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Algebraic integers as values of elliptic functions

by

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0. Introduction. In this paper, we shall deal with certain algebraic integers as values of elliptic functions constructed from the Weierstrass $\wp$-function by using infinite products (Theorem 2.2). In the process we are able to reprove the well known fact that $j(\tau)$ is an algebraic integer for an imaginary quadratic τ; our proof seems to be quite simple and elementary unlike the others ([3]–[8]). And in Section 3 we shall derive analogues (Theorem 3.2) of Berndt–Chan–Zhang's results, which could be a generalization in the case of m even. In the last section, we explore some algebraic properties of values of the Weierstrass $\wp$-function and Fricke functions.

1. Infinite product formulas for the Weierstrass $\wp$-function. Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathfrak{h}$) be a lattice and $z \in \mathbb{C}$. The *Weierstrass $\wp$-function* (relative to Λ_τ) is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\},$$

and the *Eisenstein series of weight* $2k$ (for Λ_τ and $k > 1$) is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}} \omega^{-2k}.$$

We shall use the notations $\wp(z)$ and G_{2k} instead of $\wp(z;\Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ has been fixed.

Then the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}$$

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and, for all $z \in \mathbb{C} - \Lambda_\tau$ we obtain the equation

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6.$$

As is customary, the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau),$$

where

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6.$$

Moreover, we have the following proposition at hand which will be useful in extracting infinite product expressions.

PROPOSITION 1.1 ([4], [8]). *Let* $p = e^{\pi i \tau}$.

(1) $\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 \prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n-1})^8.$

(2) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 \prod_{n=1}^{\infty}(1-p^{2n})^4(1-p^{2n-1})^8.$

(3) $\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right) = 16\pi^2 p \prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n})^8.$

Now, for simplicity we set

$$C := \prod_{n=1}^{\infty}(1-p^n), \qquad D := \prod_{n=1}^{\infty}(1+p^n), \qquad S := \prod_{n=1}^{\infty}(1-p^{2n}),$$

$$T := \prod_{n=1}^{\infty}(1+p^{2n-1}), \quad U := \prod_{n=1}^{\infty}(1+p^{2n}), \quad V := \prod_{n=1}^{\infty}(1-p^{2n-1}).$$

We then readily check that

$$CD = S, \quad TU = D, \quad SUVT = S, \quad VUT = 1. \tag{1.0}$$

By definition

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24} = g_2(\tau)^3 - 27g_3(\tau)^2,$$

which is the discriminant of the cubic polynomial

$$4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau) = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

On the other hand, we know ([9]) that the roots of this polynomial are

$$e_1 = \wp\left(\frac{1}{2}\right), \quad e_2 = \wp\left(\frac{\tau}{2}\right), \quad e_3 = \wp\left(\frac{\tau+1}{2}\right).$$

Thus, we have

$$\wp\left(\frac{1}{2}\right) + \wp\left(\frac{\tau}{2}\right) + \wp\left(\frac{\tau+1}{2}\right) = 0,$$

$$\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{\tau+1}{2}\right)\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau+1}{2}\right)=-\frac{g_2(\tau)}{4}$$

and

$$\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau}{2}\right)\wp\left(\frac{\tau+1}{2}\right)=\frac{g_3(\tau)}{4}.$$

By the above equations and Proposition 1.1, we derive that

$$\begin{aligned}2\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{\tau+1}{2}\right)&=\wp\left(\frac{\tau}{2}\right)-\left(-\wp\left(\frac{\tau+1}{2}\right)-\wp\left(\frac{\tau}{2}\right)\right)\\&=\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{1}{2}\right)=-\pi^2S^4T^8.\end{aligned}$$

And we get the following three new identities:

$$\begin{aligned}(1.1)\quad \wp\left(\frac{\tau}{2}\right)&=\frac{1}{3}\left[\left(2\wp\left(\frac{\tau}{2}\right)+\wp\left(\frac{\tau+1}{2}\right)\right)+\left(\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{\tau+1}{2}\right)\right)\right]\\&=-\frac{\pi^2}{3}S^4(T^8+16pU^8)\\&=-\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\\&\quad\times\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8+16p\prod_{n=1}^{\infty}(1+p^{2n})^8\right),\end{aligned}$$

$$\begin{aligned}(1.2)\quad \wp\left(\frac{\tau+1}{2}\right)&=16\pi^2pS^4U^8-\frac{\pi^2}{3}S^4(T^8+16pU^8)\\&=-\frac{\pi^2}{3}S^4(T^8-32pU^8)\\&=-\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\\&\quad\times\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^8-32p\prod_{n=1}^{\infty}(1+p^{2n})^8\right)\end{aligned}$$

and

$$\begin{aligned}(1.3)\quad \wp\left(\frac{1}{2}\right)&=\pi^2S^4T^8-\frac{\pi^2}{3}S^4(T^8+16pU^8)=\frac{\pi^2}{3}S^4(2T^8-16pU^8)\\&=\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\\&\quad\times\left(2\prod_{n=1}^{\infty}(1+p^{2n-1})^8-16p\prod_{n=1}^{\infty}(1+p^{2n})^8\right).\end{aligned}$$

Using (1.1)–(1.3) we obtain the identity for $g_2(\tau)$:

$$(1.4)\quad g_2(\tau) = -4\left[\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau}{2}\right) + \wp\left(\frac{\tau+1}{2}\right)\wp\left(\frac{\tau}{2}\right) + \wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau+1}{2}\right)\right]$$

$$= -4\left[\frac{\pi^2}{3}S^4(2T^8 - 16pU^8)\left(-\frac{\pi^2}{3}S^4(T^8 + 16pU^8)\right)\right.$$

$$+ \left(-\frac{\pi^2}{3}S^4(T^8 - 32pU^8)\right)\left(-\frac{\pi^2}{3}S^4(T^8 + 16pU^8)\right)$$

$$\left. + \left(\frac{\pi^2}{3}S^4(2T^8 - 16pU^8)\right)\left(-\frac{\pi^2}{3}S^4(T^8 - 32pU^8)\right)\right]$$

$$= \frac{4\pi^4}{3}S^8(T^{16} - 16pT^8U^8 + 256p^2U^{16})$$

$$= \frac{4\pi^4}{3}\prod_{n=1}^{\infty}(1-p^{2n})^8\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}\right.$$

$$\left. - 16p\prod_{n=1}^{\infty}(1+p^n)^8 + 256p^2\prod_{n=1}^{\infty}(1+p^{2n})^{16}\right).$$

We are also able to express $g_3(\tau)$ as

$$(1.5)\quad g_3(\tau) = 4\wp\left(\frac{1}{2}\right)\wp\left(\frac{\tau}{2}\right)\wp\left(\frac{\tau+1}{2}\right)$$

$$= 4\left(\frac{\pi^2}{3}S^4(2T^8 - 16pU^8)\right)$$

$$\times\left(-\frac{\pi^2}{3}S^4(T^8 - 32pU^8)\right)\left(-\frac{\pi^2}{3}S^4(T^8 + 16pU^8)\right)$$

$$= \frac{8\pi^6}{27}S^{12}(T^{24} - 24pT^{16}U^8 - 384p^2T^8U^{16} + 4096p^3U^{24})$$

$$= \frac{8\pi^6}{27}\prod_{n=1}^{\infty}(1-p^{2n})^{12}$$

$$\times\left(\prod_{n=1}^{\infty}(1+p^{2n-1})^{24} - 24p\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}(1+p^{2n})^8\right.$$

$$- 384p^2\prod_{n=1}^{\infty}(1+p^{2n-1})^8(1+p^{2n})^{16}$$

$$\left. + 4096p^3\prod_{n=1}^{\infty}(1+p^{2n})^{24}\right).$$

Then, we derive from (1.4) and (1.5) that

$$(1.6)\qquad \begin{aligned} j(\tau) &= 1728\frac{g_2(\tau)^3}{\Delta(\tau)} \\ &= 1728\frac{\left[\frac{4}{3}\pi^4 S^8(T^{16}-16pD^8+256p^2U^{16})\right]^3}{(2\pi)^{12}p^2S^{24}} \\ &= \frac{1}{p^2}(T^{16}-16pD^8+256p^2U^{16})^3. \end{aligned}$$

2. Some infinite products as algebraic integers. Throughout Sections 2 to 4 we shall fix the following notations: k is an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane and $\tau \in \mathfrak{h} \cap k$.

Let $\alpha = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $b \bmod d$ and $|\alpha|$ the determinant of α, and let

$$(*)\qquad \phi_\alpha(\tau) := |\alpha|^{12}\frac{\Delta\left(\alpha\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)\right)}{\Delta\left(\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)\right)} = |\alpha|^{12}d^{-12}\frac{\Delta(\alpha\tau)}{\Delta(\tau)}.$$

Then we recall the following well known fact.

PROPOSITION 2.1 ([4]). *For any $\tau \in k\cap\mathfrak{h}$, the value $\phi_\alpha(\tau)$ is an algebraic integer which divides $|\alpha|^{12}$.*

First, we consider

$$\frac{\Delta(\tau)}{\Delta(\tau/2)} = \frac{(2\pi)^{12}p^2\prod_{n=1}^{\infty}(1-p^{2n})^{24}}{(2\pi)^{12}p\prod_{n=1}^{\infty}(1-p^n)^{24}} = p\prod_{n=1}^{\infty}(1+p^n)^{24}$$

and

$$\frac{\Delta(\tau/2)}{\Delta(\tau)} = \frac{(2\pi)^{12}p\prod_{n=1}^{\infty}(1-p^n)^{24}}{(2\pi)^{12}p^2\prod_{n=1}^{\infty}(1-p^{2n})^{24}} = p^{-1}\frac{1}{\prod_{n=1}^{\infty}(1+p^n)^{24}}.$$

Put

$$\alpha_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

By $(*)$,

$$\phi_{\alpha_1}(\tau/2) = 2^{12}\frac{\Delta(\tau)}{\Delta(\tau/2)} \quad \left(= 2^{12}\frac{\eta(\tau)^{24}}{\eta(\tau/2)^{24}}\right),$$

from which we see by Proposition 2.1 that

$$(2.1)\qquad \sqrt{2}\,p^{1/24}\prod_{n=1}^{\infty}(1+p^n)$$

is an algebraic integer. Also, we have

$$\phi_{\alpha_2}(\tau) = 2^{12}\frac{1}{2^{12}}\cdot\frac{\Delta(\tau/2)}{\Delta(\tau)} \quad \left(= \frac{\eta(\tau/2)^{24}}{\eta(\tau)^{24}}\right),$$

and hence

$$p^{-1/24}\frac{1}{\prod_{n=1}^{\infty}(1+p^n)} \tag{2.2}$$

is an algebraic integer.

It follows from (1.0) that

$$\prod_{n=1}^{\infty}(1+p^{2n})(1+p^{2n-1})(1-p^{2n-1})=1.$$

So, by (2.2) and the above, we obtain

$$\begin{aligned}\frac{1}{p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}} &= \frac{\prod_{n=1}^{\infty}(1+p^{2n})^{24}(1+p^{2n-1})^{24}(1-p^{2n-1})^{24}}{p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24}}\\ &= p^{-2}\prod_{n=1}^{\infty}(1-p^{4n-2})^{24}.\end{aligned}$$

Thus we see from (2.2) that

$$p^{-1/24}\prod_{n=1}^{\infty}(1-p^{2n-1}) \tag{2.3}$$

is an algebraic integer. By (2.1) and (2.3), we claim that

$$\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1-p^{2n-1}) \tag{2.4}$$

is an algebraic integer.

Jacobi ([10, p. 470]) showed that

$$\prod_{n=1}^{\infty}(1+p^{2n-1})^8-\prod_{n=1}^{\infty}(1-p^{2n-1})^8=16p\prod_{n=1}^{\infty}(1+p^{2n})^8,$$

which we can now easily check by using Proposition 1.1.

Multiplying both sides in Jacobi's relation by $p^{-1/3}$, we derive from (2.1) and (2.3) that

$$p^{-1/24}\prod_{n=1}^{\infty}(1+p^{2n-1}) \tag{2.5}$$

is an algebraic integer.

Combining (2.1) and (2.5) we see that

$$\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1+p^{2n-1}) \tag{2.6}$$

is also an algebraic integer.

By (1.6), (2.1) and (2.5) we are led to

$$j(\tau)^{1/3} = -16p^{1/3}\prod_{n=1}^{\infty}(1+p^n)^8 + 256p^{4/3}\prod_{n=1}^{\infty}(1+p^{2n})^{16} + p^{-2/3}\prod_{n=1}^{\infty}(1+p^{2n-1})^{16}, \tag{2.7}$$

from which we can reprove the well known fact ([3]–[8]) that $j(\tau)$ is an algebraic integer. Observe that, in the above, we used the fact that $\sqrt{2}\,p^{1/12}\prod_{n=1}^{\infty}(1+p^{2n})$ is also an algebraic integer, which can be readily deduced from (2.1).

On the other hand, we know by (1.1) that

$$\frac{-3\wp(\tau/2)}{\pi^2\eta(\tau)^4} = \frac{\prod_{n=1}^{\infty}(1-p^{2n})^4((1+p^{2n-1})^8 + 16p(1+p^{2n})^8)}{p^{1/3}\prod_{n=1}^{\infty}(1-p^{2n})^4} = p^{-1/3}\prod_{n=1}^{\infty}(1+p^{2n-1})^8 + 16p^{2/3}\prod_{n=1}^{\infty}(1+p^{2n})^8.$$

We then conclude from (2.1) and (2.5) that

$$\frac{3}{\pi^2}\cdot\frac{\wp(\tau/2)}{\eta(\tau)^4} \tag{2.8}$$

is an algebraic integer. Also, it follows from (1.4) that

$$\begin{aligned}\frac{3g_2(\tau)}{4\pi^4\eta(\tau)^8} &= \frac{\prod_{n=1}^{\infty}(1-p^{2n})^8(1+p^{2n-1})^{16} - 16p\prod_{n=1}^{\infty}(1-p^{2n})^8(1+p^n)^8}{p^{2/3}\prod_{n=1}^{\infty}(1-p^{2n})^8} \\ &\quad + \frac{256p^2\prod_{n=1}^{\infty}(1-p^{2n})^8(1+p^{2n})^{16}}{p^{2/3}\prod_{n=1}^{\infty}(1-p^{2n})^8} \\ &= p^{-2/3}\prod_{n=1}^{\infty}(1+p^{2n-1})^{16} - 16p^{1/3}\prod_{n=1}^{\infty}(1+p^n)^8 \\ &\quad + 256p^{4/3}\prod_{n=1}^{\infty}(1+p^{2n})^{16}.\end{aligned}$$

Thus we find again by (2.1) and (2.5) that

$$\frac{3}{4\pi^4}\cdot\frac{g_2(\tau)}{\eta(\tau)^8} \tag{2.9}$$

is an algebraic integer. And we deduce from (1.5) that

$$\frac{27g_3(\tau)}{\pi^6\eta(\tau)^{12}} = 8p^{-1}\prod_{n=1}^{\infty}(1+p^{2n-1})^{24} - 192\prod_{n=1}^{\infty}(1+p^n)^8(1+p^{2n-1})^8$$
$$- 3072p\prod_{n=1}^{\infty}(1+p^n)^8(1+p^{2n})^8 + 32768p^2\prod_{n=1}^{\infty}(1+p^{2n})^{24},$$

from which we conclude by (2.1), (2.5) and (2.6) that

$$\frac{27}{\pi^6}\cdot\frac{g_3(\tau)}{\eta(\tau)^{12}} \tag{2.10}$$

is an algebraic integer.

By Proposition 1.1, (2.1), (2.3) and (2.5), we derive that

$$\frac{\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{1}{2}\right)}{\pi^2\eta(\tau)^4},\quad \frac{\wp\left(\frac{\tau+1}{2}\right)-\wp\left(\frac{1}{2}\right)}{\pi^2\eta(\tau)^4}\quad \text{and}\quad \frac{\wp\left(\frac{\tau+1}{2}\right)-\wp\left(\frac{\tau}{2}\right)}{\pi^2\eta(\tau)^4} \tag{2.11}$$

are algebraic integers. Also, it follows from (2.8) and (2.11) that

$$\frac{3}{\pi^2}\cdot\frac{\wp((\tau+1)/2)}{\eta(\tau)^4}\quad \text{and}\quad \frac{3}{\pi^2}\cdot\frac{\wp(1/2)}{\eta(\tau)^4} \tag{2.12}$$

are algebraic integers.

We summarize (2.1) to (2.12) as follows.

THEOREM 2.2. *Let* $\tau \in k\cap\mathfrak{h}$. *Then*

(a) $\sqrt{2}\,p^{1/24}\prod_{n=1}^{\infty}(1+p^n)$, $\quad p^{-1/24}\frac{1}{\prod_{n=1}^{\infty}(1+p^n)}$, $\quad p^{-1/24}\prod_{n=1}^{\infty}(1-p^{2n-1})$,

$\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1-p^{2n-1})$, $\quad p^{-1/24}\prod_{n=1}^{\infty}(1+p^{2n-1})$ *and*

$\sqrt{2}\prod_{n=1}^{\infty}(1+p^n)(1+p^{2n-1})$

are algebraic integers.

(b) $j(\tau)$, $\quad \frac{3}{\pi^2}\cdot\frac{\wp\left(\frac{\tau}{2}\right)}{\eta(\tau)^4}$, $\quad \frac{3}{\pi^2}\cdot\frac{\wp\left(\frac{\tau+1}{2}\right)}{\eta(\tau)^4}$, $\quad \frac{3}{\pi^2}\cdot\frac{\wp\left(\frac{1}{2}\right)}{\eta(\tau)^4}$, $\quad \frac{3}{4\pi^4}\cdot\frac{g_2(\tau)}{\eta(\tau)^8}$,

$\frac{27}{\pi^6}\cdot\frac{g_3(\tau)}{\eta(\tau)^{12}}$, $\quad \frac{\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{1}{2}\right)}{\pi^2\eta(\tau)^4}$, $\quad \frac{\wp\left(\frac{\tau+1}{2}\right)-\wp\left(\frac{1}{2}\right)}{\pi^2\eta(\tau)^4}$ *and*

$\frac{\wp\left(\frac{\tau+1}{2}\right)-\wp\left(\frac{\tau}{2}\right)}{\pi^2\eta(\tau)^4}$

are algebraic integers.

The Gel'fond–Schneider theorem says that $e^{\pi\alpha} = (-1)^{-i\alpha}$ is transcendental whenever $i\alpha$ is algebraic of degree at least 2 over $\mathbb{Q}$ ([8], p. 142). This facts yields that $p = e^{\pi i\tau}$ is transcendental. Therefore, we have

COROLLARY 2.3. *Let $\tau \in k \cap \mathfrak{h}$. Then*

$$\prod_{n=1}^{\infty}(1+p^n), \quad \prod_{n=1}^{\infty}(1-p^{2n-1}) \quad \text{and} \quad \prod_{n=1}^{\infty}(1+p^{2n-1})$$

are transcendental numbers.

3. Approach to $\phi(\tau)$. Let

$$\phi(\tau) := \phi(e^{\pi i\tau}) = \frac{\eta((\tau+1)/2)^2}{\eta(\tau+1)} = \prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n}) = \theta_3(0,\tau).$$

Here we refer to [2] for the last equality. Berndt, Chan and Zhang showed in [1] the following proposition by using three of Ramanujan's modular equations, values of certain class invariants of Ramanujan, representations for quotients of values of ϕ in terms of class invariants and the theta-transformation formula.

In this section we shall derive certain analogues of their results purely in terms of infinite products, which is a generalization in the case of m even.

PROPOSITION 3.1. *Let m and n be positive integers. Then $\phi(mni)/\phi(ni)$ is algebraic. Furthermore, if m is odd, then $\sqrt{2m}\,\phi(mni)/\phi(ni)$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\,\phi(mni)/\phi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

By (2.5), the value

$$\frac{\phi(\tau)}{\eta(\tau)} = \frac{\prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n})}{p^{1/12}\prod_{n=1}^{\infty}(1-p^{2n})} = p^{-1/12}\prod_{n=1}^{\infty}(1+p^{2n-1})^2 \tag{3.1}$$

is an algebraic integer. Observe that it can also be written as $\theta_3(0,\tau)/\eta(\tau)$.

By considering the identity $VUT = 1$ from (1.0), that is,

$$\prod_{n=1}^{\infty}(1+p^{2n})(1+p^{2n-1})(1-p^{2n-1}) = 1$$

we have

$$\begin{aligned}\frac{\eta(\tau)}{\phi(\tau)} &= p^{1/12}\prod_{n=1}^{\infty}(1-p^{2n})\frac{1}{\prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n})}\\ &= p^{1/12}\prod_{n=1}^{\infty}(1-p^{2n})\frac{\prod_{n=1}^{\infty}(1+p^{2n})^2(1+p^{2n-1})^2(1-p^{2n-1})^2}{\prod_{n=1}^{\infty}(1+p^{2n-1})^2(1-p^{2n})}\\ &= p^{1/12}\prod_{n=1}^{\infty}(1+p^{2n})^2(1-p^{2n-1})^2.\end{aligned}$$

Thus the value

$$2\frac{\eta(\tau)}{\phi(\tau)} \tag{3.2}$$

is an algebraic integer by (2.1) and (2.3).

Let r, s, u, v be positive integers with $(r, s) = (u, v) = 1$. Then $\frac{r}{s}\tau$ and $\frac{u}{v}\tau$ are still imaginary quadratic.

By (3.1) and (3.2) we see that

$$\frac{\phi\left(\frac{s}{r}\tau\right)}{\eta\left(\frac{s}{r}\tau\right)} \cdot 2\frac{\eta(\tau)}{\phi(\tau)} = 2\frac{\phi\left(\frac{s}{r}\tau\right)}{\phi(\tau)} \cdot \frac{\eta(\tau)}{\eta\left(\frac{s}{r}\tau\right)}$$

and

$$2\frac{\phi(\tau)}{\phi\left(\frac{u}{v}\tau\right)} \cdot \frac{\eta\left(\frac{u}{v}\tau\right)}{\eta(\tau)}$$

are algebraic integers.

Let

$$\alpha_{r/s} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \qquad \alpha_{v/u} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

Then by the above and Proposition 2.1 we derive that

$$2\frac{\phi\left(\frac{r}{s}\tau\right)}{\phi(\tau)} \cdot \frac{\eta(\tau)}{\eta\left(\frac{r}{s}\tau\right)} \cdot \phi_{\alpha_{r/s}}(\tau)^{1/24} = 2\sqrt{r}\,\frac{\phi\left(\frac{r}{s}\tau\right)}{\phi(\tau)} \tag{3.3}$$

and

$$2\frac{\phi(\tau)}{\phi\left(\frac{u}{v}\tau\right)} \cdot \frac{\eta\left(\frac{u}{v}\tau\right)}{\eta(\tau)} \cdot \phi_{\alpha_{v/u}}\left(\frac{u}{v}\tau\right)^{1/24} = 2\sqrt{v}\frac{\phi(\tau)}{\phi\left(\frac{u}{v}\tau\right)} \tag{3.4}$$

are algebraic integers.

Therefore, we have the following theorem.

THEOREM 3.2. *Let τ be any imaginary quadratic and r, s, u, v be positive integers such that $(r, s) = (u, v) = 1$. Then $4\sqrt{rv}\,\phi\left(\frac{r}{s}\tau\right)/\phi\left(\frac{u}{v}\tau\right)$ is an algebraic integer dividing $\sqrt{rsuv}$. In particular, $2\sqrt{r}\,\phi\left(\frac{r}{s}\cdot\frac{u}{v}\tau\right)/\phi\left(\frac{u}{v}\tau\right)$ and $2\sqrt{v}\,\phi\left(\frac{r}{s}\tau\right)/\phi\left(\frac{u}{v}\cdot\frac{r}{s}\tau\right)$ are algebraic integers dividing $\sqrt{rs}$ and $\sqrt{uv}$, respectively.*

REMARK 3.3. Theorem 3.2 ensures that, in fact, the algebraic integer $\sqrt{2m}\,\phi(mni)/\phi(ni)$ (respectively, $2\sqrt{m}\,\phi(mni)/\phi(ni)$) in Proposition 3.1 divides $\sqrt{m/2}$ (resp., $\sqrt{m}$) when m is odd (resp., even).

In a similar way, when working with the matrices $\left(\begin{smallmatrix} 1 & j \\ 0 & n \end{smallmatrix}\right)$ $(0 \le j \le n-1)$, we derive that $\phi\left(\frac{\tau}{n}\right)/\phi(\tau), \ldots, \phi\left(\frac{\tau+n-1}{n}\right)/\phi(\tau)$ are algebraic numbers; hence $\phi\left(\frac{\tau}{n}\right) \ldots \phi\left(\frac{\tau+n-1}{n}\right)/\phi(\tau)^n$ is an algebraic number. This implies that there exists a polynomial $f(X) = a_0X^l + a_1X^{l-1} + \ldots + a_l$ in $\mathbb{Q}[X]$ satisfying

$$f\left(\frac{\phi\left(\frac{\tau}{n}\right)\ldots\phi\left(\frac{\tau+n-1}{n}\right)}{\phi(\tau)^n}\right) = 0.$$

Therefore we get an equation

$$a_l\phi(\tau)^{nl} + \ldots + a_0\left(\phi\left(\frac{\tau}{n}\right)\ldots\phi\left(\frac{\tau+n-1}{n}\right)\right)^l = 0,$$

which leads us to the following

THEOREM 3.4. *Let n be any positive integer. Then $\phi(\tau)$ is integral over*

$$\mathbb{Q}\left[\phi\left(\frac{\tau}{n}\right)\phi\left(\frac{\tau+1}{n}\right)\ldots\phi\left(\frac{\tau+n-1}{n}\right)\right].$$

4. Approach to the Weierstrass $\wp$-function and some modular functions. Let us consider $\wp(\tau/2)$ for $\tau/2 \notin \Lambda_\tau$ where Λ_τ is the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Let r, v be positive odd integers and s, u any positive integers such that $(r, s) = (u, v) = 1$. We then see that $\frac{r}{2s}\tau$, $\frac{v}{2u}\tau$ are imaginary quadratic and $\frac{r}{2s}\tau$, $\frac{v}{2u}\tau \notin \Lambda_\tau$. By Proposition 2.1 and Theorem 2.2,

$$\frac{\wp\left(\frac{r}{2s}\tau\right)}{\wp\left(\frac{v}{2u}\tau\right)} = \frac{\wp\left(\frac{r}{2s}\tau\right)}{\pi^2\eta\left(\frac{r}{s}\tau\right)^4} \cdot \frac{\pi^2\eta\left(\frac{v}{u}\tau\right)^4}{\wp\left(\frac{v}{2u}\tau\right)} \cdot \frac{\eta\left(\frac{r}{s}\tau\right)^4}{\eta(\tau)^4} \cdot \frac{\eta(\tau)^4}{\eta\left(\frac{v}{u}\tau\right)^4}$$

is an algebraic number. Thus we get the following

THEOREM 4.1.

$$\frac{\wp\left(\frac{r}{2s}\tau\right)}{\wp\left(\frac{v}{2u}\tau\right)}$$

is an algebraic number, where τ, r, s, u and v are defined as above.

Let

$$f_0(z;\tau) = -2^7 3^5\frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp(z, \Lambda_\tau)$$

be the first Weber function for $\tau \in \mathfrak{h}$ and $z \in k$. Having fixed the integer $N > 1$, for r, s in $\mathbb{Z}$ not both divisible by N, let

$$f_{r,s}(\tau) = f_0\left(\frac{r\tau+s}{N};\tau\right)$$

be the Fricke function.

THEOREM 4.2. *Let $\tau \in \mathfrak{h} \cap k$ and r, s be positive integers not both divisible by N for fixed integer $N > 1$. Then every element $g(\tau)$ in $\Omega(j(\tau), f_{r,s}(\tau))$ is an algebraic number, where Ω is a field of algebraic numbers. In particular, any $g(\tau) \in \mathbb{Z}[j(\tau), 32r^2 f_{r,s}(\tau)]$ is an algebraic integer.*

Proof. We can derive that

$$\begin{aligned} f_{r,s}(\tau) &= f_0\left(\frac{r\tau+s}{N};\tau\right) \\ &= -2^7 3^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp\left(\frac{r\tau+s}{N};\tau\right) \\ &= -\frac{1}{8}\cdot\frac{3g_2(\tau)}{4\pi^4\eta(\tau)^8}\cdot\frac{27g_3(\tau)}{\pi^6\eta(\tau)^{12}}\cdot\frac{3\wp\left(\frac{r\tau+s}{N}\right)}{\pi^2\eta\left(\frac{2r\tau+2s}{N}\right)^4}\cdot\frac{\eta\left(\frac{2r\tau+2s}{N}\right)^4}{\eta(\tau)^4}. \end{aligned}$$

Put $\alpha_{2r/N} = \left(\begin{smallmatrix} 2r & 2s \\ 0 & N \end{smallmatrix}\right)$ with $2s$ mod N. It follows from Proposition 2.1 that

$$\phi_{\alpha_{2r/N}}(\tau)^{1/6} = \left(\frac{1}{N^{12}}|2rN|^{12}\frac{\eta\left(\frac{2r\tau+2s}{N}\right)^{24}}{\eta(\tau)^{24}}\right)^{1/6} = (2r)^2\frac{\eta\left(\frac{2r\tau+2s}{N}\right)^4}{\eta(\tau)^4}$$

is an algebraic integer.

We then conclude by Theorem 2.2 that $32r^2 f_{r,s}(\tau)$ is an algebraic integer. Therefore the theorem follows.

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