theorem 5.2. First, we prove Proposition 5.1, which gives an estimate (in the scalar case) for our Whittaker function in the complement of M_ϵ. This uses the results of Section 4, of course. The combination of Proposition 5.1 with Corollary 3.5 gives us Theorem 5.2. What is used in this section is effective convexity estimates in $\mathbb{C}^n$. In particular, an estimate is obtained coming from a multi-variable Cauchy integral, over different polydisks.

In Section 6, we record several long remarks about Theorem 5.2. Particularly, in Section 6.1, we discuss the estimates of Section 5 from a $\mathbb{C}^n$ analysis perspective. We wished to show the reader that these are standard convexity estimates within $\mathbb{C}^n$. We have drawn a figure (Figure 5) that shows all of this analysis for a rank 2 example. We also discuss what happens in higher rank; some geometry is different (the proofs above are valid, though) if $n \geq 3$. In Section 6.2, we discuss the modifications necessary to extend our main result to complex groups. In Section 6.3, we give a short sketch that the main result is trivial if $n = 1$. (The type of convexity we use in Section 5 does not exist in $\mathbb{C}^1$.)

In Section 7, we discuss an application of this result, in reference to simplifying the proof of the known result of the boundedness in vertical strips of Langlands–Shahidi L-functions originally proved in Gelbart and Shahidi [6]. At the beginning of Section 7, we give credit where it is due of this result. In Section 7.1, we review what we will need from the Langlands–Shahidiv method. Due to the references for this method, our notation here differs from the first six sections. In Section 7.2, we give in Theorem 7.1 a proof of this boundedness result, using our Theorem 5.2, and an important result of Müller [17].

For the benefit of the reader, in Section 3, we have drawn four figures to help illustrate our analysis. Some of this was to help with our definitions of sets in $\mathbb{C}^n$. We also wished to help illustrate what is happening geometrically with this convexity estimate. All five figures picture the real projection of the same rank 2 example, though not all are drawn to the same scale.

1. Preliminaries. In this section we discuss some general assumptions and structure theory of our group G. This is since some structure theory is needed to define the type of Whittaker function we are studying. We also discuss some notation used in this paper, as compared to [9].

Let us describe the Whittaker functions in [9]. Suppose G is a Chevalley group defined over $\mathbb{Q}$. That is, we assume G is split over $\mathbb{Q}$ and is reductive. In this paper, we will be interested in only the real or complex points of G. (Jacquet [9] also considers the p-adic and adelic points.) Let us first assume G is *real* (i.e., $G = G(\mathbb{R})$, and G is of course split over $\mathbb{R}$). The case where G is complex will be described in Section 6.2.

Let $\mathfrak{g}$ be the Lie algebra of $G = G(\mathbb{R})$. Let NAK be an Iwasawa decomposition of this G. Here, K is a maximal compact subgroup, N is a full unipotent subgroup, and A is split. Write $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{k}$ for the Lie algebras of N, A, and K respectively. As in Shahidi [20], let ψ denote the set of roots of

$\mathfrak{g}$ with respect to $\mathfrak{a}$. Let Δ, ψ^+, and ψ^- be the simple, positive and negative roots respectively. Then

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \psi} \mathfrak{g}_\alpha$$

for root spaces $\mathfrak{g}_\alpha$. We may assume those $\mathfrak{g}_\alpha$ for $\alpha \in \psi^+$ generate $\mathfrak{n}$. Let us define $\rho = \frac{1}{2}\sum_{\alpha\in\psi^+} \alpha$.

Let $\mathfrak{a}^*_{\mathbb{C}}$ denote the dual of the complexification of $\mathfrak{a}$. Let λ denote the complex multi-variable of $\mathfrak{a}^*_{\mathbb{C}}$. We have $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^n$ for n the dimension of $\mathfrak{a}$. We assume $n \geq 2$. (The case $n = 1$ is trivial, and will be discussed in Section 6.3.) Let us take a^λ to be defined as $a^\lambda = e^{\lambda \log(a)}$, where $\log : A \to \mathfrak{a}$ is the inverse of the exponential map. Let $M_0 = \mathcal{Z}_K(\mathfrak{a})$ denote the centralizer in K of $\mathfrak{a}$ so that $P = M_0AN$ is the Langlands decomposition of a minimal parabolic subgroup P. Thus, G also has the factorization $G = PK$.

Let W be the Weyl group of G. For each $\alpha \in \Delta$ let w_α be the corresponding reflection. Let us define $T = M_0A$. Then T is a maximal torus, and we assume W acts on T in the standard way. Let us define w_l to be the longest element of W. We write l_0 for the length of w_l, and V for the unipotent subgroup opposite N. In other words, $V = w_l N w_l^{-1}$. The Lie algebra of V is generated by those $\mathfrak{g}_\alpha$ for $\alpha \in \psi^-$.

Let $\mathcal{D}$ be a unitary representation of K on a finite-dimensional Hilbert space $\mathcal{H}$. (So $\mathcal{H}$ is a complex vector space.) Let us denote the inner product and norm on $\mathcal{H}$ by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}}$, respectively. We will denote a vector in $\mathcal{H}$ as v°, to distinguish between elements of V.

Let $\eta \in \hat{M_0}$, i.e., a unitary character of M_0. Let $P(\mathcal{D},\eta)$ be the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{H}(\mathcal{D},\eta)$ consisting of all vectors $v^\circ \in \mathcal{H}$ so that

$$\eta(m_o)v^\circ = \mathcal{D}(m_o)v^\circ \quad \text{for all } m_o \in M_0. \tag{1.1}$$

(In [9], this condition also involves N, since N has nontrivial intersection with K in the p-adic case. In the archimedean case, this does not concern us.)

For any element $g \in G$ written as $g = m_0ank$ let us define the function $L_{\mathcal{D},\eta}(g,\lambda)$ (on $G \times \mathbb{C}^n$, that operates on $\mathcal{H}$), with values in $\mathcal{H}(\mathcal{D},\eta)$, as

$$L_{\mathcal{D},\eta}(g,\lambda) = \eta(m_0)\mathcal{D}(k)a^{\lambda+\rho}P(\mathcal{D},\eta). \tag{1.2}$$

Having $P(\mathcal{D},\eta)$ here is a technical necessity. We have different actions of M_0 and K with $M_0 \subset K$. The factor $P(\mathcal{D},\eta)$ ensures these actions are compatible. (Further, since there is no dependence on N, and M_0 normalizes N, there is no ambiguity in the definition of $L_{\mathcal{D},\eta}$.)

We can view (1.2) as defining a section of a principal series representation of G, in a natural way. Let $v^\circ, \tilde{v}^\circ \in \mathcal{H}$, and let us define

$$f(g,\lambda) = \langle L_{\mathcal{D},\eta}(g,\lambda)v^\circ, \tilde{v}^\circ\rangle_{\mathcal{H}}.$$

Then $f(g,\lambda)$ satisfies

$$f(m_0ang,\lambda) = \eta(m_0)a^{\lambda+\rho}f(g,\lambda) \tag{1.3}$$

for all $m_0 \in M_0$, $a \in A$, $n \in N$, and $g \in G$. In this sense, $f(g,\lambda)$ is a scalar-valued function, but is a *section* of a principal series representation of G. Further, $f(g,\lambda)|_K$ (i.e., we restrict g to K) does not depend on λ (the section is *flat*). More importantly, $f(g,\lambda)|_K$ is K-finite. Thus, (1.2) is the vector-valued version of K-finite functions $f(g,\lambda)$ that satisfy (1.3).

Let χ be a *generic* character of V. Let V_α denote the connected subgroup of V with Lie algebra $\mathfrak{g}_\alpha$. Here χ being generic on V means the restriction of χ to V_α is nontrivial for each $-\alpha \in \Delta$.

We can now define the Whittaker function $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ associated to the data $L_{\mathcal{D},\eta}(g,\lambda)$ and χ. For λ in the positive Weyl chamber it is defined by

$$E_{\mathcal{D},\eta,\chi}(g,\lambda) = \int\limits_V L_{\mathcal{D},\eta}(vg,\lambda)\chi(v)\,dv. \tag{1.4}$$

This is still operator-valued (acts on $\mathcal{H}$). For any fixed g, the theorem of Gindikin and Karpelevich gives absolute convergence of this integral for λ in the positive Weyl chamber $B(V)$.

As explained in Shahidi [20, p. 103], in all of our estimates, we may assume G is semisimple and simply connected, by passing to the simply connected cover of the derived group of G. This is due to the fact that the homomorphism from the simply connected cover of the derived group of G to the derived group of G is an isomorphism on V; $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ is by definition, for appropriate λ, an absolutely convergent integral on V.

REMARK 1.1 (Notation). The following four paragraphs refer to the first six sections. We will be referring to Jacquet [9] many times. Let us remark that many notations that are standard now are somewhat different than in [9]. Specifically, K denotes a field in [9]. Further, M is a maximal compact subgroup and A is a maximal torus in [9]. This differs from our notation, where K is the maximal compact subgroup, A is the split component, and M_0A is a maximal torus. Further, the group action of a principal series representation is on the left, in [9]. For us, the action is on the right. This reverses the order of subgroups in the Iwasawa decomposition, in comparison to [9].

Further, we have adopted some notations of Shahidi [20]. For example, η is a character of M_0 while χ is the generic character. There are some differences to [20]; for us N is a full unipotent subgroup of G (the unipotent radical of M_0AN). In [20], this is denoted as U. For us χ is a character on V, but in [20], χ is a character on U.

Further, we use a capital letter R (along with other parameters) to denote the meromorphic operator appearing in a functional equation of

$E_{\mathcal{D},\eta,\chi}(g,\lambda)$ (see equations (2.4) and (3.2)). Aside from the parameters, the same operator is denoted by the capital Russian letter "che" in [9], and by Υ in [20].

There will be many definitions of constants and parameters below. We should mention that c and d will denote specific global constants depending only on G. Other constants and parameters will depend upon ϵ, which could be any positive number. Specifically, r_1 and r_2 below are specifically defined parameters depending upon other parameters, including ϵ. We will use the phrase "for all j" to mean "for all $j \in \{1,\dots,n\}$".

Section 7 does not use the same notation of the previous six sections. This was done to more accurately follow Langlands–Shahidi references. We hope there is no confusion. All the necessary notation for the proof of Theorem 7.1 in Section 7.2 can be found in Section 7.1.

1.1. Coordinates. In this section we set up a coordinate system on $\mathfrak{a}^*_{\mathbb{C}}$ which uses the fundamental weights as a basis. Since we will also be using the "natural" coordinate system, due to the simple Euclidean properties of the Weyl chambers, we prove a small lemma (Lemma 1.2 below) which relates the two norms of these coordinate systems.

Let us set up our coordinates in $\mathfrak{a}^*_{\mathbb{C}}$. First let $\{\alpha_j\}_{j=1}^n$ be an ordering of the roots in Δ. Now, there is already a natural inner product, $\langle\cdot,\cdot\rangle$, on the Euclidean space $\mathfrak{a}^*_{\mathbb{R}} \cong \mathbb{R}^n$, which is induced from the Killing form on $\mathfrak{a}_{\mathbb{R}}$. This is the standard inner product on $\mathbb{R}^n$ (symmetric and positive definite) in which the roots of Δ are embedded with specific geometric characteristics coming from the Dynkin diagram of G. Since each w_α, $\alpha \in \Delta$, acts as a reflection, it follows that $\langle\cdot,\cdot\rangle$ is invariant by W. Without loss of generality, let us assume we have an orthonormal basis $\mathbf{e} = \{e_1,\dots,e_n\}$ of $\mathfrak{a}^*_{\mathbb{R}}$.

Let us define $\Lambda_i \in \mathfrak{a}^*_{\mathbb{R}}$ by the relations $\langle \Lambda_{j_1}, \alpha_{j_2}\rangle = 0$ for $j_1 \neq j_2$, and $\langle \Lambda_j, \alpha_j\rangle = \frac{1}{2}\langle \alpha_j, \alpha_j\rangle$. (These are the fundamental weights.) The *positive Weyl chamber* (denoted by $B(V)$) is of course given by

$$B(V) = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \langle \Re\lambda, \alpha_j\rangle > 0 \ \forall \alpha_j\}.$$

Now $\Lambda = \{\Lambda_1,\dots,\Lambda_n\}$ is a basis of $\mathfrak{a}^*_{\mathbb{R}}$. It follows that Λ is a basis (over $\mathbb{C}$) of $\mathfrak{a}^*_{\mathbb{C}}$. Let us denote $\lambda = \sum_{j=1}^n s_j \Lambda_j \in \mathfrak{a}^*_{\mathbb{C}}$ by $\lambda = (s_1,\dots,s_n) \in \mathbb{C}^n$. This will be our second coordinate system for $\mathfrak{a}^*_{\mathbb{C}}$. In all that follows, the parenthesis notation $\lambda = (s_1,\dots,s_n)$ is with respect to the Λ basis. Now, the positive chamber in the Λ basis is given by $B(V) = \{(s_1,\dots,s_n) \in \mathbb{C}^n \mid \Re s_j > 0 \ \forall j\}$.

It will be useful to use the simple (Euclidean) geometric properties of the Weyl chambers. (For example, Λ is not orthogonal with respect to the given inner product.) Thus, Euclidean distance will be used, along with the $\mathbf{e}$ basis. We need to relate the $\mathbb{C}^n$-modulus of a given point in the Λ basis to the $\mathbb{C}^n$-modulus in the $\mathbf{e}$ basis. A simple inequality will suffice.

Let us first define norms in both the Λ and the $\mathbf{e}$ bases. The $\mathbf{e}$ basis already has a Euclidean norm, over $\mathfrak{a}_{\mathbb{R}}^*$. This is standard. If $\lambda \in \mathfrak{a}_{\mathbb{R}}^* \cong \mathbb{R}^n$, we can put $\|\lambda\| = \sqrt{\langle \lambda, \lambda \rangle}$. It follows that $\|\lambda_1 - \lambda_2\|$ is the Euclidean distance between $\lambda_1, \lambda_2 \in \mathfrak{a}_{\mathbb{R}}^*$. If $\lambda = \sum_{j=1}^n x_j e_j$ with $x_j \in \mathbb{R}$, since $\mathbf{e}$ is orthonormal, we have $\|\lambda\|^2 = \sum_{j=1}^n x_j^2$. We need to extend this definition to $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^n$.

This is easy, and standard. Suppose $\lambda_1, \lambda_2 \in \mathfrak{a}_{\mathbb{C}}^*$. Then, for each $j = 1, 2$, $\lambda_j = \Re\lambda_j + i\Im\lambda_j$, where $\Re\lambda_j, \Im\lambda_j \in \mathfrak{a}_{\mathbb{R}}^*$. Let us define

$$\langle \lambda_1, \lambda_2 \rangle_{\mathbb{C}} = \langle \Re\lambda_1, \Re\lambda_2 \rangle + \langle \Im\lambda_1, \Im\lambda_2 \rangle + i\langle \Re\lambda_2, \Im\lambda_1 \rangle - i\langle \Re\lambda_1, \Im\lambda_2 \rangle.$$

One can check this definition extends $\langle \cdot, \cdot \rangle$ to a positive definite inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^n$, with the correct complex linear properties. We can now define the norm $\|\cdot\|_{\mathbb{C}^n}$ as follows. We define $\|\lambda\|_{\mathbb{C}^n} = \sqrt{\langle \lambda, \lambda \rangle_{\mathbb{C}}}$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. It follows $\|\lambda\|_{\mathbb{C}^n}$ is the $\mathbb{C}^n$-modulus of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ taken in the $\mathbf{e}$ basis induced from $\mathfrak{a}_{\mathbb{R}}^*$. For $\lambda = \sum_{j=1}^n x_j e_j$ with $x_j \in \mathbb{C}$ we have $\|\lambda\|_{\mathbb{C}^n}^2 = \sum_{j=1}^n |x_j|^2$. Further, the Cauchy–Schwarz inequality holds with this norm and inner product on $\mathfrak{a}_{\mathbb{C}}^*$.

Defining a norm for $\mathfrak{a}_{\mathbb{C}}^*$ in the Λ basis is also easy. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we can write $\lambda = (s_1, \ldots, s_n)$ uniquely, where $s_j \in \mathbb{C}$. Let us define

$$|\lambda|_{\Lambda} = \sqrt{\sum_{j=1}^n |s_j|^2},$$

where $|s_j|$ denotes the $\mathbb{C}$-modulus of s_j. It can easily be checked $|\lambda|_{\Lambda}$ is an actual norm.

Lemma 1.2. *There exists a constant c so that*

$$\|\lambda\|_{\mathbb{C}^n} \leq c|\lambda|_{\Lambda} \quad \text{and} \quad |\lambda|_{\Lambda} \leq c\|\lambda\|_{\mathbb{C}^n} \qquad \text{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Here, c depends only on G.

Clearly, such a c must satisfy $c \geq 1$. We will not need the minimum c with the above properties, and we will assume later that $c > 1$. This lemma is a standard argument in comparing norms of different bases. We will show the first inequality directly and the other indirectly.

Proof. To prove the first inequality, as above, take $\lambda = \sum_{j=1}^n s_j \Lambda_j$. Then

$$\begin{aligned}
\|\lambda\|_{\mathbb{C}^n}^2 = \langle \lambda, \lambda \rangle_{\mathbb{C}} &= \sum_{j=1}^n |s_j|^2 \|\Lambda_j\|^2 + \sum_{j_1 < j_2} (s_{j_1}\overline{s_{j_2}} + s_{j_2}\overline{s_{j_1}}) \langle \Lambda_{j_1}, \Lambda_{j_2} \rangle \\
&\leq \Big(\sum_{j=1}^n |s_j| \cdot \|\Lambda_j\| \Big)^2 \leq (\max_j \|\Lambda_j\|)^2 \Big(\sum_{j=1}^n |s_j| \Big)^2 \\
&\leq n(\max_j \|\Lambda_j\|)^2 \Big(\sum_{j=1}^n |s_j|^2 \Big) = n(\max_j \|\Lambda_j\|)^2 |\lambda|_{\Lambda}^2.
\end{aligned}$$

The first inequality here uses the Cauchy–Schwarz inequality for the inner product $\langle \Lambda_{j_1}, \Lambda_{j_2}\rangle_{\mathbb{C}}$ on $\mathbb{C}^n$. (Notice this reduces to $\langle \Lambda_{j_1}, \Lambda_{j_2}\rangle$ since each Λ_j is in $\mathfrak{a}^*_{\mathbb{R}}$.) The second inequality is trivial. The third inequality uses the Cauchy–Schwarz inequality for sums with n terms. We have shown the first inequality of the lemma, where we could take $c = \sqrt{n}\max_j \|\Lambda_j\|$.

To prove the second inequality, suppose there is no $c > 0$ such that $|\lambda|_\Lambda \le c\|\lambda\|_{\mathbb{C}^n}$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Then there is a sequence of points $\lambda_h \in \mathfrak{a}^*_{\mathbb{C}}$ with $|\lambda_h|_\Lambda > h\|\lambda_h\|_{\mathbb{C}^n}$ for $h \in \mathbb{N}$. Both norms scale properly, that is, for any real constant $\xi \ge 0$ we have $|\xi\lambda|_\Lambda = \xi|\lambda|_\Lambda$ and $\|\xi\lambda\|_{\mathbb{C}^n} = \xi\|\lambda\|_{\mathbb{C}^n}$. By rescaling, we may thus assume $|\lambda_h|_\Lambda = 1$ for every h. Now the set $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid |\lambda|_\Lambda = 1\}$ is compact. Thus, $\{\lambda_h\}$ has a convergent subsequence $\{\lambda_{h_k}\}$. Put $\lim_{k\to\infty} \lambda_{h_k} = \lambda_\infty$. Then $|\lambda_\infty|_\Lambda = 1$ but $\|\lambda_\infty\|_{\mathbb{C}^n} = 0$. Clearly this cannot happen, which shows the second inequality. ∎

1.2. Finite order properties. In this section, we record some simple finite order properties of functions. We record a lemma (Lemma 1.3) about such functions on $\mathbb{C}^n$. Lemma 1.3 is analytically trivial, but since it is used many times below, we prove two of the conclusions.

Let $\epsilon > 0$. From known properties of the classical Γ function, we have

$$\frac{1}{|\Gamma(s)|} = O_\epsilon(e^{|s|^{1+\epsilon}}) \tag{1.5}$$

for all $s \in \mathbb{C}$, where the constant depends only on ϵ. Further, if we restrict to the half-plane $\Re s \ge 1/2$, then $\Gamma(s)$ is holomorphic, and (by Stirling's formula)

$$|\Gamma(s)| \le O_\epsilon(e^{|s|^{1+\epsilon}}), \tag{1.6}$$

once again with the constant depending on ϵ.

We extend this type of estimate to $\mathbb{C}^n$ in the obvious way. Specifically, suppose Ω is a subset of $\mathbb{C}^n$. Suppose the complex-valued function $f(\lambda)$ is either smooth on Ω, or holomorphic in a neighborhood of Ω. (Continuity issues on the boundary of Ω will not come up for us, if f is not holomorphic.) We define

$$f(\lambda) = O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}}) \quad \text{for } \lambda \in \Omega$$

to mean $|f(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon}}$ is bounded on Ω for all $\epsilon > 0$, with the bound depending only on ϵ. Clearly this estimate only has substance if Ω is unbounded.

We will use the following (trivial) principle.

LEMMA 1.3. *Suppose $f_1(\lambda), f_2(\lambda)$ are two functions satisfying the assumptions above, as well as the estimate $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$ on some set $\Omega \subset \mathbb{C}^n$ for any $\epsilon > 0$. Then the product function $(f_1 \cdot f_2)(\lambda)$ is also $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$. Suppose $f_3(\lambda)$ satisfies the continuity assumptions above on Ω. If f_3 is either $O_\epsilon(e^{|\xi\lambda|_\Lambda^{1+\epsilon}})$ or $O_\epsilon(e^{\xi|\lambda|_\Lambda^{1+\epsilon}})$, for some constant $\xi \ge 0$, then it is also $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$*

on Ω, with the constant depending also on ξ. Suppose $f_4(\lambda)$ also satisfies the assumptions above as well as the estimate $O_\epsilon(e^{|\lambda+\lambda_0|_\Lambda^{1+\epsilon}})$ where λ_0 is a fixed vector in $\mathbb{C}^n$. Then f_4 is also $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$, with the constant also depending on λ_0.

The conclusions here are similar, and we will show the first two.

Proof. Let $\epsilon > 0$. Then $\epsilon/2 > 0$, and by definition $|f_1(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon/2}}$ and $|f_2(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon/2}}$ are bounded on Ω, with the bound depending only on $\epsilon/2$. On the set consisting of all λ in Ω with $2 \le |\lambda|_\Lambda^{\epsilon/2}$, we have

$$|f_1 \cdot f_2(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon}} \le |f_1(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon/2}} \cdot |f_2(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon/2}}.$$

This shows $|f_1 \cdot f_2(\lambda)|e^{-|\lambda|_\Lambda^{1+\epsilon}}$ is bounded on this subset of Ω. This leaves the set of all $\lambda \in \Omega$ with $2 \ge |\lambda|_\Lambda^{\epsilon/2}$. So, $|\lambda|_\Lambda \le 2^{2/\epsilon}$ (λ is contained in the intersection of Ω with a compact set), and we can just plug this into the particular estimate $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$ (taking into account the constant that depends on ϵ) for each of f_1 and f_2. Taking the product of the maximum constants from the two different estimates gives the estimate on Ω.

Let us assume f_3 satisfies the continuity assumptions above, as well as the estimate $O_\epsilon(e^{|\xi\lambda|_\Lambda^{1+\epsilon}})$ on Ω for all positive ϵ. The claim is trivial if $\xi \le 1$, so let us assume $\xi > 1$. Let $\epsilon > 0$. Then $\epsilon/2 > 0$ and f_3 is $O_{\epsilon/2}(e^{|\xi\lambda|_\Lambda^{1+\epsilon/2}})$ on Ω. When $\lambda \in \Omega$ and $|\lambda|_\Lambda \ge \xi^{2/\epsilon+1}$ we have $e^{|\xi\lambda|_\Lambda^{1+\epsilon/2}} \le e^{|\lambda|_\Lambda^{1+\epsilon}}$. Further, on the intersection of Ω with λ such that $|\lambda|_\Lambda \le \xi^{2/\epsilon+1}$, we deduce f_3 is bounded, since this region is contained in a compact set. Clearly this bound depends upon f_3, ξ, and ϵ. ∎

Using the same reasoning, we remark that if $f(\lambda)$ satisfies the estimate $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$ on Ω then it also satisfies the estimate $O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}})$ on Ω (and vice-versa). This of course uses Lemma 1.2.

2. Reduction to SL_2**.** The key ingredient for the proof of holomorphy of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ for λ in all of $\mathbb{C}^n$ (this is the main result of [9], Theorem 3.4) is that the functional equation, corresponding to $\lambda \mapsto w_{\alpha_j}\lambda$, mimics the functional equation of an SL_2 Whittaker function. What is particularly important for us from [9] is Lemma 3.2, Proposition 3.3, Theorem 3.4, and some results from Sections 1 and 4. (Technically, $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ only has a functional equation under $\lambda \mapsto w_{\alpha_j}\lambda$ if the character χ restricted to $V_{-\alpha_j}$ is nontrivial. Thus in the wording of Theorem 3.4 of [9], holomorphy of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ extends to the convex hull generated by $B(V)$ and all reflections w_{α_j} so that $\chi|_{V_{-\alpha_j}}$ is nontrivial. Since we are assuming χ is generic, this is all of $\mathbb{C}^n$.)

To obtain effective estimates, we need to follow the proof in [9] closely. What becomes relevant, is how $\mathcal{D}$ decomposes when restricted to various subgroups of K, each of which is isomorphic to SO(2). The functional equation is in some sense "vector-valued" on the different isotypic subspaces. This was not really needed in [9]. Indeed, many results in [9] assume these restrictions are irreducible. It is only stated that the results do carry over to the general case (see p. 255 of [9]). This does not change the basic ideas for us, neither the effective estimates nor convexity estimates. Rather, we need to be careful in the bookkeeping of these decompositions.

Therefore, in Section 2.1 we record some basic representation-theoretic facts for $\mathrm{SL}_2(\mathbb{R})$ and SO(2). Further, we record in particular the meromorphic operator appearing in the functional equation of an $\mathrm{SL}_2(\mathbb{R})$ Whittaker function. In Section 2.2 we record information on the meromorphic operator appearing in the functional equation of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$, under $\lambda \mapsto w_{\alpha_j}\lambda$. It is here that we see the decomposition of $\mathcal{D}$ under different subgroups of K. These subgroups (and the functional equation) come from the embedding $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow G$, for each root $\alpha_j \in \Delta$.

In Section 2.3, we review Lemma 3.2 of [9] closely. In particular, we create a function C_j with effective estimates, which makes Lemma 3.2 effective, even if $\mathcal{D}$ reduces on restriction. In Section 2.4 we review the proof of Proposition 3.3 of [9]. This is done for the benefit of the reader. Our review uses the effective version of Lemma 3.2 (from the previous section), so that the proof will be easy to modify to make it effective. This is done in Section 3.2. We also briefly discuss the proof of Theorem 3.4 of [9] using Proposition 3.3 of [9].

2.1. SL_2 **facts.** In this section, we record some representation-theoretic facts about $\mathrm{SL}_2(\mathbb{R})$ and SO(2). In particular, we record some specifics about the meromorphic projection operators appearing in the functional equation of an $\mathrm{SL}_2(\mathbb{R})$ Whittaker function.

First, we will take the Cartan involution on $\mathfrak{sl}_2$ (the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ which consists of 2×2 real matrices of trace zero) to be negative transpose. With respect to this involution, the maximal compact subgroup is given by $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for real θ. Let us denote this matrix by $\tau(\theta)$. We denote the split component by A_0; it consists of diagonal matrices with positive entries. If $\mathfrak{a}_0$ denotes the corresponding Lie algebra, then $\mathfrak{a}_0$ consists of diagonal matrices with trace zero. The unipotent subgroup is denoted by N_0, and is generated by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for real x. One can see easily that $\mathcal{Z}_{\mathrm{SO}(2)}(\mathfrak{a}_0) = \pm I$.

Suppose δ is an irreducible representation of SO(2) on a complex vector space. Since $\mathrm{SO}(2) \cong \mathbb{R}/\mathbb{Z}$ is abelian, it is well known the complex space has dimension one, and $\delta(\tau(\theta)) = e^{2\pi i\theta q_\delta}$ for a unique $q_\delta \in \mathbb{Z}$.

Let $\mathfrak{d}$ be a unitary representation of SO(2) on the finite-dimensional Hilbert space $\mathfrak{H}$. Then we can write $\mathfrak{H} = \bigoplus_\delta \mathfrak{h}^\delta$, an orthogonal direct sum, where $\mathfrak{h}^\delta$ denotes the δ-isotypic subspace of $\mathfrak{H}$. Here the (finite) sum is over irreducible representations δ of SO(2), and $\mathfrak{h}^\delta$ is the subspace of vectors transforming according to δ. As above, each δ is indexed by a unique integer q_δ. Let $\mathfrak{y}$ be a character on $\mathcal{Z}_{\mathrm{SO}(2)}(\mathfrak{a}_0) = \pm I$. Let $\mathfrak{H}_\mathfrak{y}$ denote the subspace of vectors $v^\circ \in \mathfrak{H}$ that satisfy

$$\mathfrak{d}(m)v^\circ = \mathfrak{y}(m)v^\circ \quad \text{for all } m \in \mathcal{Z}_{\mathrm{SO}(2)}(\mathfrak{a}_0).$$

In the real case (which we are now assuming) $\mathfrak{y}$ is determined by $\mathfrak{y}(-I) = (-1)^\varepsilon$, with $\varepsilon \in \{0,1\}$. Let $P(\mathfrak{d},\mathfrak{y})$ denote the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{H}_\mathfrak{y}$. Then clearly, if q_δ (corresponding to δ) and ϵ have the same parity, $P(\mathfrak{d},\mathfrak{y})$ fixes every vector (acts as the scalar 1) of $\mathfrak{h}^\delta$. Otherwise, if δ and ε have different parity, $P(\mathfrak{d},\mathfrak{y}) = 0$ on $\mathfrak{h}^\delta$.

For each δ occurring in the direct sum above, let q_δ be the corresponding integer. For each δ, let us define the function, for $s \in \mathbb{C}$,

$$\phi_\delta(s) = (-\mathrm{sgn}(q_\delta))^{q_\delta} \pi^s \frac{\Gamma\big(\frac{1-s+|q_\delta|}{2}\big)}{\Gamma\big(\frac{1+s+|q_\delta|}{2}\big)}, \tag{2.1}$$

where Γ denotes the classical gamma function. Finally, we can define the operator $R(\mathfrak{d},\mathfrak{y},s)$ on $\mathfrak{H}$ (with coefficients meromorphic in $s \in \mathbb{C}$) as follows:

$$R(\mathfrak{d},\mathfrak{y},s) = \bigoplus_\delta \phi_\delta(s) P(\mathfrak{d},\mathfrak{y}). \tag{2.2}$$

Specifically, if we write any vector $v \in \mathfrak{H}$ as $v = \sum_\delta v_\delta$ according to the orthogonal direct sum $\mathfrak{H} = \bigoplus_\delta \mathfrak{h}^\delta$, then $R(\mathfrak{d},\mathfrak{y},s)v = \sum \phi_\delta(s)P(\mathfrak{d},\mathfrak{y})v_\delta$.

Remark 2.1 ($R(\mathfrak{d},\mathfrak{y},s)$ notation). Our operator $R(\mathfrak{d},\mathfrak{y},s)$ here is the same as in Shahidi [20] and Jacquet [9] when the field is $\mathbb{R}$. However, in [9] we first see it in Corollary 1.10, where our R is replaced by the capital Russian letter “che”. In [20, equation (2.2.4)], Υ replaces our R.

2.2. $\mathrm{SL}_2 \hookrightarrow G$**.** In this section, we discuss the specifics of the meromorphic projection operator appearing in the functional equation of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$. What is relevant here is how the representation $\mathcal{D}$ of K reduces when restricted to different subgroups K_j of K. So, we also need to review the embedding of $\mathrm{SL}_2(\mathbb{R})$ into G for each root $\alpha \in \Delta$.

For each $\alpha \in \psi^+$, let N_α denote the connected subgroup of N with Lie algebra $\mathfrak{g}_\alpha$. Suppose $\alpha_j \in \Delta$. Let G^{α_j} denote the algebraic group (a subgroup of G) generated by N_{α_j} and $V_{-\alpha_j}$. With our assumption that G is simply connected, we have for each $j \in \{1,\dots,n\}$ an isomorphism

$$\mathfrak{X}_j : \mathrm{SL}_2(\mathbb{R}) \to G^{\alpha_j}.$$

For each $j \in \{1, \ldots, n\}$, let K_j denote the image under $\mathfrak{X}_j$ of $\mathrm{SO}(2)$. Then $K_j = K \cap G^{\alpha_j}$. Similarly, let us put $M_0^j = M_0 \cap G^{\alpha_j}$. Then M_0^j is the image of $\pm I$. It can be easily seen that N_{α_j} and $V_{-\alpha_j}$ are the images of N_0 and of the lower triangular unipotent matrices respectively. Let A_j denote $A \cap G^{\alpha_j}$. Then A_j is the image of A_0.

Let $X^*(M_0A)$ denote the rational characters of the maximal torus M_0A, i.e., algebraic group homomorphisms $\vartheta : M_0A \to \mathbb{R}^*$. It is well known the fundamental weights Λ_j are a basis for the $\mathbb{Z}$-module $X^*(M_0A)$. Since M_0 is compact, it follows the image of M_0 under any such ϑ is contained in the subgroup $\{\pm 1\}$ of $\mathbb{R}^*$. We can thus factor $\eta \in \hat{M}_0$ as follows: for $m \in M_0$,

$$\eta(m) = \prod_j \eta_j(\Lambda_j(m)),$$

where each character η_j maps $\{\pm 1\} \to \{\pm 1\}$.

We know χ restricted to $V_{-\alpha_j}$ is nontrivial. We can thus write

$$\chi\big[\mathfrak{X}_j\big(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\big)\big] = e^{2\pi i x \mu_j}$$

for a unique nonzero $\mu_j \in \mathbb{R}$.

Recall $\mathcal{D}$ is a finite-dimensional representation of K on the (complex) Hilbert space $\mathcal{H}$. We have no reducibility assumptions on $\mathcal{D}$. Let $\mathcal{H}_j$ denote the subspace of $\mathcal{H}$ generated by all vectors of the form $\mathcal{D}(k)v^\circ$ where $k \in K_j$ and $v^\circ \in \mathcal{H}(\mathcal{D}, \eta)$. Let $\mathcal{D}_j$ be the representation $\mathcal{D} \circ \mathfrak{X}_j$ of $\mathrm{SO}(2)$ on $\mathcal{H}_j$. Let $\mathcal{H}_j(\eta_j)$ denote the subspace consisting of all vectors $v^\circ \in \mathcal{H}_j$ so that

$$\eta_j(\Lambda_j(m))v^\circ = \mathcal{D}(m)v^\circ \quad \text{for all } m \in M_0^j. \tag{2.3}$$

Let $P(\mathcal{D}_j, \eta_j)$ denote the orthogonal projection of $\mathcal{H}_j$ onto $\mathcal{H}_j(\eta_j)$. (Let us identify the domain of η_j as defined above with $\pm I = \mathcal{Z}_{\mathrm{SO}(2)}(\mathfrak{a}_0)$ in the obvious manner.)

It is easy to see by the definitions that $\mathcal{H}_j$ is invariant under the action of $\mathrm{SO}(2)$ by $\mathcal{D}_j$. Thus, we can write $\mathcal{H}_j = \bigoplus_\delta \mathcal{V}_j^\delta$, where $\mathcal{V}_j^\delta$ denotes the δ-isotypic subspace of $\mathcal{H}_j$. Here, the sum is over irreducible representations δ of $\mathrm{SO}(2)$, and $\mathcal{V}_j^\delta$ denotes the (invariant) subspace of $\mathcal{H}_j$ transforming according to δ under the action of $\mathrm{SO}(2)$ by $\mathcal{D}_j$. Since $\mathcal{H}_j$ is finite-dimensional, this direct sum is finite, and each $\mathcal{V}_j^\delta$ is finite-dimensional. One can see using a standard argument that $\bigoplus_\delta \mathcal{V}_j^\delta$ is an orthogonal direct sum with respect to the inner product on $\mathcal{H}$. This can be shown using the fact that $\mathcal{D}$ is unitary (in particular $\mathcal{D}$ restricted to K_j), as well as the particular form of irreducible representations of $\mathrm{SO}(2)$, as described above. Clearly, $\mathcal{H}(\mathcal{D}, \eta) \subset \mathcal{H}_j(\eta_j) \subset \mathcal{H}_j$ by construction. Thus, if $v^\circ \in \mathcal{H}(\mathcal{D}, \eta)$, we can write $v^\circ = \sum_\delta v_\delta^\circ$ uniquely, with each $v_\delta^\circ \in \mathcal{V}_j^\delta$, according to this orthogonal decomposition.

For each δ occurring in the direct sum $\mathcal{H}_j = \bigoplus_\delta \mathcal{V}_j^\delta$, let q_δ be the associated integer. We define the operator $R_j(\mathcal{D}, \eta, s)$ on $\mathcal{H}$ (with coefficients meromorphic in $s \in \mathbb{C}$) as follows. Recall

$$\mathcal{H}(\mathcal{D}, \eta) \subset \mathcal{H}_j(\eta_j) \subset \mathcal{H}_j \subset \mathcal{H}.$$

First, we require $R_j(\mathcal{D}, \eta, s)(\mathcal{H}_j^\perp) = 0$. Recall the subspace $\mathcal{H}_j$ is the space for the representation $\mathcal{D}_j$ on SO(2). Further, we have the projection $P(\mathcal{D}_j, \eta_j)$ onto $\mathcal{H}_j(\eta_j)$. With notation (in particular $\phi_\delta(s)$) as above, we define $R_j(\mathcal{D}, \eta, s)$ on $\mathcal{H}_j$ to be $R(\mathcal{D}_j, \eta_j, s)$, where in the notation of (2.2), we have $\mathfrak{d} = \mathcal{D}_j$, $\mathfrak{y} = \eta_j$, and $\mathfrak{H} = \mathcal{H}_j$. More specifically, on $\mathcal{H}_j$ we put

$$R_j(\mathcal{D}, \eta, s) = \bigoplus_\delta \phi_\delta(s) P(\mathcal{D}_j, \eta_j), \tag{2.4}$$

where $\phi_\delta(s)$ is as in (2.1).

On the first page of Section 3 of [9] the representation $\mathcal{D}_j$ of SO(2) on $\mathcal{H}_j$ is constructed exactly as we have done above (we have used slightly different notation). Jacquet [9] then states that the restriction of $P(\mathcal{D}, \eta)$ to $\mathcal{H}_j$ is the projection $P(\mathcal{D}_j, \eta_j)$ onto $\mathcal{H}_j(\eta_j)$, by Lemma 1.2 of [9].

In what follows, due to the functional equation, we will also be interested in estimating $E_{\mathcal{D}, \eta^w, \chi}(g, \lambda)$. Here, $w \in W$ and η^w denotes the character on M_0 given by $\eta^w(m) = \eta(wmw^{-1})$. Notice that $\mathcal{V}_j^\delta$ depends on η, since the underlying generating subspace is $\mathcal{H}(\mathcal{D}, \eta)$.

Let $w \in W$. Similar to the above, we can write $\eta^w(m) = \prod_j \eta_j^w(\Lambda_j(m))$ for $m \in M_0$ and each η_j^w is a character on $\{\pm 1\}$. We also define $\mathcal{H}(\mathcal{D}, \eta^w)$ similarly to the above. We let $\mathcal{H}_j^w$ denote the subspace of $\mathcal{H}$ generated by $\mathcal{D}(k)v^\circ$ for $k \in K_j$ and $v^\circ \in \mathcal{H}(\mathcal{D}, \eta^w)$. Let $\mathcal{D}_j^w$ be the representation of SO(2) on $\mathcal{H}_j^w$. With a similar definition to the above, $\mathcal{H}_j^w$ decomposes as $\bigoplus_\delta \mathcal{V}_j^\delta(w)$. Let $\mathcal{H}_j^w(\eta_j^w)$ be the subset of vectors $v^\circ \in \mathcal{H}_j^w$ such that $\eta_j^w(\Lambda_j(m))v^\circ = \mathcal{D}(m)v^\circ$ for all $m \in M_0^j$. Finally, we denote the projection of $\mathcal{H}_j^w$ onto the subspace $\mathcal{H}_j^w(\eta_j^w)$ as $P(\mathcal{D}_j^w, \eta_j^w)$.

2.3. Lemma 3.2 of [9]. In this section, we review Lemma 3.2 of Jacquet [9], in the case where the field is $\mathbb{R}$. We actually create a function C_j below, which we have effective estimates on, that makes this lemma effective. The bookkeeping that started in the last section, how $\mathcal{D}$ decomposes when restricted to K_j, is necessary to obtain an effective estimate for this lemma. Due to the functional equation under $\lambda \mapsto w_{\alpha_j}(\lambda)$ (see (2.7) below), in estimating $E_{\mathcal{D},\eta,\chi}$ at λ, we will need an estimate of $E_{\mathcal{D},\eta^{w_{\alpha_j}},\chi}$ at $w_{\alpha_j}(\lambda)$. We accomplish this by keeping track of estimates for $E_{\mathcal{D},\eta^w,\chi}$ for all $w \in W$.

Let us suppose $\lambda \in B(V)$ and $w \in W$. For $\alpha_j \in \Delta$ and for each $g \in G$ let us define

$$E_{V_{-\alpha_j},\mathcal{D},\eta^w}(g,\lambda) = \int\limits_{V_{-\alpha_j}} L_{\mathcal{D},\eta^w}(vg,\lambda)\chi(v)\,dv.$$

With this definition, $E_{V_{-\alpha_j},\mathcal{D},\eta^w}(g,\lambda)$ is operator-valued; it still acts on $\mathcal{H}$. Absolute convergence is ensured, for $\lambda \in B(V)$. In fact, if we restrict $E_{V_{-\alpha_j},\mathcal{D},\eta^w}$ to $g \in G^{\alpha_j}$, by pulling back to $\mathrm{SL}_2(\mathbb{R})$ by $\mathfrak{X}_j^{-1}$, we have a one-dimensional Whittaker function, with data as follows (cf. equation (3.1.6) of [9]). The complex parameter is $s_j \in \mathbb{C}$, the K-type is (contained in) $\bigoplus_\delta \mathcal{V}_j^\delta(w)$ under the action of $\mathrm{SO}(2)$ by $\mathcal{D}_j^w$, the unitary character on $\mathcal{Z}_{\mathrm{SO}(2)}(\mathfrak{a}_0)$ is η_j^w, and the generic character on $V_{-\alpha_j}$ is determined by μ_j. The normalization of ρ inside a one-variable integral works precisely because s_j is in the Λ_j direction.

In particular, we have a functional equation extending holomorphy to $\Re s_j \leq 0$. This means $E_{V_{-\alpha_j},\mathcal{D},\eta^w}$ (when restricted to G^{α_j}) can be extended analytically to the convex closure of $B(V) \cup w_{\alpha_j}(B(V))$. In other words, holomorphy extends from $B(V)$ to the reflection of $B(V)$ by w_{α_j}, and also includes the α_j-wall, which is the common wall of these two chambers. Crucial to Jacquet [9] is that this functional equation for $E_{V_{-\alpha_j},\mathcal{D},\eta^w}$, corresponding to reflection of λ by w_{α_j}, remains valid for all $g \in G$, not just G^{α_j}. This is the computation of Lemma 3.2 of [9].

Before stating this lemma, let us briefly recall more notation from [9]. Let us further record some trivial estimates in the following paragraph.

For each j and for each w, let each δ be so that the projection (under $P(\mathcal{D}_j^w,\eta_j^w)$) of $\mathcal{V}_j^\delta(w)$ onto $\mathcal{H}_j^w(\eta_j^w)$ is nonzero, and let us define the function $C_{j,\delta}^w(s)$ for $s \in \mathbb{C}$ as

$$C_{j,\delta}^w(s) = \frac{\pi^{\frac{1}{2}(q_\delta^w + \Re s + 1)}}{\left|\Gamma\left(\frac{|q_\delta^w| + s + 1}{2}\right)\right|}.$$

Here q_δ^w is the integer corresponding to δ. Notice $C_{j,\delta}^w(s) \geq 0$, and is C^∞ on $\mathbb{C}$, but of course not holomorphic. From the discussion above (including equation (1.5) and an estimate similar to Lemma 1.3 for functions on $\mathbb{C}^1$), $C_{j,\delta}^w(s) = O_{j,\delta,\epsilon}(e^{|s|^{1+\epsilon}})$, with the constant depending on ϵ and q_δ^w, and thus on δ and j. Let us define the function

$$C_j(s) = \max_w(\max_\delta C_{j,\delta}^w(s)).$$

Then clearly, $C_j(s)$ satisfies the estimate $O_{\epsilon,j}(e^{|s|^{1+\epsilon}})$. Here, the constant depends on ϵ, and all the possible $\mathrm{SO}(2)$-types δ occurring in $\bigoplus_\delta \mathcal{V}_j^\delta(w)$; and so depends implicitly on K_j, so on j. We record this:

$$C_j(s) = O_\epsilon(e^{|s|^{1+\epsilon}}) \tag{2.5}$$

for all $s \in \mathbb{C}$, with the constant depending only on ϵ, and essentially on the group G and $\mathcal{D}$, since $\mathcal{D}$ is finite-dimensional.

For $\lambda \in B(V)$, let us define

$$E_{V_{-\alpha_j}}(g,\lambda) = \int\limits_{V_{-\alpha_j}} L_{1,1}(vg,\lambda)\,dv. \tag{2.6}$$

The notation $L_{1,1}$ simply means we are taking a function $L_{\mathcal{D},\eta}$, as described above, but we set both the K-type and the character on M_0 to be trivial. Since the relevant projection operator is the identity, in *this* situation, we view $L_{1,1}(g,\lambda)$ and $E_{V_{-\alpha_j}}(g,\lambda)$ to be complex-valued functions on $G \times B(V)$.

We can now state what we will need from Lemma 3.2 of [9]:

LEMMA 2.2 (Lemma 3.2 of [9]). *Let $\mu_j \neq 0$ be as above. Let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$.*

(ii) *For $g \in G$, we have*

$$E_{V_{-\alpha_j},\mathcal{D},\eta}(g,\lambda) \\ = \overline{\eta}_j(\operatorname{sgn}(\mu_j))|\mu_j|^{s_j} E_{V_{-\alpha_j},\mathcal{D},w_{\alpha_j}(\eta)}(g, w_{\alpha_j}(\lambda)) R_j(\mathcal{D},\eta,s_j).$$

(iii) *For $b > 0$, and all $w \in W$, there exists a constant B so that*

$$|E_{V_{-\alpha_j},\mathcal{D},\eta^w}(g,\lambda)v^\circ|_{\mathcal{H}} \leq BC_j(s_j)E_{V_{-\alpha}}(g,\lambda_b)$$

in the region $|\Re s_j| \leq b$ and $\lambda_b = \Re\lambda - \frac{1}{2}(\Re s_j - b)\alpha_j$.

Notice that $w_{\alpha_j}(\eta) = \eta^{w_{\alpha_j}}$. Our contribution here is that we have an effective estimate for $C_j(s_j)$ (estimate (2.5) above), even if $\mathcal{D}$ is reducible. We will refer to this lemma below either as Lemma 2.2, or the effective version of Lemma 3.2 of [9].

Proof of Lemma 2.2. To see this result, we are using the explicit computations of Section 4 of [9] for $L_\varphi(\eta,s)$, for the *privileged* function φ, over $\mathbb{R}$. (See Section 1 of [9] for L_φ.) Actually, B and $C_j(s_j)$ come from the maximum principle (see Sections 1 and 3 of [9]). B does not depend on $\mathcal{D}$ or any η^w. This lemma needs the results of Section 1 of [9], most of which are stated assuming that $\mathcal{D}_j^w$ is irreducible. In Section 1 of [9], it is pointed out that results do extend to the (finite) reducible case. We see this here, in that the function C_j depends upon the possible SO(2)-types δ appearing in the relevant projection of $\bigoplus \mathcal{V}_j^\delta(w)$. ∎

Notice that, in part (iii), λ appears on the left hand side, but on the right, C_j is only a function of s_j. Further, in Λ coordinates, λ_b is exactly a shift of $\Re\lambda$ in the α_j direction, until the s_j coordinate becomes b. If $\Re s_j < 0$, so that $\lambda \in w_{\alpha_j}(B(V))$, the map $\lambda \mapsto \lambda_b$ is a reflection of $\Re\lambda$ (though not necessarily an isometry in either of our coordinate systems) through the α_j-wall.

2.4. Proposition 3.3 of [9]. In this section, we briefly review the proof of Proposition 3.3 of Jacquet [9] in the case where the underlying field is $\mathbb{R}$. This was put in for the benefit of the reader, so it will be easy to see how to modify this proof to obtain effective estimates. We do use the effective version of Lemma 3.2 of [9] from the previous section.

Let us state Proposition 3.3 of [9]: *Let Ψ_j denote the convex closure* (*or hull*) *of $B(V)\cup w_{\alpha_j}(B(V))$. Then holomorphy of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ extends to Ψ_j. Further, for any $\lambda\in\Psi_j$, we have the functional equation*

$$E_{\mathcal{D},\eta,\chi}(g,\lambda)=\overline{\eta}_j(\mathrm{sgn}(\mu_j))|\mu_j|^{s_j}E_{\mathcal{D},w_{\alpha_j}(\eta),\chi}(g,w_{\alpha_j}(\lambda))R_j(\mathcal{D},\eta,s_j). \tag{2.7}$$

For the benefit of the reader, in the remainder of this section, we sketch the proof of this proposition. Clearly, Ψ_j consists of the two chambers $B(V)$ and $w_{\alpha_j}(B(V))$, and the common wall between them, the α_j-wall. Fix $g\in G$. (In [9], g is allowed to vary in a compact set, for uniform convergence issues on G. This is relevant for continuity in the g variable, using Sections 1 and 3 of [9]. We will not need this.)

For each $\alpha_j\in\Delta$, let V^j be the subgroup of V generated by all negative roots distinct from $-\alpha_j$. Then V factors as $V=V^jV_{-\alpha_j}$. It is known the Haar measure on V is the product of the Haar measures of these two subgroups V^j and $V_{-\alpha_j}$ (of course, suitably normalized).

Let us assume first $\lambda\in B(V)$. Due to the absolute convergence of $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ as an integral over V, using the fact that V factors as above, and that χ is a character on V, we can write

$$E_{\mathcal{D},\eta,\chi}(g,\lambda)=\int\limits_{V^j}E_{V_{-\alpha_j},\mathcal{D},\eta}(v^jg,\lambda)\chi(v^j)\,dv^j. \tag{2.8}$$

Using this integral expression for $\lambda\in B(V)$, we want to apply Lemma 3.2 of [9] to extend holomorphy to Ψ_j, with a functional equation.

Let $b>0$, and $v^\circ\in\mathcal{H}$ with $|v^\circ|_{\mathcal{H}}\le 1$. Let us assume Ω is a compact set of Ψ_j, such that if $\lambda\in\Omega$, then $|\Re s_j|\le b$. With λ_b defined as above, we (trivially) have $\lambda_b\in B(V)$. By Lemma 2.2 above (the effective version of Lemma 3.2 of [9]), we have

$$|E_{V_{-\alpha_j},\mathcal{D},\eta}(v^jg,\lambda)v^\circ|_{\mathcal{H}}\le BC_j(s_j)E_{V_{-\alpha}}(v^jg,\lambda_b). \tag{2.9}$$

If we restrict λ to Ω, we can replace the bound $BC_j(s_j)$ above by just a constant B', since C_j is smooth and Ω is compact.

Now, let us suppose $\lambda\in\Omega$. Equation (2.9) gives the estimate

$$|E_{V_{-\alpha_j},\mathcal{D},\eta}(v^jg,\lambda)v^\circ|_{\mathcal{H}}\le B'E_{V_{-\alpha}}(v^jg,\lambda_b)$$

for $\lambda\in\Omega$. Notice that $E_{V_{-\alpha_j},\mathcal{D},\eta}(v^jg,\lambda)$ has been analytically continued to $\Omega \subset \Psi_j$. Up to the factor $\chi(v^j)$, this is the integrand in (2.8) initially defined only for $\lambda \in B(V)$. Now, we integrate $E_{V_{-\alpha_j},\mathcal{D},\eta}(v^j g, \lambda)\chi(v^j)$ (which is now analytic on Ω and satisfies the same bounds as above) over V^j. This gives

$$|E_{\mathcal{D},\eta,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} \le B' \int\limits_{V^j} E_{V_{-\alpha_j}}(v^j g, \lambda_b)\, dv^j$$

for $\lambda \in \Omega$. Since $\lambda_b \in B(V)$, this integral converges absolutely. In particular, $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ has been analytically continued to $\lambda \in \Omega$. Further, convergence is uniform on Ω.

Now, *any* point $\lambda \in \Psi_j$ can be written as $|\Re s_j| \le b$ for $\lambda \in \Omega$ for some positive b and some compact set $\Omega \subset \Psi_j$. It follows $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ has analytic continuation to Ψ_j (which, recall, is the convex closure of $w_{\alpha_j}(B(V)) \cup B(V)$). Further, the functional equation (under $\lambda \mapsto w_{\alpha_j}\lambda$) for $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ comes from (and is the same as) the functional equation for $E_{V_{-\alpha_j},\mathcal{D},\eta}(g,\lambda)$ here.

The proof of Theorem 3.4 of [9] is now easy, in the case χ is generic. In this case, we iterate the functional equation until all chambers are covered. Lemma 3.4.2 of [9] guarantees us that at each step, the meromorphic operator R_j is holomorphic. In this way, we cover all (open) chambers, and all walls between chambers in $\mathbb{C}^n$. This set is connected. The points that the functional equation cannot reach are the points which are on the intersection of walls of chambers. These are now covered, by applying the theorem of Hartogs.

3. Effective estimates. A slight variant of the above sketch of Proposition 3.3 of [9] will be suitable for our purposes. Recall we are fixing $g \in G$.

In the following three subsections we prove many results which are effective versions of results from [9]. In Section 3.1 we prove an effective estimate (Lemma 3.1) for $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ in $B_\epsilon(V)$. Here $B_\epsilon(V)$ is *most* of the positive Weyl chamber (which we denote by $B(V)$), depending on $\epsilon > 0$. In Section 3.2, we prove effective estimates (Lemma 3.2 and Corollary 3.3) which correspond to Proposition 3.3 of [9]. Section 3.2 also uses the effective version (Lemma 2.2 above) of Lemma 3.2 of [9]. In Section 3.3, we prove effective estimates (Proposition 3.4 and Corollary 3.5) which correspond to Theorem 3.4 of [9]. Indeed, Corollary 3.5 is an estimate on M_ϵ. Here M_ϵ is *most* of $\mathbb{C}^n$. More specifically, the complement of M_ϵ is contained in a small region (depending on ϵ) surrounding the intersection of walls of all Weyl chambers.

Due to the functional equation, we need to keep track of estimates of $E_{\mathcal{D},\eta^w,\chi}$ for $w \in W$. An estimate for $E_{\mathcal{D},\eta,\chi}(g,\lambda)$, for $\lambda \notin M_\epsilon$, will come in Section 5. This will need the geometry of Section 4.

3.1. Effective Lemma 3.2 of [9]. In this section, we prove an effective estimate (Lemma 3.1) for the Whittaker function $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ in a set that is *close* to the positive Weyl chamber.

Let us define $B_\epsilon(V)$ to be the closure of the set of points in $B(V)$ that are at least a distance of ϵ away from the walls of $B(V)$. Here, we are taking real Euclidean distance. We will be precise with coordinates, immediately below.

For each j, and any $\epsilon > 0$, let us define the function

$$b_j(\epsilon) = 2\epsilon/\|\alpha_j\|.$$

In coordinates, using the defining properties of the Λ_j, along with the Euclidean properties of the given inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}_{\mathbb{R}}^*$, we have

$$B_\epsilon(V) = \{\lambda \mid \Re s_j \geq b_j(\epsilon)\ \forall j\}.$$

See Figure 1 below, where the example of the real projection of rank 2 is sketched.

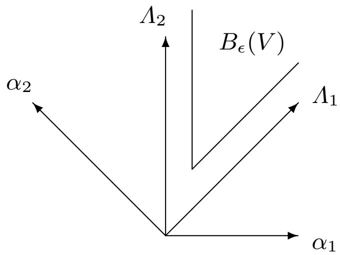

Fig. 1

LEMMA 3.1. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$. For $\lambda \in B_\epsilon(V)$, and all $w \in W$, we have $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{|\Lambda|_\Lambda^{1+\epsilon}})$. The bound depends on ϵ, g, and G.*

Proof. This follows from the original integral expression for $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$. Let $\lambda \in B_\epsilon(V)$. Then $\Re s_j \geq b_j(\epsilon)$. Using the integral expression

$$E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ = \int\limits_V L_{\mathcal{D},\eta^w}(vg,\lambda)v^\circ \cdot \chi(v)\,dv,$$

we can obtain a trivial bound by taking the $\mathcal{H}$-norm inside the integral.

In doing this, $\mathcal{D},\eta^w,\chi$ all become trivial and λ becomes $\Re\lambda$. Now, this new integral factors into rank one intertwining integrals, by the theorem of Gindikin and Karpelevich. Let us assume (for this paragraph) that g, which is fixed, equals the identity, i.e., $g = e$. Since all data are trivial, each factor is (up to a constant) of the form $\Gamma(t)/\Gamma(t+1/2)$ with $t \in \mathbb{R}$ depending on $\Re\lambda$. (This is an explicit computation. See Theorem 6.14 of Helgason [8]. In this computation, $g = e$ is a technical necessity. Note this reference treats a more general case, and uses different notation.) The number of factors appearing is equal to the dimension of $\mathfrak{n}$. We may thus associate each factor to a root in ψ^+. The t corresponding to a particular factor α depends upon how the root $\alpha \in \psi^+$ is written as a sum of nonnegative roots from Δ. In *all* factors, we have $t \gg \epsilon$, with the constant depending only on G. Combining the facts that $\Gamma(t)/\Gamma(t+1/2) \to 0$ as $t \to \infty$, and $\Gamma(t)/\Gamma(t+1/2) \sim 1/\sqrt{\pi}t$ as $t \to 0$, we have the estimate $|E_{\mathcal{D},\eta^w,\chi}(e,\lambda)v^\circ|_{\mathcal{H}} \ll 1/\epsilon^{\dim \mathfrak{n}}$ for $\lambda \in B_\epsilon(V)$. So, actually $|E_{\mathcal{D},\eta^w,\chi}(e,\lambda)v^\circ|_{\mathcal{H}}$ is bounded for $\lambda \in B_\epsilon(V)$. This shows the lemma in the case $g = e$. So, the computation of Theorem 6.14 of [8] gives an effective upper bound for the divergence rate of the trivial estimate of $E_{\mathcal{D},\eta^w,\chi}(e,\lambda)$ as λ approaches any wall of $B(V)$.

Let us now consider the case where g is still fixed, but $g \neq e$. Let us write $g = w_l \tilde{g} w_l^{-1}$. (Recall w_l is the longest element in W, and $V = w_l N w_l^{-1}$. We assume $w_l \in K$.) Let us write

$$\tilde{g} = \tilde{n}\tilde{a}\tilde{m}_0\tilde{k} \quad \text{in terms of the Iwasawa decomposition } NAM_0K.$$

(Since M_0 commutes with A and normalizes N, we can write our minimal parabolic subgroup M_0AN as NAM_0.) Since $M_0 \subset K$, $\tilde{m}_0$ and $\tilde{k}$ may not be unique. So, we have

$$g = v(g)a'm_0'(w_l\tilde{k}w_l^{-1}),$$

where clearly $w_l\tilde{k}w_l^{-1} \in K$, $v(g) = w_l\tilde{n}w_l^{-1} \in V$, $a' \in A$, and $m_0' \in M_0$, since $a'm_0'$ equals the maximal torus element $\tilde{a}\tilde{m}_0$ acted on by w_l. Let $A(h)$ denote the projection of any element $h \in G$ to the split component part, when h is written in its Iwasawa decomposition. Following the method above, it follows we are interested in the integral

$$\int_V (A(vg))^{\Re\lambda+\rho}\,dv = \int_V (A(v \cdot v(g)a'))^{\Re\lambda+\rho}\,dv.$$

Since V is unipotent, $v(g)$ is absorbed into the measure, by a change of variable without changing the integral. By the change of variable $v \mapsto a'v(a')^{-1}$, the above integral is

$$(a')^{-2\rho} \int_V (A(a'v))^{\Re\lambda+\rho}\,dv.$$

Now, clearly, any element $v \in V$ has an N component in terms of its Iwasawa decomposition. Since A normalizes N, we have easily $A(a'v) = a'A(v)$. So, the above expression is

$$(a')^{\Re\lambda-\rho} \int_V (A(v))^{\Re\lambda+\rho}\,dv.$$

This integral is exactly the computation of the last paragraph, and so we know it is bounded, with the constant depending on ϵ and G. It follows that $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)|_{\mathcal{H}} = O_\epsilon(e^{\log(a')\Re\Lambda})$, with the constant depending on ϵ, G, and a' (and so on g).

For $0 < \epsilon' < \epsilon$ we have $B_\epsilon(V) \subset B_{\epsilon'}(V)$, so we can apply Lemma 1.3 on $B_\epsilon(V)$. This gives the estimate, i.e.,

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{|\Lambda|_\Lambda^{1+\epsilon}}),$$

with the constant depending on ϵ, G, and g, under the assumption $|v^\circ|_{\mathcal{H}} \leq 1$. ■

3.2. Effective Proposition 3.3 of [9]. In this section, we prove effective estimates for $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ (Lemma 3.2 and Corollary 3.3) which parallel Proposition 3.3 of [9]. Here, the effective version of Lemma 3.2 of [9] is crucial.

For each each simple root α_j, let us define $B_\epsilon^j(V)$ to be the subset of Ψ_j with the following properties. For each j, recall $b_j(\epsilon) = 2\epsilon/\|\alpha_j\|$. For λ to be in $B_\epsilon^j(V)$, we require that $\lambda \in \Psi_j$, $|\Re s_j| \leq b_j(\epsilon)$, and that $\lambda_{b_j(\epsilon)}$ is on the boundary of $B_\epsilon(V)$ closest (using real Euclidean distance) to the α_j-wall. Using the defining properties of $\{\Lambda_j\}$, the fact that $|\Re s_j| \leq b_j(\epsilon)$ automatically shows that the Euclidean distance from $\Re\lambda$ to the α_j-wall is $\leq \epsilon$, but this alone does not guarantee $\lambda_{b_j(\epsilon)} \in B_\epsilon(V)$.

Let us define $\tilde{B}_\epsilon(V) = \bigcup_j B_\epsilon^j(V)$. See Figure 2, where we have sketched $\tilde{B}_\epsilon(V)$ for our previous rank 2 example. Note this is still the real projection of $\mathfrak{a}_\mathbb{C}^*$. Here $\Re s_1$ and $\Re s_2$ are drawn in the directions of Λ_1 and Λ_2 respectively. Moreover, $\tilde{B}_\epsilon(V)$ consists of the union of the "shaded" regions. In Section 4, we will give a precise description in coordinates of each $B_\epsilon^j(V)$.

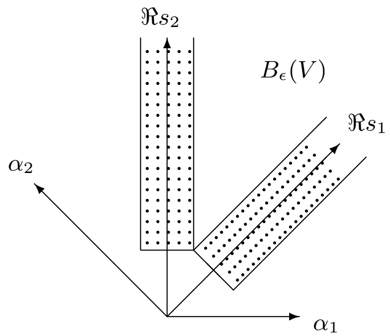

Fig. 2

Lemma 3.2. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$. Then, for $\lambda \in \tilde{B}_\epsilon(V)$, and all $w \in W$, we have the estimate*

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{2|\lambda|_\Lambda^{1+\epsilon}}).$$

Here, the constant depends on ϵ, g, G, $\mathcal{D}$, and χ.

Proof. To see this, assume $\lambda \in \tilde{B}_\epsilon(V)$, and consider once again the integral representation

$$E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ = \int\limits_{V^j} E_{V_{-\alpha_j},\mathcal{D},\eta^w}(v^j g,\lambda)v^\circ \cdot \chi(v^j)\,dv^j. \tag{3.1}$$

(Since $\lambda \in \tilde{B}_\epsilon(V)$, there is a particular (possibly not unique) α_j-wall so that the real Euclidean distance from $\Re\lambda$ to this α_j-wall is minimal. We assume the j above is from this wall.) By part (iii) of the effective version of Lemma 3.2 of [9] (Lemma 2.2 above), we know

$$|E_{V_{-\alpha_j},\mathcal{D},\eta^w}(v^j g,\lambda)v^\circ|_{\mathcal{H}} \le B_j C_j(s_j) E_{V_{-\alpha}}(v^j g,\lambda_b).$$

Here $\lambda_b = \lambda_{b_j(\epsilon)}$. This estimate is valid on *all* of $\tilde{B}_\epsilon(V)$, not just on a compact subset of Ψ_j used in [9], which is relevant for uniform convergence issues, for continuity in the G variable. A constant B appears in the original lemma, which depends upon $b_j(\epsilon)$ (for us), as well as on μ_j. We have denoted this constant by B_j, and we see the dependence is on ϵ, G, and χ (since μ_j is defined by χ).

Let us replace the integrand in (3.1) with this estimate. Since χ has modulus 1, we have

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} \le (\max_j B_j)(\max_j C_j(s_j)) \int\limits_{V^j} E_{V_{-\alpha}}(v^j g,\lambda_b)\,dv^j.$$

Since we are taking $\max_j C_j(s_j)$, it does not matter which $B^j_\epsilon(V)$ the point λ is in, for $\lambda \in \tilde{B}_\epsilon(V)$. Recall that the components of λ_b are real, and by definition, $\lambda_b = \lambda_{b_j(\epsilon)}$ is on the boundary of $B_\epsilon(V)$. Recall that in the integral defining $E_{V_{-\alpha_j}}$ (see (2.6) above) *all* data are trivial. Consequently, when we integrate $E_{V_{-\alpha_j}}(v^j g,\lambda_b)$, we obtain the *exact* estimate used in Lemma 3.1. Thus, by Lemma 3.1, this integral term is $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$, since $\lambda_{b_j(\epsilon)} \in B_\epsilon(V)$. Here the constant depends on ϵ, g, and G.

By estimate (2.5) above, the $\max_j C(s_j)$ term here is $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$, with the constant depending only on ϵ, G, and $\mathcal{D}$. Combining these results shows the lemma. Notice we have picked up a dependence on χ in our estimate. Since $\mathcal{D}$ restricted to M_0 is a direct sum of characters, if η^w is not a combination of some of these characters, then $\mathcal{H}(\mathcal{D},\eta^w)$ is 0. Consequently, we can absorb any dependence on η^w into a dependence on $\mathcal{D}$. ∎

Let us define $M_\epsilon(V) = (\overline{B(V)} \cap \tilde{B}_\epsilon(V)) \cup B_\epsilon(V)$. Here $\overline{B(V)}$ denotes the closure of $B(V)$. In coordinates (in the Λ basis), $\overline{B(V)} = \{\lambda \mid \Re s_j \ge 0\ \forall j\}$. In Section 4, we will need to be more precise with the description of coordinates of $M_\epsilon(V)$ and of each $\overline{B(V)} \cap B^j_\epsilon(V)$.

See Figure 3 below, where we have drawn $M_\epsilon(V)$. Note this figure is magnified slightly over the previous figures. Notice also that *most* of the walls of $B(V)$ are included in $M_\epsilon(V)$.

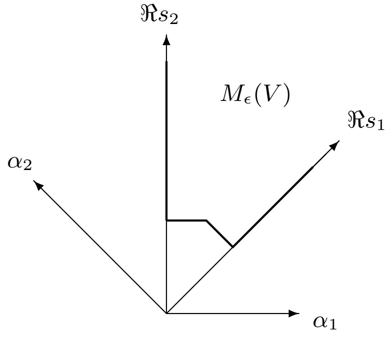

Fig. 3

If $0 < \epsilon' < \epsilon$ then $M_\epsilon(V) \subset M_{\epsilon'}(V)$, so we may apply Lemma 1.3 to $M_\epsilon(V)$. Combining Lemmas 3.1 and 3.2 and applying Lemma 1.3 gives us:

COROLLARY 3.3. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$. For $\lambda \in M_\epsilon(V)$, and all $w \in W$, we have $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$. Here the constant depends on ϵ, g, G, $\mathcal{D}$, and χ.*

3.3. Effective Theorem 3.4 of [9]. In this section, we prove effective estimates for $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ (Proposition 3.4 and Corollary 3.5) that parallel Theorem 3.4 of [9]. This is done, of course, by iterating the functional equation. We actually obtain an estimate on the set M_ϵ (see definition below) which is *most* of $\mathbb{C}^n$.

We must now obtain estimates for $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ in the other chambers. For this, we of course need the functional equation. With our assumption χ is generic (and g is fixed), let us state what we need from Theorem 3.4 of [9]: *Holomorphy of $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ extends to all of $\mathbb{C}^n$. Further, the functional equation, under $\lambda \mapsto w_{\alpha_j}\lambda$, is for any $\lambda \in \mathbb{C}^n$,*

$$(3.2)\quad E_{\mathcal{D},\eta^w,\chi}(g,\lambda) = \overline{\eta_j^w}(\operatorname{sgn}(\mu_j))|\mu_j|^{s_j} E_{\mathcal{D},w_{\alpha_j}(\eta^w),\chi}(g,w_{\alpha_j}(\lambda))R_j(\mathcal{D},\eta^w,s_j).$$

(Notice this is the same as (2.7) but with no restriction on λ.)

We need a bound on the operator $R_j(\mathcal{D},\eta^w,s_j)$. In the following few paragraphs, we briefly record some trivial estimates.

Put $s = s_j$. For $w \in W$, $R_j(\mathcal{D},\eta^w,s)$ is defined above; see the discussion surrounding (2.4). We must recall how $R_j(\mathcal{D},\eta^w,s)$ was constructed, so we refer to (2.1) and (2.2) (with data $\mathfrak{d} = \mathcal{D}_j^w$, $\mathfrak{y} = \eta_j^w$, and $\mathfrak{H} = \mathcal{H}_j^w$). We see by (2.1) that

$$|\phi_\delta^w(s)| = \left|\Gamma\left(\frac{1-s+|q_\delta^w|}{2}\right)\right| \frac{\pi^{\Re s}}{\left|\Gamma\left(\frac{1+s+|q_\delta^w|}{2}\right)\right|}.$$

Here ϕ_δ^w denotes ϕ_δ, where δ is an irreducible representation appearing in $\mathcal{H}_j^w = \bigoplus_\delta \mathcal{V}_j^\delta(w)$. Each δ depends on j and w. Further, q_δ^w is the integer corresponding to each such δ. The first Γ factor here is $O_{j,w,\delta,\epsilon}(e^{|s|^{1+\epsilon}})$ in the half-plane $\Re s \leq 0$, using (1.6). Obviously, the constant will depend on j, w, ϵ, and q_δ^w (and so on δ). The fraction on the right above is $O_{j,w,\delta,\epsilon}(e^{|s|^{1+\epsilon}})$, valid for $s \in \mathbb{C}$ in view of (1.5). Once again, the constant depends on j, w, ϵ, and δ. Both estimates here are using estimates similar to Lemma 1.3 for functions on $\mathbb{C}^1$.

Let us define

$$\Phi_j(s) = \max_w(\max_\delta |\phi_\delta^w(s)|),$$

the inner maximum being taken over all δ in the direct sum $\mathcal{H}_j^w = \bigoplus \mathcal{V}_j^\delta(w)$, so that $\mathcal{V}_j^\delta(w)$ does not project to zero under $P(\mathcal{D}_j^w, \eta_j^w)$. It follows, for $\Re s \leq 0$, that

$$\Phi_j(s) = O_\epsilon(e^{|s|^{1+\epsilon}}), \tag{3.3}$$

by the above paragraph. The constant depends on ϵ, G, and $\mathcal{D}$.

For each $w \in W$ and for each j, let γ_j^w denote the number of isotypic subspaces $\mathcal{V}_j^\delta(w)$ occurring in the direct sum $\mathcal{H}_j^w = \bigoplus \mathcal{V}_j^\delta(w)$. Let us put $\gamma = \max_j(\max_w \gamma_j^w)$. From (3.3), we have, by estimating trivially (if $v^\circ \in \mathcal{H}$ with $|v^\circ|_\mathcal{H} \leq 1$),

$$|R_j(\mathcal{D}, \eta^w, s_j)v^\circ|_\mathcal{H} = \gamma O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}}) \tag{3.4}$$

if $\Re s_j \leq 0$. Notice the right hand side above does not depend on j. (We have done this just for convenience later.) Further, since γ depends on $\mathcal{D}$ and K, we can absorb this into the constant in the O_ϵ term. Moreover, γ depends on η, but is bounded above depending on $\mathcal{D}$ alone, since $\mathcal{D}$ is finite-dimensional. So, we see the dependence is once again on ϵ, G, and $\mathcal{D}$.

We would like to use the functional equation (3.2), Corollary 3.3, and estimate (3.4) to obtain an estimate for all of $\mathbb{C}^n$. This is the estimate generated by acting on $M_\epsilon(V)$ by the entire Weyl group. We will do this, but the set generated this way contains holes. We will obtain an estimate on the remaining region by an effective computation using convexity in $\mathbb{C}^n$.

Let us define $M_\epsilon = \bigcup_{w\in W} w(M_\epsilon(V))$. Let us put $\mu = \max_j\{|\mu_j|, |\mu_j|^{-1}\}$. Recall the notion of the *length* of a Weyl element: $l(w)$ is the smallest number of simple reflections w_{α_j} needed to write w as a product of these w_{α_j}. Let us extend this notion to M_ϵ as follows. For $\lambda \in M_\epsilon$, let us put $l(\lambda) = \min\{l(w) \mid w^{-1}(\lambda) \in M_\epsilon(V)\}$. If $\lambda \in M_\epsilon(V)$, then $l(\lambda) = 0$. Recall l_0 denotes the length of the longest Weyl element. For integers q between 0 and l_0, define B^q to be $\bigcup_w w(M_\epsilon(V))$ where the union is over all elements w of length q. Then $B^0 = M_\epsilon(V)$.

In Figure 4 below, the real projection of our rank 2 example is sketched. Here, M_ϵ consists of all of $\mathbb{R}^2$, with the (open) octagon at the origin removed. Since M_ϵ is a tube domain, the octagon is the real projection of the complement in $\mathbb{C}^2$ of M_ϵ, i.e., $\mathbb{C}^2 \setminus M_\epsilon$. We discuss briefly the simple geometry of what happens in higher rank after our final figure, in Section 6.1. (If $n \geq 3$ then the real projection of $\mathbb{C}^n \setminus M_\epsilon$ is not compact.)

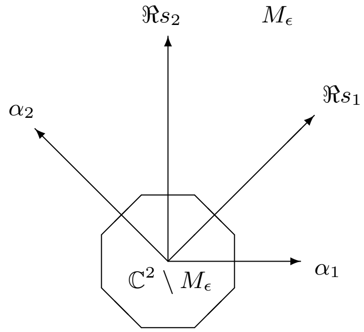

Fig. 4

PROPOSITION 3.4. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$. Then for $\lambda \in M_\epsilon$ and all $w \in W$, we have*

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = \mu^{\frac{c^{2l(\lambda)}-1}{c^2-1}|\lambda|_\Lambda} \cdot O_\epsilon(e^{(c^{2(l(\lambda)+1)}|\lambda|_\Lambda)^{1+\epsilon}}).$$

Here the constant depends on ϵ, g, G, $\mathcal{D}$, and χ.

Proof. This can be seen using induction on $l(\lambda)$, in a similar way to [9]. Since $c > 1$, the inductive step ($\lambda \in B^0$) is taken care of by Corollary 3.3. (We also see a dependence of the constant, from Corollary 3.3.) Let us assume the result is true for all $\lambda \in B^q$.

Let us assume now $\lambda \in M_\epsilon$ with $l(\lambda) = q + 1$. Let us assume λ is in a chamber, and not on a wall. Then there exists $w' \in W$ with $l(w') = q + 1$ and $(w')^{-1}(\lambda) \in M_\epsilon(V) \cap B(V)$.

Since $l(w') = q+1$, by definition there exist j and $w'' \in W$ with $l(w'') = q$ such that we can write $w' = w_{\alpha_j} w''$. By Lemma 3.4.2 of [9], we have $\Re s_j \leq 0$. Further, $l(w_{\alpha_j}(\lambda)) = q$ since $w_{\alpha_j}(\lambda) \in w''(B(V))$; within the Weyl group, since each w_{α_j} is a reflection, $w_{\alpha_j}^2 = 1$. Thus $(w_{\alpha_j})^{-1} = w_{\alpha_j}$. We can now use the functional equation (3.2). (This equation in general is not scalar-valued. The estimate (3.4) and the estimate of Corollary 3.3 are using the norm in $\mathcal{H}$.)

By the inductive hypothesis (recall $l(w_{\alpha_j}(\lambda)) = q$), the functional equation (3.2), and (3.4) (in which $\Re s_j \leq 0$ is crucial), we have, estimating trivially,

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} \le \mu^{|\lambda|_\Lambda}\big(\mu^{\frac{c^{2q}-1}{c^2-1}|w_{\alpha_j}(\lambda)|_\Lambda} \cdot O_\epsilon(e^{(c^{2(q+1)}|w_{\alpha_j}(\lambda)|_\Lambda)^{1+\epsilon}})\big).$$

Here we are applying the inductive hypothesis to the character $w_{\alpha_j}(\eta^w) = \eta^{w_{\alpha_j}\cdot w}$, with $w_{\alpha_j}(\lambda) \in B^q$. Since the given inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}_{\mathbb{R}}^*$ is invariant under W, so is $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $\mathfrak{a}_{\mathbb{C}}^*$. Thus,

$$|w_{\alpha_j}(\lambda)|_\Lambda \le c\|w_{\alpha_j}(\lambda)\|_{\mathbb{C}^n} = c\|\lambda\|_{\mathbb{C}^n} \le c^2|\lambda|_\Lambda,$$

by two applications of Lemma 1.2. Inserting these estimates above, we have

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} \le \mu^{(\frac{c^{2q}-1}{c^2-1}c^2+1)|\lambda|_\Lambda} \cdot O_\epsilon(e^{(c^{2(q+1)}c^2|\lambda|_\Lambda)^{1+\epsilon}}).$$

This is exactly the estimate of the proposition, with $\lambda \in B^{q+1}$. The only restriction for $\lambda \in M_\epsilon$, with $l(\lambda) = q+1$, is that we assumed λ was in a chamber, not on a wall. This restriction can now be removed, just by the continuity of $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)|_{\mathcal{H}}$, since we know $E_{\mathcal{D},\eta^w,\chi}(g,\lambda)$ is holomorphic on all of $\mathbb{C}^n$. Consequently, the proposition holds for $\lambda \in M_\epsilon$ with $l(w) = q+1$, and the full result follows by induction. ∎

As a consequence, we have:

COROLLARY 3.5. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \le 1$. Then for $\lambda \in M_\epsilon(V)$, and all $w \in W$, we have $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$. Here the constant depends on ϵ, g, G, $\mathcal{D}$, and χ.*

Proof. To see this, recall l_0 denotes the length of the longest element of W. By Proposition 3.4, since $c > 1$, it follows that an upper bound for $|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}}$ for $\lambda \in M_\epsilon$ is

$$O_\epsilon\big(e^{[(2(l_0+1))^{1+\epsilon}+\frac{c^{2l_0}-1}{c^2-1}\log\mu]|\lambda|_\Lambda^{1+\epsilon}}\big),$$

where the constant is the same as in Corollary 3.3. Now, l_0 depends on W, hence G. Trivially, c depends on G. Moreover, μ depends on χ and V, hence G. Consequently, the constant term in brackets depends on ϵ, χ, and G.

Now, if $0 < \epsilon' < \epsilon$ we have $M_\epsilon \subset M_{\epsilon'}$ and so we can apply Lemma 1.3 to the set M_ϵ. Thus by Lemma 1.3, the estimate above can be absorbed into an estimate of the form $O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$, for $\lambda \in M_\epsilon$. ∎

Since $|\lambda|_\Lambda \le c\|\lambda\|_{\mathbb{C}^n}$ by Lemma 1.2, we will also have

$$|E_{\mathcal{D},\eta^w,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}})$$

using the comments following Lemma 1.3. This is under the same assumptions as above: $\lambda \in M_\epsilon$, g is fixed, and all $w \in W$. The constant is not the same as in Corollary 3.5, but will still depend on ϵ, g, G, $\mathcal{D}$, and χ.

4. Simple geometry of $\mathfrak{a}_{\mathbb{C}}^*$. In this section, we prove results (Lemmas 4.1 and 4.2) concerning the geometry of M_ϵ, and in particular how the convex closure of M_ϵ contains $\mathbb{C}^n \setminus M_\epsilon$. Recall we have defined above

$$M_\epsilon(V) = (\overline{B(V)} \cap \tilde{B}_\epsilon(V)) \cup B_\epsilon(V) \quad \text{and} \quad M_\epsilon = \bigcup_{w\in W} w(M_\epsilon(V)).$$

Recall Corollary 3.5 above gives us the desired estimate for $E_{\mathcal{D},\eta,\chi}(g,\lambda)$, with $\lambda \in M_\epsilon$.

Thus, we need an estimate in $\mathbb{C}^n \setminus M_\epsilon$. It turns out the real projection of this set is contained within a small Euclidean distance of a finite number of hyperplanes in $\mathbb{R}^n$. This is the conclusion of Lemma 4.1 below, where of course ϵ is relevant. So, in essence, the real projection of M_ϵ is already *most* of $\mathbb{R}^n$.

We still need to be precise in how the convex closure of M_ϵ contains $\mathbb{C}^n \setminus M_\epsilon$. We will be using simple Euclidean geometry for Lemma 4.1, that is, M_ϵ is invariant under W. Lemma 4.2 gives us a precise description of this. It is needed for an effective estimate for $|E_{\mathcal{D},\eta,\chi}(g,\lambda)v^\circ|_{\mathcal{H}}$, for $\lambda \in \mathbb{C}^n \setminus M_\epsilon$, in the following section.

In what follows, it will be useful to have a precise description in coordinates of $M_\epsilon(V)$, and thus of $\overline{B(V)} \cap \tilde{B}_\epsilon(V)$. Now

$$\overline{B(V)} \cap \tilde{B}_\epsilon(V) = \overline{B(V)} \cap \bigcup_j B_\epsilon^j(V) = \bigcup_j (\overline{B(V)} \cap B_\epsilon^j(V)).$$

Let us therefore describe in coordinates each $\overline{B(V)} \cap B_\epsilon^j(V)$.

Thus, fix j_1 and suppose $\lambda \in \overline{B(V)} \cap B_\epsilon^{j_1}(V)$. Recall Ψ_{j_1} is the convex closure of $B(V)$ and $w_{\alpha_{j_1}}(B(V))$. From the above, if $\lambda \in B_\epsilon^{j_1}(V)$, we have first $\lambda \in \Psi_{j_1}$, with the restrictions

$$|\Re s_{j_1}| \le b_{j_1}(\epsilon) \quad \text{and} \quad \lambda_{b_{j_1}(\epsilon)} \in B_\epsilon(V).$$

Recall

$$\lambda_{b_{j_1}(\epsilon)} = \Re\lambda + \tfrac{1}{2}(b_{j_1}(\epsilon) - \Re s_{j_1})\alpha_{j_1}.$$

Now, we can write

$$\alpha_{j_1} = \sum_{j=1}^n \beta_j(j_1)\Lambda_j, \quad \text{with the } \textit{real} \text{ coefficients} \quad \beta_j(j_1) = \frac{2\langle\alpha_{j_1},\alpha_j\rangle}{\|\alpha_j\|^2}.$$

Consequently,

$$\lambda_{b_{j_1}(\epsilon)} = \sum_{j=1}^n \left(\Re s_j + (b_{j_1}(\epsilon) - \Re s_{j_1})\frac{\langle\alpha_{j_1},\alpha_j\rangle}{\|\alpha_j\|^2}\right)\Lambda_j.$$

(Notice the coefficient of Λ_{j_1} is $b_{j_1}(\epsilon)$.) The condition $\lambda_{b_{j_1}(\epsilon)} \in B_\epsilon(V)$ then forces

$$\Re s_j + (b_{j_1}(\epsilon) - \Re s_{j_1})\frac{\langle\alpha_{j_1},\alpha_j\rangle}{\|\alpha_j\|^2} \ge b_j(\epsilon) \quad \text{for each } j \ne j_1.$$

Thus, we can finally write

$$(4.1)\qquad \lambda \in \overline{B(V)} \cap B_\epsilon^{j_1}(V) \Rightarrow\ 0 \le \Re s_{j_1} \le b_{j_1}(\epsilon) \text{ and}$$
$$\Re s_j \ge b_j(\epsilon) - (b_{j_1}(\epsilon) - \Re s_{j_1})\frac{\langle \alpha_{j_1}, \alpha_j\rangle}{\|\alpha_j\|^2}\ \forall j \ne j_1.$$

Let us proceed with a few more definitions. For $j_1 \ne j_2$, let Q_{j_1,j_2} denote the intersection of the α_{j_1}-wall and α_{j_2}-wall of $B(V)$. Let us define

$$F = \bigcup_{w\in W} w(Q_{j_1,j_2}),$$

where the union is also over all $j_1 \ne j_2$. Now, each $\Re Q_{j_1,j_2}$ is a hyperplane of $\mathbb{R}^n$ of corank 2. (This is why the geometry is a little trivial in the $n = 2$ case.) There are $\binom{n}{2}$ such hyperplanes $\Re Q_{j_1,j_2}$. It follows that the number of corank 2 hyperplanes that $\Re F$ consists of is *trivially* bounded by $|W|\binom{n}{2}$, where $|W|$ denotes the order of W.

For $\lambda \in \mathbb{C}^n$, and each $r > 0$, let us define

$$D_r(\lambda) = \{\nu \in \mathbb{C}^n \mid \|\Re\nu - \Re\lambda\| \le r\}.$$

Then $D_r(\lambda)$ is a tube domain. It consists of all points in $\mathbb{C}^n$ that project to the ball in $\mathbb{R}^n$ of radius r (in Euclidean distance) centered about $\Re\lambda$. (Recall $\|\Re\nu - \Re\lambda\|$ is the real Euclidean distance between $\Re\lambda$ and $\Re\nu$.) If H is any subset of $\mathbb{C}^n$, let

$$D_r(H) = \{\nu \in \mathbb{C}^n \mid \exists h \in H \text{ with } \nu \in D_r(h)\},$$

with the obvious similar definition for $D_r^{\Lambda}(H)$.

Let us define the constant

$$d = \max_j \frac{1}{\|\alpha_j\|}.$$

Clearly, d depends only on G.

Lemma 4.1. *$\mathbb{C}^n \setminus M_\epsilon$ is contained in $D_{5\epsilon d}(F)$.*

Proof. To see this, first notice that M_ϵ and F are invariant by W. Since we are only concerned with real Euclidean distance using D_r, we need only show the lemma for $\lambda \in \overline{B(V)} \setminus M_\epsilon(V)$. (Real Euclidean distance is also invariant by W.) In this lemma, we will be using the Λ basis, where $\lambda = (s_1, \ldots, s_n)$ in coordinates.

Let us assume $\lambda \in \overline{B(V)}$, but $\lambda \notin M_\epsilon(V)$. Then $\lambda \notin B_\epsilon(V)$. By the definition above, there exists j_1 with $0 \le \Re s_{j_1} < b_{j_1}(\epsilon)$. Recall the definition of $M_\epsilon(V)$ and $B_\epsilon^j(V)$. Since $\lambda \notin M_\epsilon(V)$, we have $\lambda \notin B_\epsilon^j(V) \cap \overline{B(V)}$ for all j. If we look at coordinates for what this means for j_1, by (4.1) we see there exists $j_2 \ne j_1$ with

$$0 \le \Re s_{j_2} < b_{j_2}(\epsilon) - (b_{j_1}(\epsilon) - \Re s_{j_1})\frac{\langle \alpha_{j_1}, \alpha_{j_2}\rangle}{\|\alpha_{j_2}\|^2}.$$

Notice the term on the right here is greater than or equal to $b_{j_2}(\epsilon)$ since $0 \le \Re s_{j_1} < b_{j_1}(\epsilon)$ and $\langle \alpha_{j_1}, \alpha_{j_2}\rangle \le 0$.

Using the Cauchy–Schwarz inequality in $\mathbb{R}^n$ for the given inner product $\langle\cdot,\cdot\rangle$, in addition to the facts that $\Re s_{j_1} \ge 0$ and $b_{j_1}(\epsilon) = 2\epsilon/\|\alpha_{j_1}\|$, one can easily show

$$b_{j_2}(\epsilon) - (b_{j_1}(\epsilon) - \Re s_{j_1})\frac{\langle \alpha_{j_1}, \alpha_{j_2}\rangle}{\|\alpha_{j_2}\|^2} < 2b_{j_2}(\epsilon).$$

Summarizing, if $\lambda \in \overline{B(V)}$ and $\lambda \notin M_\epsilon(V)$, there exist j_1 and j_2 ($j_1 \ne j_2$) such that

$$0 \le \Re s_j < b_{j_1}(\epsilon) \quad \text{and} \quad 0 \le \Re s_j < 2b_{j_2}(\epsilon).$$

Let $\tilde\lambda \in \overline{B(V)}$ be obtained from λ by setting the j_1 and j_2 coordinates (in the Λ basis) zero, and keeping all other coordinates fixed. Then

$$\|\Re\lambda - \Re\tilde\lambda\| < \sqrt{b_{j_1}(\epsilon)^2 + 4b_{j_2}(\epsilon)^2} < 5\epsilon d.$$

Since $\tilde\lambda \in Q_{j_1,j_2}$ (by construction), $\tilde\lambda \in F$, and the proof is complete. ∎

Recall the orthonormal basis $\mathbf{e} = \{e_1, \ldots, e_n\}$. If H is any subset of $\mathbb{C}^n$, for each j, let us write

$$H + \mathbb{R}e_j = \{\lambda + re_j \mid \lambda \in H \text{ and } r \in \mathbb{R}\}.$$

For each j, let us define $F_j = F + \mathbb{R}e_j$. Trivially, $\Re Q_{j_1,j_2} + \mathbb{R}e_j$ is a hyperplane in $\mathbb{R}^n$ of corank either 1 or 2, depending on whether e_j is contained in $\Re Q_{j_1,j_2}$. It is easy to show that $\Re F_j$ is contained in $|W|\binom{n}{2}$ hyperplanes of corank 1 in $\mathbb{R}^n$.

Let us define

$$r_1 = 2^{n-1}n^2(n-1)|W|\frac{5\epsilon d + 2n}{\pi^{n/2}/\Gamma(n/2+1)}.$$

The denominator here, $\pi^{n/2}/\Gamma(n/2+1)$, is relevant because it is the $\mathbb{R}^n$-volume of the unit ball in $\mathbb{R}^n$ (in the $\mathbf{e}$ basis).

Lemma 4.2. *If $\lambda \in \mathbb{C}^n$ and $r > \max\{r_1, 5\epsilon d + 2n\}$, there exists $z \in \mathbb{C}^n$ with $\|\lambda - z\|_{\mathbb{C}^n} < r$ and $D_{2n}(z) + \mathbb{R}e_j \subset M_\epsilon$ for all j.*

Notice first that λ has *no* restriction here; that is, λ could be any point in $\mathbb{C}^n$.

Proof of Lemma 4.2. The result can be seen from simple geometry and crude combinatorics. Recall $D_r(\lambda)$ above is a tube domain consisting of all points in $\mathbb{C}^n$ that project to the (open) ball of Euclidean radius r and center $\Re\lambda$; $\Re D_r(\lambda)$ is this ball.

Let us denote $\mathbb{R}^n$-volume (i.e., Lebesgue measure on $\mathfrak{a}^*_{\mathbb{R}}$) by vol. Now, $\mathrm{vol}(\Re D_r(\lambda))$ is $\pi^{n/2}r^n/\Gamma(n/2+1)$. If H denotes any hyperplane of $\mathbb{R}^n$ of corank greater than or equal to 1, clearly (for $r > t > 0$)

$$\mathrm{vol}(\Re D_r(\lambda) \cap \Re D_t(H)) \le 2t(2r)^{n-1}.$$

Recall $\Re F_j$ is contained in $|W|\binom{n}{2}$ hyperplanes of corank 1. Thus $\bigcup_j \Re F_j$ consists of $n \cdot \binom{n}{2}|W|$ such hyperplanes. It follows that

$$\operatorname{vol}\Big(\Re D_{5\epsilon d+2n}\Big(\bigcup_j \Re F_j\Big) \cap \Re D_r(\lambda)\Big) \le 2^{n-1} r^{n-1}(5\epsilon d+2n)(n^2(n-1)|W|)$$

when $r > 5\epsilon d + 2n$, which we are assuming.

Using the above inequality, when $r > r_1$, we have

$$\operatorname{vol}\Big(\Re D_{5\epsilon d+2n}\Big(\bigcup_j \Re F_j\Big) \cap \Re D_r(\lambda)\Big) < \Re D_r(\lambda) = \frac{\pi^{n/2} r^n}{\Gamma(n/2+1)},$$

and so there exists a $z \in \Re D_r(\lambda)$ with $z \notin \Re D_{5\epsilon d+2n}(\Re F_j)$ for all j. By the definitions and properties of Euclidean distance, we deduce that

$$\Re D_{2n}(z) \cap (\Re D_{5\epsilon d}(\Re F) + \mathbb{R}e_j) = \emptyset \quad \text{for all } j.$$

Since all $\mathbb{C}^n$ sets here are tube domains, we may assume $z \in D_r(\lambda)$ with $z \notin D_{5\epsilon d+2n}(F_j)$ for all j, with no restriction on $\Im z$.

By the definitions, it is easy to see this means

$$(D_{2n}(z) + \mathbb{R}e_j) \cap D_{5\epsilon d}(F) = \emptyset \quad \text{for all } j.$$

We have no restriction on $\Im z$ yet. Thus, for each j, $D_{2n}(z)+\mathbb{R}e_j \subset \mathbb{C}^n$ is contained in the complement of $D_{5\epsilon d}(F)$. By Lemma 4.1, we have $D_{2n}(z)+\mathbb{R}e_j \subset M_\epsilon$ for each j.

The final conclusion of the lemma is verified, with no restriction on $\Im z$, for $z \in D_r(\lambda)$. All that is left to show is that $\|\lambda - z\|_{\mathbb{C}^n} < r$. Since $z \in D_r(\lambda)$, this follows trivially if we now specify $\Im z = \Im \lambda$, and the proof is complete. ∎

5. $\mathbb{C}^n$ analysis. In this section, we prove our main result, Theorem 5.2. However, we first prove an effective estimate for $E_{\mathcal{D},\eta,\chi}(g,\lambda)$ (this is Proposition 5.1), for $\lambda \notin M_\epsilon$, using the geometric results of the previous section, along with convexity in $\mathbb{C}^n$. Proposition 5.1 and Corollary 3.5 easily give us Theorem 5.2. If $\lambda \notin M_\epsilon$, we obtain an estimate at λ, by actually expanding around the point z, given by Lemma 4.2. An effective convexity estimate is easy to obtain, since λ and z are close to each other, within a distance (in the $\|\cdot\|_{\mathbb{C}^n}$ norm) independent of λ.

Let us define

$$r_2 = \max\{r_1, 5\epsilon d + 2n\} + 1.$$

Then $r_2 > 5$, since $n \ge 2$.

Suppose $\lambda \notin M_\epsilon$. For the rest of this section, λ is *fixed*. Moreover, ν will be our complex multi-variable, and we will be using the **e** basis. Then $\nu = \sum_{j=1}^n \nu_j e_j$ where $\nu_i \in \mathbb{C}$.

Let $v^\circ, \tilde{v}^\circ \in \mathcal{H}$, and for the rest of this section, let us put

$$f(\nu) = \langle E_{\mathcal{D},\eta,\chi}(g,\nu)v^\circ, \tilde{v}^\circ\rangle_{\mathcal{H}}.$$

Without loss of generality, let us assume $|v^\circ|_{\mathcal{H}} \le 1$ and $|\tilde{v}^\circ|_{\mathcal{H}} \le 1$. We have been doing this in using v° prior to this point.

PROPOSITION 5.1. *Suppose $g \in G$ is fixed, and $\lambda \notin M_\epsilon$, v°, and $\tilde{v}^\circ$ are as immediately above. Then, given any $\epsilon > 0$, we have*

$$|f(\lambda)| = O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}}).$$

The constant here depends on ϵ, g, G, $\mathcal{D}$, and χ, but not on $\lambda \notin M_\epsilon$.

Defining f as above allows us to use a *scalar-valued* multiple Cauchy integral to get at the coefficients of f. This is done just for convenience.

Proof of Proposition 5.1. Recall $\lambda \notin M_\epsilon$ is fixed.

By Lemma 4.2, there exists $z \in \mathbb{C}^n$ with

$$\|\lambda - z\|_{\mathbb{C}^n} < r_2 \quad \text{and} \quad D_{2n}(z) + \mathbb{R}e_j \subset M_\epsilon \quad \text{for all } j.$$

As above, let $z = \sum_{j=1}^n z_j e_j$ be the coordinate representation of z in the $\mathbf{e}$ basis, where $z_j \in \mathbb{C}$. From this point on, we also assume that z is fixed.

For each j, let us define the polydisk (with ν as the complex multi-variable)

$$\mathfrak{C}(j) = \{\nu \in \mathbb{C}^n \mid |\nu_j - z_j| < r_2^{2n} \text{ and } |\nu_i - z_i| < 2 \text{ for all } i \ne j\}.$$

Recall $\nu = \sum_{j=1}^n \nu_j e_j$. Notice that z is the center of each polydisk. It is easy to see that for each j, $\mathfrak{C}(j)$ and its closure are contained in $D_{2n}(z) + \mathbb{R}e_j$. Consequently, they are contained in M_ϵ, by Lemma 4.2.

Now, f is holomorphic at $\nu = z$, by Theorem 3.4 of [9]. Thus, it has a multiple Taylor series. Then

$$f(\nu) = \sum_{i_1,\ldots,i_n \ge 0} c_{i_1,\ldots,i_n} (\nu_1 - z_1)^{i_1} \cdots (\nu_n - z_n)^{i_n}. \tag{5.1}$$

We know this series must converge absolutely on all of $\mathbb{C}^n$, but we do not have an estimate yet at $\nu = \lambda$, since $\lambda \notin M_\epsilon$.

For each j, we do know, by the multi-variable Cauchy integral, that

$$c_{i_1,\ldots,i_n} = \frac{1}{(2\pi i)^n} \int_{\partial\mathfrak{C}(j)}\!\!\cdots\!\int \frac{f(\zeta_1,\ldots,\zeta_n)}{(\zeta_1 - z_1)^{i_1+1} \cdots (\zeta_n - z_n)^{i_n+1}}\, d\zeta_1 \cdots d\zeta_n$$

for all j. Here $\partial\mathfrak{C}(j)$ denotes the obvious poly*circle*, the boundary of $\mathfrak{C}(j)$. The point we take advantage of here is that $\partial\mathfrak{C}(j) \subset M_\epsilon$, and here we have an estimate for f.

By the remarks following Corollary 3.5, $|f(\zeta)| = O_\epsilon(e^{\|\zeta\|_{\mathbb{C}^n}^{1+\epsilon}})$ for $\zeta \in \partial\mathfrak{C}(j)$. Consequently,

$$|c_{i_1,\ldots,i_n}| \le \frac{\max_{\zeta\in\partial\mathfrak{C}(j)} O_\epsilon(e^{\|\zeta\|_{\mathbb{C}^n}^{1+\epsilon}})}{(2\pi)^n} \int_{\partial C(j)}\!\!\cdots\!\int \frac{d|\zeta_1| \cdots d|\zeta_n|}{\prod_{t=1}^n |\zeta_t - z_t|^{i_t+1}}.$$

This integral can easily be evaluated. Using the definition of $\mathfrak{C}(j)$, we see

$$|c_{i_1,\dots,i_n}| \le \frac{\max_{\zeta\in\partial\mathfrak{C}(j)} O_\epsilon(e^{\|\zeta\|_{\mathbb{C}^n}^{1+\epsilon}})}{2^{\sum_{t\ne j} i_t}(r_2^{2n})^{i_j}}.$$

The constant in the O_ϵ term here is from Corollary 3.5.

Now,

$$\|\zeta\|_{\mathbb{C}^n} = \|\lambda + (z-\lambda) + (\zeta - z)\|_{\mathbb{C}^n} \le \|\lambda\|_{\mathbb{C}^n} + \|z-\lambda\|_{\mathbb{C}^n} + \|\zeta - z\|_{\mathbb{C}^n},$$

and so

$$\|\zeta\|_{\mathbb{C}^n} \le \|\lambda\|_{\mathbb{C}^n} + r_2 + \sqrt{r_2^{4n} + 4(n-1)} \quad \text{for } \zeta \in \partial\mathfrak{C}(j).$$

It follows by Lemma 1.3 that

$$|c_{i_1,\dots,i_n}| \le \frac{2^{i_j} O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}})}{2^{\sum_{t=1}^n i_t}(r_2^{2n})^{i_j}}. \tag{5.2}$$

The constant in the O_ϵ term here depends on the constants from Corollary 3.5, as well as from Lemma 1.3, but in particular is independent of λ. (So, the dependence is once again on ϵ, g, G, $\mathcal{D}$, and χ, since r_2 depends on n, d, and ϵ; hence on G and ϵ.)

This estimate (5.2) is valid for *all* j. So, multiplying each estimate and taking nth roots, we arrive at

$$|c_{i_1,\dots,i_n}| \le \frac{O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}})}{(2^{(n-1)/n} r_2^2)^{\sum_{t=1}^n i_t}}. \tag{5.3}$$

Let us now estimate $f(\lambda)$. Now $\|\lambda - z\|_{\mathbb{C}^n} \le r_2$. By (5.1), we have

$$|f(\lambda)| \le \sum_{i_1,\dots,i_n\ge 0} |c_{i_1,\dots,i_n}| r_2^{\sum_{t=1}^n i_t}.$$

Inserting estimate (5.3) for $|c_{i_1,\dots,i_n}|$, we have

$$|f(\lambda)| \le O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}}) \sum_{i_1,\dots,i_n\ge 0} \left(\frac{1}{2^{(n-1)/n} r_2}\right)^{\sum_{t=1}^n i_t}.$$

By the definitions, $r_2 > 5$, and this gives us

$$|f(\lambda)| < O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}}) \sum_{i_1,\dots,i_n\ge 0} 5^{-\sum_{t=1}^n i_t}.$$

This series converges, depending only on n, and we have finally shown

$$|f(\lambda)| \le O_\epsilon(e^{\|\lambda\|_{\mathbb{C}^n}^{1+\epsilon}}),$$

where the constant depends on ϵ, g, G, $\mathcal{D}$, and χ, but not on λ. By the comments after Lemma 1.3 it follows that $|f(\lambda)| = O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}})$ for $\lambda \notin M_\epsilon$, with the constant also depending on ϵ, g, G, $\mathcal{D}$, and χ. ∎

Clearly, combining Proposition 5.1 and Corollary 3.5 gives us the main result of this paper.

THEOREM 5.2. *Suppose $g \in G$ is fixed, and let $v^\circ \in \mathcal{H}$ with $|v^\circ|_{\mathcal{H}} \leq 1$. Given any $\epsilon > 0$, we have*

$$|E_{\mathcal{D},\eta,\chi}(g,\lambda)v^\circ|_{\mathcal{H}} = O_\epsilon(e^{|\lambda|_\Lambda^{1+\epsilon}}) \quad \text{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

The constant here depends on ϵ, g, G, $\mathcal{D}$, and χ.

6. Remarks. In the following three subsections, we record several remarks about Theorem 5.2. In Section 6.1 we discuss the convexity analysis in $\mathbb{C}^n$ that was used in Section 5. In particular, we have drawn a figure which demonstrates all of this analysis. This figure is the real projection of the rank 2 example that we have been previously using. We also discuss the geometry in higher rank. In Section 6.2 we sketch the modifications necessary to extend our main result to complex groups. In Section 6.3 we show the trivial case of our result for $n = 1$. (This was technically left out above.)

6.1. $\mathbb{C}^n$-analysis remarks. In this section, we wish to show how our simple $\mathbb{C}^n$-analysis above is a simple case of the general principle that holomorphy in $\mathbb{C}^n$ extends to the convex closure of a domain. We have drawn a figure below (Figure 5) which shows the real projection of all of this analysis, for a rank 2 example. We also discuss what happens geometrically in higher rank.

For a several complex variables reference, we use the book of Bochner and Martin [1]. We will state an easily proved adaptation of Theorem 10 from Section 5 of Chapter 5 of [1] (p. 93).

Fix $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Denote by $\mathfrak{C}(z, \varrho_j)$ the polydisk

$$\{\lambda = (s_1, \ldots, s_n) \in \mathbb{C}^n \mid |z_j - s_j| < \varrho_j\} \quad \text{for } j = 1, \ldots, n.$$

THEOREM. *Let $f(s_1, \ldots, s_n)$ be holomorphic in the n polydisks $\mathfrak{C}(z, \varrho_{i,j})$ for $i = 1, \ldots, n$. Then f has a holomorphic continuation to the union of all polydisks of the form $\mathfrak{C}(z, \varrho_{j,\theta})$ where*

$$\log \varrho_{j,\theta} = \sum_i \theta_i \log \varrho_{i,j},$$

the continuation being effected by the Taylor series expansion for f at z. Here, $\theta_i \geq 0$ and $\sum \theta_i = 1$.

(See [1]. The theorem in [1] only assumes there are two polydisks. The extension from 2 to n polydisks is very easy to see. The geometry of the tube domain M_ϵ is very simple. This is why the analysis above is rather easy, and is why a more difficult effective version of Hartogs' theorem is not needed.)

For $\mathfrak{C}(j)$ above, we have (for each j) $\varrho_{j,j} = r_2^2$ and $\varrho_{i,j} = 2$ for $i \neq j$. Let us take the particular convex combination $\theta_i = 1/n$ for all i. For each j, let us put $\varrho_j = \varrho_{j,\theta_j}$. Then

$$\log \varrho_j = \frac{n-1}{n} \log 2 + \frac{1}{n} \log(r_2^{2n}), \quad \text{so} \quad \varrho_j = 2^{(n-1)/n} r_2^2.$$

In particular, taking nth roots of the product of estimates from formula (5.2) (one term for each j) corresponds to the combination $\theta_i = 1/n$. This is the reason for the appearance of the $2^{(n-1)/n} r_2^2$ term in (5.3). Further, our $\mathfrak{C}(j)$ were defined conveniently so that ϱ_j here does not depend on j.

In Figure 5 below, the real projection of a picture for this analysis is shown following our rank 2 example. (Note the scale is different than in the previous figures.)

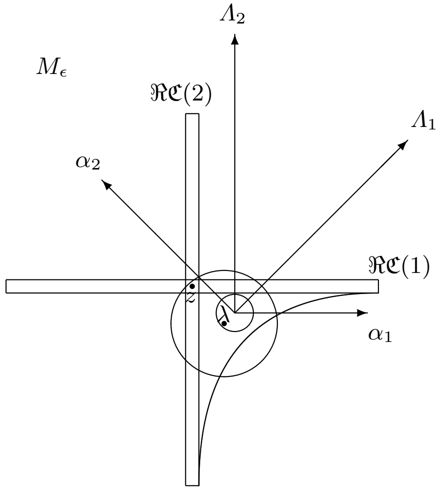

Fig. 5

Now, the octagon from Figure 4 is contained in the small circle, which is centered at the origin. The small circle has radius $5\epsilon d$. By Lemma 4.1, if we stay outside of this small circle, we are in M_ϵ. Now, λ is a point in this small circle, and $\lambda \notin M_\epsilon$. We draw a circle around λ of radius r_2, and we find a point z within this larger circle with the following properties. (Technically z and λ in the picture should be labeled $\Re z$ and $\Re\lambda$.) We construct polydisks $\mathfrak{C}(1)$ and $\mathfrak{C}(2)$, both with center z, that are contained within M_ϵ. We take e_1 to be in the α_1 direction and e_2 to be in the Λ_2 direction. Notice the real projections of these polydisks are rectangles parallel to the e_j directions. Now λ is contained in the *logarithmic* convex closure of these polydisks; this is the basis for our simple estimate. We have denoted this containment by a simple curve connecting one corner of $\Re\mathfrak{C}(1)$ to a corner of $\Re\mathfrak{C}(2)$. Obtaining an estimate at λ

comes from the fact that we have an estimate in the polydisks, which stay within M_ϵ.

In higher rank, the convexity idea is the same, but some geometry is different. In particular, we have n polydisks centered at a point z, within r_2 of $\lambda \notin M_\epsilon$. The point λ is contained in the logarithmic convex closure of these polydisks, which is the basis for the estimate. However, the geometry is not as simple as in the rank 2 case.

Specifically, if $n \geq 3$, then $\Re F$ is *not* compact, and will be a union of corank 2 hyperplanes in $\mathbb{R}^n$. For example, in Figure (15.8) on page 215 of Fulton and Harris [4], a positive Weyl chamber for $\mathrm{SL}_4(\mathbb{R})$ is drawn. Here $n = 3$, and $\Re F$ is a union of lines in $\mathbb{R}^3$. One can see from this picture in [4] that $\bigcup_j \Re F_j$ actually cuts up any M_ϵ into pieces. This was the reason for our volume considerations in Section 4.

6.2. Complex groups. In this section, we briefly discuss the modifications necessary to extend the analysis above, and the result of Theorem 5.2, to complex groups. In the above analysis, we have kept some notation and results for the reader's convenience; they actually simplify in the real case. This is so that the complex case can be checked easily, as described below. (For example, in the real case, η is real, and so there is no need for the bar in $\overline{\eta_j}$, appearing in equation (3.2).)

We suppose G is a Chevalley group (split and reductive) defined over $\mathbb{Q}$ and we now consider the complex points, $G(\mathbb{C})$. By passing to the simply connected cover of the derived group of G, as before, we can assume this group is simply connected, and has semisimple rank $n \geq 2$.

Now the structure theory of Section 1 is entirely similar. Let us assume M_0, A, N, and K have the same definitions as above. One difference is that M_0 can be much bigger than in the real case. Let us assume $\mathcal{D}$, $\mathcal{H}$, η, $P(\mathcal{D}, \eta)$, and $\mathcal{H}(\mathcal{D}, \eta)$ are defined as above.

First, Lemma 3.1 carries over easily, since our reference [8] for the main computation has also included the case of complex groups. For Lemma 3.2 and Corollary 3.3 to carry over, we need an effective version of Lemma 3.2 of [9] for complex groups. This can also be obtained easily. For this, we need to look at the computation in Section 5 of [9]. By the form of $L_\varphi(s)$, we see we can create the functions C_j as in Section 2.3 above, with the same finite order estimate. For the analogue of Proposition 3.4 and Corollary 3.5, we need to look at projection operators of SU(2).

If we look at the relevant projection theory for SU(2) (as we did for SO(2) in Section 2.1), many things are similar. The theory is not quite as trivial as for SO(2), but still very well understood. We will again encounter meromorphic functions $\phi_\delta(s)$ as in equation (2.1) above. Once again, however, these all have a finite order one estimate, if $\Re s \leq 0$, with the constants depending on the SU(2)-types. This can be seen from the specific computation in Section 5 of [9]. This means that when applying the functional equation $\lambda \mapsto w_{\alpha_j}\lambda$, the meromorphic operator $R_j(\mathcal{D}, \eta, s_j)$ retains a finite order bound, in the same way as was used in Proposition 3.4. So, Proposition 3.4 and Corollary 3.5 carry over easily.

The rest of the analysis also carries over very well. In particular, the geometry from Section 4 and the analysis from Section 5 will be the same. We obtain the theorem for split reductive complex groups, and leave the details to the reader.

6.3. The case $n = 1$. There is no convexity in $\mathbb{C}^1$, in the sense that we have used above. However, if $n = 1$, with our assumptions on the group G, we are only considering $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$. For either situation, the result is essentially trivial.

In particular, suppose $E_{\mathcal{D},\eta,\chi}(g, s)$ is our Whittaker function, with $s \in \mathbb{C}$, and g is fixed. By Lemma 3.1, we have an order one bound in $\Re s \geq 1$. By the functional equation, we have an order one bound in $\Re s \leq -1$. This is due to the ratio of Γ factors appearing in the $\phi_\delta(s)$ terms, regardless of whether we are over $\mathbb{R}$ or $\mathbb{C}$. In the strip $|\Re s| \leq 1$, we can use the effective version of Lemma 3.2 of [9] (Lemma 2.2 above). In Lemma 2.2, we found a particular estimate for $C_1(s)$, which is order one. This is if the field is $\mathbb{R}$. By the discussion in the previous section, if the field is $\mathbb{C}$, effective estimates (which are also order one) can be found by the same process, looking at $L_\varphi(s)$, in Section 5 of [9].

7. An application to L-functions. In this section, we use Theorem 5.2 to prove that automorphic L-functions appearing in the Langlands–Shahidi method are bounded in vertical strips, away from their (possibly finite number of) poles. This is Theorem 7.1 in Section 7.2. We refer the reader to the Introduction of McKee [15] (and to the references there) for the importance of this result in using a converse theorem in the proof of cases of functoriality.

In Section 7.1 we recall much of the setup of the Langlands–Shahidi method. We only review what we will need. For example we do not cover the constant term calculation. A concise review of results from this method can be found in Kim's notes [2]. For simplicity, we work only over $\mathbb{Q}$; there is, however, no obstruction in working over a general number field. Since Jacquet [9] is our main reference for the above estimates, it is necessary that our group be split. We apologize to the reader for not using in Sections 7.1 and 7.2 the same notation from the previous six sections. This is done so that our notation more accurately matches the notation from Langlands–Shahidi references.

We must point out that Theorem 7.1 was first proved in Gelbart and Shahidi [6] and essentially reproved in Gelbart and Lapid [5], by a more direct procedure. (In [5], the completed L-functions, without the archimedean factors, are proven to be meromorphic of finite order. This is only one step away from boundedness in vertical strips.) Granting Theorem 5.2, we view our Theorem 7.1 as a simplified proof of the original work of [6]. As in [5], we do not need to use the delicate complex analysis result, Matsaev's theorem, that [6] depends on. The analogous result of Gelbart and Lapid [5] (Theorem 2 of Section 3 of [5]) is actually more general, since one does not need to assume the automorphic cuspidal representation π on $\mathbf{M}(\mathbb{A}_{\mathbb{Q}})$ (see Section 7.1) is generic. We would like to point out that the original work that went in to our Theorem 5.2 (McKee [14]) predates [5] by several years, but was never published. Further, we would like to emphasize that our Theorem 7.1 and the analogous results in [5] would not be possible without the estimates of Müller [17]. The work [17] was not available to Gelbart and Shahidi [6]; they used estimates in an earlier work of Müller [16], which also was indispensable for their result.

7.1. Langlands–Shahidi preliminaries. In this section we review what we need from the Langlands–Shahidi method. We do not keep the same notation from the previous six sections.

Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{Q}$, which is reductive and split. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$. Let $|\cdot|_p$ denote the p-adic valuation of an element in $\mathbb{Q}_p$. (If $p=\infty$ this is regular absolute value on $\mathbb{R}$.) Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Then we can write $\mathbf{B}=\mathbf{TU}$ with $\mathbf{T}$ a split (over $\mathbb{Q}$) torus and $\mathbf{U}$ a full unipotent radical of $\mathbf{B}$. Let $\mathbf{P}=\mathbf{MN}$ be the Levi decomposition of a maximal subgroup containing $\mathbf{B}$. Here $\mathbf{T}\subset\mathbf{M}$ and $\mathbf{N}\subset\mathbf{U}$. We write B_p, G_p, M_p, N_p, P_p, U_p, and T_p for the corresponding groups over $\mathbb{Q}_p$. For every prime p (including $p=\infty$), a maximal compact subgroup K_p of G_p can be selected so that $\mathbf{K}(\mathbb{A})=\bigotimes_p K_p$, where for almost all finite p we have $K_p=G(\mathbb{Z}_p)$. We write $B=\mathbf{B}(\mathbb{A})$, $G=\mathbf{G}(\mathbb{A})$, and so on. Then $G=PK$.

We define $\mathbf{A}$ to be the split component of the center of $\mathbf{M}$, so that $\mathbf{A}\subset\mathbf{T}$. If H denotes any $\mathbb{Q}$-group we define $X(H)_{\mathbb{Q}}$ to be the $\mathbb{Q}$-rational characters of H. We denote by $\mathfrak{a}$ the real Lie algebra of $\mathbf{A}$ which is $\mathrm{Hom}(X(\mathbf{A})_{\mathbb{Q}},\mathbb{R})$. We denote the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$, which is $X(\mathbf{M})_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{R}=X(\mathbf{A})_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{R}$. The complex dual of $\mathfrak{a}$ is denoted by $\mathfrak{a}^*_{\mathbb{C}}$ and it is $\mathfrak{a}^*\otimes_{\mathbb{R}}\mathbb{C}$.

Now the embedding $X(\mathbf{M})_{\mathbb{Q}}\subset X(\mathbf{M})_{\mathbb{Q}_p}$ induces a map $\mathrm{Hom}(X(\mathbf{A})_{\mathbb{Q}_p},\mathbb{R})=\mathfrak{a}_p\to\mathfrak{a}$. There exists a homomorphism

$$H_M : M\to\mathfrak{a}\quad\text{defined by}\quad \exp\langle\chi,H_M(m)\rangle=\prod_p|\chi(m_p)|_p.$$

Here $m = (m_p)$, $\chi \in X(\mathbf{M})_{\mathbb{Q}}$, and the product is over all primes p including $p = \infty$. We extend H_M to G by making it trivial on N and K. We denote this extension by H_P. We call H_P a *Harish-Chandra homomorphism*, and we refer the reader to Gelbart and Shahidi [6] or Kim's notes [2] for more information.

Let Σ denote the set of $\mathbb{Q}$-roots of $\mathbf{T}$. Then $\Sigma = \Sigma^+ \cup \Sigma^-$ where Σ^+ denotes the set of positive roots (which generate $\mathbf{U}$) and Σ^- denotes the set of negative roots. Let W be the Weyl group; it acts on $\mathbf{T}$ in the usual manner. Let $\Delta \subset \Sigma^+$ denote the set of simple roots. Since $\mathbf{P}$ is maximal, there exists a unique root $\alpha \in \Delta$ of $\mathbf{N}$. Let $\rho_{\mathbf{P}}$ denote half the sum of the roots generating the Lie algebra of $\mathbf{N}$. Let us define

$$\tilde{\alpha} = \frac{\rho_{\mathbf{P}}}{\langle \rho_{\mathbf{P}}, \alpha \rangle} \in \mathfrak{a}^*.$$

The inner product above comes from the Killing form. We identify $\mathbb{C}$ with a subspace of $\mathfrak{a}^*_{\mathbb{C}}$ by $s \in \mathbb{C} \leftrightarrow s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$.

Since $\mathbf{P}$ is maximal, it is generated by $\theta = \Delta \setminus \{\alpha\}$. There exists a unique Weyl element $w_0 \in W$ such that $w_0(\theta) \subset \Delta$ and $w_0(\alpha) \in \Sigma^-$. We define $\mathbf{P}'$ to be the maximal parabolic generated by $w_0(\theta)$. It then has a Levi decomposition $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ where $\mathbf{N}' \subset \mathbf{U}$.

Let π be an automorphic unitary cuspidal representation of $\mathbf{M}(\mathbb{A})$. Then a function ϕ in the space of π is an element of $L^2_0(Z_{\mathbf{M}}(\mathbb{A})\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A}))$. This means ϕ transforms according to a grossencharacter, by right translation of an element in $Z_{\mathbf{M}}(\mathbb{A})$, the center of $\mathbf{M}$, and is square integrable on $Z_{\mathbf{M}}(\mathbb{A})\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})$. Furthermore ϕ is cuspidal, meaning integrals of ϕ (right translated by any element in $\mathbf{M}$) over any unipotent radical of $\mathbf{M}(\mathbb{A})$ vanish, and the space cut out by right translation of ϕ is irreducible. Let $I(s,\pi) = I(s\tilde{\alpha},\pi)$ be the parabolically induced space

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}\, \pi \otimes \exp(\langle s\tilde{\alpha}, H_P(\cdot)\rangle) \otimes 1$$

where H_P is the Harish-Chandra homomorphism above. Let $f_s \in I(s,\pi)$. We assume f_s is K_∞-finite. If $f_s = \phi \exp(\langle s\tilde{\alpha} + \rho_{\mathbf{P}}, H_P(\cdot)\rangle)$ we define the Eisenstein series

$$E(s,\pi,\phi,g) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} \phi(\gamma g) \exp(\langle s\tilde{\alpha} + \rho_{\mathbf{P}}, H_P(\gamma g)\rangle). \tag{7.1}$$

Now $\mathbf{U}/[\mathbf{U},\mathbf{U}] = \prod_{\alpha\in\Delta} U_\alpha$. Let $\psi = \bigotimes \psi_p$ be a generic character on $\mathbf{U}(\mathbb{Q})\backslash\mathbf{U}(\mathbb{A})$. This means each ψ_p is nontrivial on each U_α. Put $\mathbf{U}_{\mathbf{M}} = \mathbf{U} \cap \mathbf{M}$, the unipotent radical of $\mathbf{M}$. Let us define $\psi_{\mathbf{M}} = \psi|_{\mathbf{U}_{\mathbf{M}}}$. Then $\psi_{\mathbf{M}}$ is generic on $\mathbf{U}_{\mathbf{M}}$. For a function ϕ in the space of π (as above) we define the Whittaker function

$$W_\phi(g) = \int_{\mathbf{U}_{\mathbf{M}}(\mathbb{Q})\backslash\mathbf{U}_{\mathbf{M}}(\mathbb{A})} \phi(ug)\overline{\psi_{\mathbf{M}}(u)}\, du.$$

From now on, we assume there exists (and we fix) a generic ψ as above so that the $\psi_{\mathbf{M}}$-Whittaker function W_ϕ does not vanish.

Now there is a nonunique factorization $\pi = \bigotimes \pi_p$ where π_p is a unitary irreducible representation of M_p for all p. Further, for almost all finite p, π_p is spherical. It is well known that in this case π_p injects into an induced representation on M_p (a principal series), where the induced data comes from an unramified quasi-character χ_p on the maximal torus T_p of M_p, over $\mathbb{Q}_p$. (The quasi-character is also given by the Satake isomorphism.)

For H any reductive $\mathbb{Q}$-group, let ${}^LH$ denote the L-group of H, i.e., the Langlands dual group. Put ${}^LH_p$ for the L-group of H over $\mathbb{Q}_p$. Suppose r is a finite-dimensional complex representation of ${}^LH$. There is a natural homomorphism ${}^LH_p \to {}^LH$ and we put r_p for the composition with this representation; r_p is then a representation for ${}^LH_p$.

Now ${}^LM$ acts on ${}^L\mathfrak{n}$, the (complex) Lie algebra of ${}^LN$ by adjoint action. Let r denote this action; we decompose r into its irreducible constituents $\bigoplus_{j=1}^m r_j$ on ${}^L\mathfrak{n} = \bigoplus_{j=1}^m V_j$. It is known that each V_j consists of all roots $\beta^\vee$ in ${}^L\mathfrak{n}$ so that $\langle \tilde{\alpha}, \beta^\vee \rangle = j$. For each finite p such that π_p is unramified we let t_p be the semisimple conjugacy class in ${}^LT_p$ given by the Satake isomorphism. (Let $(r_j)_p$ denote the corresponding representation on ${}^LM_p$.) Let ω_p be a uniformizer in $\mathbb{Z}_p$. (Clearly we can take $\omega_p = p$.) Then $\chi_p \circ \beta^\vee(\omega_p) = \beta^\vee(t_p)$. For each such j, the local Langlands L-function is then given by

$$L(s, \pi_p, (r_j)_p) = \prod_{\langle \tilde{\alpha}, \beta^\vee \rangle = j} L(s, \chi_p \circ \beta^\vee),$$

where $L(s, \chi_p \circ \beta^\vee) = (1 - \chi_p \circ \beta^\vee(\omega_p) p^{-s})^{-1}$. The product above is of course over all roots $\beta^\vee$ in ${}^L\mathfrak{n}$.

Let S be a finite set of primes (including $p = \infty$) such that for all $p \notin S$, π_p and ψ_p are unramified. For each such j, we then define the partial L-function

$$L_S(s, \pi, r_j) = \prod_{p \notin S} L(s, \pi_p, (r_j)_p).$$

With π and ψ as above, and the Eisenstein series defined as (7.1), we have a nonconstant term

$$E_\psi(s, \pi, \phi, g) = \int_{\mathbf{U}(\mathbb{Q}) \backslash \mathbf{U}(\mathbb{A})} E(s, \pi, \phi, ug) \overline{\psi(u)} \, du. \tag{7.2}$$

By computations of Shahidi (see Kim's notes [2]), we have

$$E_\psi(s, \pi, \phi, e) = \frac{\prod_{p \in S} W_{s,p}(e)}{\prod_{j=1}^m L_S(1 + js, \pi, r_j)}. \tag{7.3}$$

(We have used Kim's parameterization of Satake parameters [2], so there are no contragredients of r_j terms.) Here we have factored $f_s = \bigotimes_p f_{s,p} \in$

$\bigotimes_p I(s,\pi_p)$, so that for almost all finite p, $f_{s,p} = f_p^0$ is the unique K_p-fixed function, normalized by $f_p^0(e) = 1$. For finite $p \in S$, we have

$$W_{s,p}(e) = \int\limits_{N'(\mathbb{Q}_p)} \overline{\psi(n')}\lambda_{M_p}(f_{s,p}(w_0^{-1}n'))\,dn'$$

where λ_{M_p} is the local ψ_{M_p}-Whittaker functional. We will deal with $W_{s,\infty}(e)$, the archimedean Whittaker function, in the next section.

7.2. Application to automorphic L-functions. In this section we show how Theorem 5.2 can be used to prove the boundedness in vertical strips of Langlands–Shahidi L-functions. This is our Theorem 7.1 below. This of course uses Müller [17], which is the real heavy machinery.

With π, r_j and other notation as above, let $L(s,\pi,r_j)$ be the *completed* automorphic L-function attached to π (for $s \in \mathbb{C}$), where the local factors for finite $p \in S$ are defined as in Shahidi [24]. It has been some time since Gelbart and Shahidi [6] appeared, and much has happened. By the work of H. Kim and W. Kim, Assumption 2.1 in [6] has been proved in many cases (see [10]–[12]). Indeed, it has been reduced to the standard module conjecture and Shahidi's conjecture on tempered L-functions which is now proved in general by Heiermann and Opdam [7]. With this assumption on normalized intertwining operators proved, this means (as shown in [6]) that each (completed) $L(s,\pi,r_j)$ has only a finite number of poles in $\mathbb{C}$. Further, as explained in [6], we can assume π is normalized so that all potential poles are real.

Given a real closed interval I and $\varepsilon > 0$, following [6] we let $T_{\varepsilon,I}$ be the set of all $z \in \mathbb{C}$ such that $\Re(z) \in I$ and $|\Im(z)| > \varepsilon$.

Our application is a simplification of the proof of the following:

THEOREM 7.1. *For each $L(s,\pi,r_j)$, $1 \le j \le m$, there exists $\varepsilon > 0$ such that $L(s,\pi,r_j)$ is bounded in every $T_{\varepsilon,I}$.*

In other words, away from its poles, each $L(s,\pi,r_j)$ is bounded in vertical strips. Let us reiterate that this was proved in Gelbart and Shahidi [6], and essentially reproved in Gelbart and Lapid [5].

Proof of Theorem 7.1. By Müller [17, Theorem 0.2], $E(s,\pi,\phi,g)$ is meromorphic of finite order for $s \in \mathbb{C}$. More specifically, there exists an entire function $q(s)$ of finite order such that for any compact set $\Omega \in \mathbf{G}(\mathbb{A})$ there exist $c_1, n > 0$ such that

$$|q(s)E(s,\pi,\phi,g)| = O(e^{c_1|s|^n}) \quad \text{ for all } g \in \Omega.$$

Here the constant in the O term as well as c_1 will depend on Ω, but the orders of $q(s)$ and n only depend on $\mathbf{G}$ (see [17]). Since $\mathbf{U}(\mathbb{Q}) \setminus \mathbf{U}(\mathbb{A})$ is compact and ψ has modulus one, using (7.2) we see that

$$q(s)E_\psi(s,\pi,\phi,e)$$

is entire of finite order.

Using (7.3) we see that

$$\frac{\prod_{p\in S} W_{s,p}(e)}{\prod_{j=1}^m L_S(1+js,\pi,r_j)}$$

is meromorphic of finite order. According to [6] local data can be chosen for each finite $p \in S$ so that each such $W_{s,p}(e)$ is a constant (independent of $s \in \mathbb{C}$). By Casselman's subrepresentation theorem,

$$\pi_\infty \hookrightarrow I(\tilde{\nu},\eta_0),$$

a principle series representation of a minimal parabolic subgroup. Here $\tilde{\nu} = s\tilde{\alpha} + \nu_0$ is in the complex dual of the Lie algebra of $\mathbf{T}$. Further, η_0 is a character of T_∞. By assumption, the image is K_∞-finite. By shifting ν_0 we may assume η_0 is a character of $T_\infty \cap K_\infty$ (in the notation of the first six sections, this was M_0). Thus, by Theorem 5.2 and Lemma 1.3, $W_{s,\infty}(e)$ is of order one. Hence

$$\prod_{j=1}^m L_S(1+js,\pi,r_j)$$

is meromorphic of finite order.

We must now consider the local L-factors for $p \in S$. For $p = \infty$ each $L(1+js,\pi_p,(r_j)_p)$ is meromorphic of finite order, since any shift of the Γ function is (as well as any power of the transcendental number π). For finite $p \in S$, by construction, each $L(1+js,\pi_p,(r_j)_p)$ is a rational function in p^{-s}, so is clearly meromorphic of finite order. Thus

$$\prod_{j=1}^m L(1+js,\pi,r_j)$$

is meromorphic of finite order.

If $m = 1$, then $r = r_1$ and $L(s,\pi,r)$ is meromorphic of finite order. Hence we can write it as $f_1(s)/f_2(s)$ where both f_1 and f_2 are entire of finite order. Since $L(s,\pi,r)$ only has a finite number of poles, by the Weierstrass product factorization, we may assume $f_2(s)$ is a polynomial. Let $\sigma_1 > 0$ be sufficiently large so that $L_S(s,\pi,r)$ converges absolutely and hence is bounded on $\Re(s) \geq \sigma_1$. Further, we assume σ_1 is sufficiently large so that each $L(s,\pi_p,r_p)$ (for $p \in S$) is holomorphic for $\Re(s) \geq \sigma_1$. By the decay of the Γ function coming from the archimedean L-factor ($p = \infty \in S$), it follows that $f_1(s)$ is bounded on $\Re(s) = \sigma_1$. (Essentially the local L-factors for finite $p \in S$ do not cause problems. They are holomorphic for $\Re(s) \geq \sigma_1$ and are periodic on vertical lines.) Let us assume that $\sigma_2 < 0$ and that

$-\sigma_2$ is sufficiently large so that $L_S(1-s,\pi,\tilde{r})$ converges absolutely and hence is bounded for $\Re(s) \leq \sigma_2$. (Further we assume $-\sigma_2$ is sufficiently large so that the local L-factors for $p \in S$ of the completed $L(1-s,\pi,\tilde{r})$ are holomorphic for $\Re(s) \leq \sigma_2$.) By the form of the ε-factor in the functional equation $L(s,\pi,r) = \varepsilon(s,\pi,r)L(1-s,\pi,\tilde{r})$ and due to the Γ function in the archimedean L-factor for $L(1-s,\pi,\tilde{r})$ it follows that $f_1(s)$ is bounded on $\Re(s) = \sigma_2$. (Once again the local L-factors for finite $p \in S$ of $L(1-s,\pi,\tilde{r})$ do not cause any problems.) Since $f_1(s)$ is of finite order, it is bounded in the vertical strip $\sigma_2 \leq \Re(s) \leq \sigma_1$ by the theorem of Phragmén–Lindelöf. Since σ_1 and $-\sigma_2$ were sufficiently large, but otherwise arbitrary, it follows that $f_1(s)$ is bounded in all vertical strips of finite width. This is enough to prove the theorem in the case $m = 1$. In addition we see that $L(s,\pi,r)$ is meromorphic of finite order one.

If $m \geq 2$ one can use induction to reduce to the case $m = 1$. This is explained rather thoroughly in [6] (specifically the references found in [6]) and is also outlined in Kim's notes [2]. In short (for us), for each r_j with $j \geq 2$, $L(s,\pi,r_j)$ appears as the *first* L-function in some other Langlands–Shahidi situation, that is, with a different reductive split group and Levi factor. See also Shahidi [23]. ∎

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References

[1] S. Bochner and W. Martin, *Several Complex Variables*, Princeton Univ. Press, Princeton, NJ, 1948.

[2] J. Cogdell, H. Kim and M. Murty, *Lectures on Automorphic L-functions*, Fields Inst. Monogr. 20, Amer. Math. Soc., Providence, RI, 2004.

[3] L. Cohn, *Analytic Theory of the Harish-Chandra C-function*, Lecture Notes in Math. 429, Springer, Berlin, 1974.

[4] W. Fulton and J. Harris, *Representation Theory*, Grad. Texts in Math. 129, Springer, Berlin, 1996.

[5] S. Gelbart and E. Lapid, *Lower bounds for L-functions at the edge of the critical strip*, Amer. J. Math. 128 (2006), 619–638.

[6] S. Gelbart and F. Shahidi, *Boundedness of automorphic L-functions in vertical strips*, J. Amer. Math. Soc. 14 (2001), 79–101.

[7] V. Heiermann and E. Opdam, *On the tempered L-function conjecture*, Amer. J. Math., to appear.

[8] S. Helgason, *Groups and Geometric Analysis*, Math. Surveys Monogr. 83, Amer. Math. Soc., Providence, RI, 2000.

[9] H. Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France 95 (1967), 243–309.

[10] H. Kim, *On local L-functions and normalized intertwining operators*, Canad. J. Math. 57 (2005), 535–597.

[11] H. Kim and W. Kim, *On local L-functions and normalized intertwining operators II; Quasi-split groups*, in: On Certain L-Functions, Clay Math. Proc. 11, Amer. Math. Soc., Providence, RI, 2011, 265–295.

[12] W. Kim, *Square integrable representations and the standard module conjecture for general spin groups*, Canad. J. Math. 61 (2009), 617–640.

[13] R. Langlands, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math. 544, Springer, Berlin, 1976.

[14] M. McKee, *On the finite order of Whittaker functions, Eisenstein series, and automorphic L-functions*, thesis, Princeton Univ., 2003.

[15] M. McKee, *An infinite order Whittaker function*, Canad. J. Math. 61 (2009), 373–381.

[16] W. Müller, *The trace class conjecture in the theory of automorphic forms*, Ann. of Math. 130 (1989), 473–529.

[17] W. Müller, *On the singularities of residual intertwining operators*, Geom. Funct. Anal. 10 (2000), 1118–1170.

[18] G. Schiffmann, *Intégrales d'entrelacement et fonctions de Whittaker*, Bull. Soc. Math. France 99 (1971), 3–72.

[19] F. Shahidi, *Functional equation satisfied by certain L-functions*, Compos. Math. 37 (1978), 171–207.

[20] F. Shahidi, *Whittaker models for real groups*, Duke Math. J. 47 (1980), 99–125.

[21] F. Shahidi, *On certain L-functions*, Amer. J. Math. 103 (1981), 297–355.

[22] F. Shahidi, *Local coefficients as Artin factors for real groups*, Duke Math. J. 52 (1985), 973–1007.

[23] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. 127 (1988), 547–584.

[24] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. 132 (1990), 171–193.

[25] J. Shalika, A. *The multiplicity one theorem for* GL_n, Ann. of Math. 100 (1974), 171–193.

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