The distribution of real-valued $Q$-additive functions modulo 1

by

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0. Introduction. Let $q \geq 2$ be an integer. A $q$-additive function $f : \mathbb{N} \rightarrow \mathbb{C}$ is a function of the form $f(n) = \sum_{j \geq 0} f_j(a_j(n))$ where $n = \sum_{j \geq 0} a_j(n)q^j$ is the base-$q$ representation of $n$ and the “component functions” $f_j$ are functions defined on \{0, 1, \ldots, q - 1\} and satisfying $f_j(0) = 0$. These functions were introduced by A. O. Gel’fond [Ge] in 1968, and have been studied by Coquet [Co1], Delange [De3], and others. They generalize the sum-of-digits functions $s_q(n)$ with respect to base $q$.

In 1977, Coquet [Co1] generalized $q$-additive functions to more general systems of numeration. Specifically, he considered so-called mixed radix representations (also called Cantor representations) defined as follows. Let $Q = \{Q_j\}_{j \geq 0}$ be a sequence of strictly increasing positive integers with $Q_0 = 1$ such that $Q_j | Q_{j+1}$ for all $j$. Note that the sequence $Q$ is uniquely determined by the factors $q_j = Q_{j+1}/Q_j$. It is easily seen that each non-negative integer $n$ has a unique “base-$Q$” representation of the form $n = \sum_{j \geq 0} a_j(n)Q_j$, where the “digits” $a_j(n)$ satisfy $0 \leq a_j(n) < q_j$. Examples of such representations are the ordinary base-$q$ representations ($q_j = q$) as well as the factorial representation ($q_j = j + 2$), the factorial-squared representation ($q_j = (j + 2)^2$), and the doubly-geometric representations ($q_j = q^j$).

For a full discussion of these and other numeration systems see, for example, Grabner et al. [GLT] or the survey article by Fraenkel [Fr] and the references therein.

Given a mixed radix system $Q$, Coquet defined a $Q$-additive function $f : \mathbb{N} \rightarrow \mathbb{C}$ to be a function of the form $f(n) = \sum_{j \geq 0} f_j(a_j(n))$ where $n = \sum_{j \geq 0} a_j(n)Q_j$ is the base-$Q$ representation of $n$ and the component functions $f_j$ are functions defined on \{0, 1, \ldots, q_j - 1\} and satisfying $f_j(0) = 0$. 

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A simple example of a $Q$-additive function is the sum-of-digits function $s_Q(n) = \sum_{j \geq 0} a_j(n)$, which corresponds to the choice $f_j(a) = a$. This function has been studied by Kirschenhofer and Tichy [KT], among others. For recent work on general $Q$-additive functions see Manstavičius [Ma]. For generalizations of $q$-additive functions to other numeration systems see, for example, Barat and Grabner [BG].

Our main result, Theorem 1, characterizes those real-valued $Q$-additive functions that have a limit distribution (resp. uniform limit distribution) modulo 1. In order to prove this result, we consider so-called $Q$-multiplicative functions, which are defined in analogy to $Q$-additive functions as follows. A $Q$-multiplicative function is a function $g : \mathbb{N} \to \mathbb{C}$ of the form $g(n) = \prod_{j \geq 0} g_j(a_j(n))$, where $n = \sum_{j \geq 0} a_j(n) Q_j$ is the base-$Q$ representation of $n$ and the component functions $g_j$ are functions defined on $\{0, 1, \ldots, q_j - 1\}$ and satisfying $g_j(0) = 1$. These functions have been studied by Coquet [Co1] and others, usually in conjunction with work on $Q$-additive functions.

We establish mean value theorems for $Q$-multiplicative functions analogous to those of Delange and Wirsing (see, e.g., [El, Chapter 6]) for ordinary multiplicative functions.

Throughout this paper, we set $e(x) = e^{2\pi i x}$ and write $\|x\|$ to denote the distance from $x$ to the nearest integer and $\{x\}$ to denote the fractional part of $x$.

1. Statement of results. Let $Q = \{Q_j\}_{j \geq 0}$ be a mixed radix system with factors $q_j = Q_{j+1}/Q_j$. Let $f$ be a real-valued $Q$-additive function with component functions $f_j$. We say that $f$ has a limit distribution modulo 1 if there is a distribution function $F$ (i.e., $F$ is right-continuous and monotonic with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \geq 1$) such that the limit

$$\lim_{N \to \infty} \frac{1}{N} \#\{0 \leq n < N : \{f(n)\} \leq x\}$$

exists and equals $F(x)$ for every $x$ at which $F$ is continuous. We say that $f$ has a uniform limit distribution modulo 1 if this holds with $F(x) = x$ for $0 \leq x < 1$. Aside from its intrinsic interest, the study of the distribution modulo 1 of $Q$-additive functions is motivated in part by the results of Coquet [Co1] and Mendès France [MF] connecting the uniform distribution of certain $Q$-additive functions to so-called P-V numbers. Our main result is a complete characterization of real-valued $Q$-additive functions that have a limit distribution (resp. uniform limit distribution) modulo 1.

**Theorem 1.** A real-valued $Q$-additive function $f$ has a limit distribution modulo 1 if and only if, for each integer $k \neq 0$, at least one of the following conditions holds:

(i) There exists $j \geq 0$ such that $\sum_{0 \leq n < q_j} e(kf_j(n)) = 0$. 
(ii) The series
\[ \sum_{j \geq 0} \left( 1 - \frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(kf_j(n)) \right| \right) \]
diverges.

(iii) The series
\[ \sum_{j \geq 0} \left( 1 - \frac{1}{q_j} \sum_{0 \leq n < q_j} e(kf_j(n)) \right) \]
converges, and
\[ \lim_{j \to \infty} \left( \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} \| kf_j(m) \|^2 \right) = 0. \]

Furthermore, \( f \) is uniformly distributed modulo 1 if and only if, for all integers \( k \neq 0 \), at least one of conditions (i) or (ii) holds.

This result generalizes the characterization given by Kim [Ki, p. 27] for the special case of \( q \)-additive functions.

We apply Theorem 1 to derive several corollaries that deal with special cases. We first consider numeration systems in which the factors \( q_j \) are bounded. In particular, these systems include the ordinary base-\( q \) representations generated by \( q_j = q \) for all \( j \).

**Corollary 1.** Suppose the factors \( q_j \) are bounded. Then \( f \) is uniformly distributed modulo 1 if and only if, for all \( k \neq 0 \), either the series
\[ \sum_{j \geq 0} \sum_{0 \leq n < q_j} \| kf_j(n) \|^2 \]
diverges, or for some \( j \geq 0 \) we have
\[ \sum_{0 \leq n < q_j} e(kf_j(n)) = 0. \]

Let \( \alpha \in \mathbb{R} \). We call an integer-valued arithmetic function \( f \) normal if the function \( \alpha f \) is uniformly distributed modulo 1 if and only if \( \alpha \) is irrational. Coquet [Co2] showed that for any base \( q \geq 2 \), the associated sum-of-digits function \( s_q(n) \) is normal. General criteria for the normality of arithmetic functions have been given by Drmota and Tichy [DT, Section 1.4.3]. In Corollaries 2 and 3 below, we apply Theorem 1 to show that two classes of integer-valued \( Q \)-additive functions are normal.

**Corollary 2.** For any mixed radix numeration system \( Q \), the function \( s_Q(n) \), the sum of digits in the base-\( Q \) representation of \( n \), is normal.

We call a \( Q \)-additive function \( f \) completely \( Q \)-additive if there exists a function \( g : \mathbb{N} \to \mathbb{C} \) such that, for all \( j \geq 0 \) and \( 0 \leq n < q_j \), \( f_j(n) = g(n) \),
i.e., if the component functions $f_j$ are independent of $j$ on their respective domains. The following corollary generalizes a result of Drmota and Tichy [DT, Theorem 1.99].

**Corollary 3.** Suppose that the series $\sum_{j \geq 0} 1/q_j$ diverges. Let $f$ be a completely $Q$-additive, integer-valued function such that there is some integer $1 \leq a < \min_j q_j$ with $f_0(a) > 0$. Then $f$ is normal.

In the next two corollaries we investigate the normality of two particular integer-valued $Q$-additive functions that have been previously considered in [Ho, examples (c) and (a)] and [KT, Theorem 3]. These results provide examples of functions that have a non-uniform limit distribution modulo 1 as well as functions that have no limit distribution modulo 1.

**Corollary 4.** Let $\alpha$ be an irrational number. Let $M(n)$ be the number of digits in the base-$Q$ representation of $n$ which are maximal, and set $f(n) = \alpha M(n)$. Then $f$ has a limit distribution modulo 1. Moreover, the limit distribution is uniform if and only if the series $\sum_{j \geq 0} 1/q_j$ diverges.

**Corollary 5.** Let $a > 0$ be a fixed integer and let $\alpha$ be an irrational number. Let $N_a(n)$ be the number of digits $a$ in the base-$Q$ representation of $n$, and set $f(n) = \alpha N_a(n)$. Then $f$ is uniformly distributed modulo 1 if and only if $\sum_{q_j > a} 1/q_j$ diverges. The function $f$ has a non-uniform limit distribution modulo 1 if and only if $q_j \leq a$ for all but at most finitely many $j$.

2. Lemmas. The first lemma is a well known result on the distribution modulo 1 of real-valued arithmetic functions (see, e.g., [De2, p. 216]). The second assertion of the lemma is known as Weyl’s Criterion [We].

**Lemma 1.** A real-valued arithmetic function $f$ has a limit distribution modulo 1 if and only if, for each integer $k \neq 0$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e(kf(n))$$

exists. Further, the distribution is uniform modulo 1 if and only if, for each integer $k \neq 0$, the above limit is 0.

Throughout the remainder of this section, we fix a mixed radix system $Q$ with factors $\{q_j\}_{j \geq 0}$. We denote by $g$ a $Q$-multiplicative function satisfying $|g| \leq 1$ with component functions $g_j$, and define

$$\mu_j(g) = \frac{1}{q_j} \sum_{0 \leq n < q_j} g_j(n).$$

Thus, $\mu_j(g)$ is the mean value of $g_j$ on its domain $\{0, 1, \ldots, q_j - 1\}$.

The following lemma relates the mean value of $g$ on $\{0, 1, \ldots, rQ_j - 1\}$ to that of the functions $g_i$. 
Lemma 2. For \( j \geq 0 \) and any positive integer \( r \) with \( 1 \leq r \leq q_j \) we have

\[
(2.1) \quad \frac{1}{rQ_j} \sum_{n=0}^{rQ_j-1} g(n) = \left( \frac{1}{r} \sum_{n=0}^{r-1} g_j(n) \right) \left( \frac{1}{Q_j} \sum_{n=0}^{Q_j-1} g(n) \right).
\]

Moreover, for any \( j \geq 0 \),

\[
(2.2) \quad \frac{1}{Q_j} \sum_{n=0}^{Q_j-1} g(n) = \prod_{i=0}^{j-1} \mu_i(g).
\]

Proof. We first note that, since \( Q_0 = 1 \) and \( g(0) = 1 \), we have

\[
\frac{1}{Q_0} \sum_{n=0}^{Q_0-1} g(n) = g(0) = 1.
\]

Thus, relation (2.2) follows from (2.1) by applying (2.1) with \( r = q_j \) and iterating the identity. Hence it suffices to prove (2.1).

Observe that any non-negative integer \( n < rQ_j \) can be written uniquely (via the division algorithm) in the form \( n = sQ_j + t \) with \( 0 \leq t < Q_j \) and \( 0 \leq s < r \). By the \( Q \)-multiplicativity of \( g \), we have, with this representation,

\[
g(n) = g_j(s)g(t).
\]

As \( n \) runs through the set \( \{0, 1, \ldots, rQ_j - 1\} \), \( s \) and \( t \) run independently through the sets \( \{0, 1, \ldots, r-1\} \) and \( \{0, 1, \ldots, Q_j-1\} \), respectively. It follows that

\[
\frac{1}{rQ_j} \sum_{n=0}^{rQ_j-1} g(n) = \frac{1}{rQ_j} \sum_{s=0}^{r-1} \sum_{t=0}^{Q_j-1} g_j(s)g(t) = \frac{1}{r} \sum_{s=0}^{r-1} g_j(s) \frac{1}{Q_j} \sum_{t=0}^{Q_j-1} g(t),
\]

which is (2.1).

To obtain necessary and sufficient conditions for the convergence of the product in (2.2), we will use the following lemma, a proof of which can be found in many elementary texts on complex variables (see, e.g., [LR, pp. 383–384]).

Lemma 3. Let \( z_0, z_1, \ldots \) be complex numbers satisfying \( |z_j| \leq 1 \), and let \( P_i = \prod_{j=0}^i z_j \). Then \( \lim_{i \to \infty} P_i = 0 \) if and only if at least one of the following two conditions holds:

(i) There is some \( j \geq 0 \) such that \( z_j = 0 \).

(ii) \( \sum_{j=0}^\infty (1 - |z_j|) = \infty \).

Then \( \lim_{i \to \infty} P_i \) exists and is non-zero if and only if the following two conditions are both satisfied:

(iii) \( z_j \neq 0 \) for all \( j \).

(iv) \( \sum_{j=0}^\infty (1 - z_j) \) converges.
The next lemma relates the mean value of $g$ on $\{0, 1, \ldots, N - 1\}$ for general integers $N$ to the mean values of the functions $g_j$.

**Lemma 4.** Let $N$ be a positive integer and let $\sum_{j=0}^{i} a_j Q_j$, with $a_i > 0$, be the base-$Q$ representation of $N$. Then

\begin{equation}
\frac{1}{N} \sum_{n=0}^{N-1} g(n) = \sum_{j=0}^{i} \frac{a_j Q_j}{N} \left( \prod_{m=j+1}^{i} g_m(a_m) \right) \left( \frac{1}{a_j} \sum_{n=0}^{a_j-1} g_j(n) \right) \left( \prod_{k=0}^{j-1} \mu_k(g) \right),
\end{equation}

where empty products and empty sums are to be interpreted as 1 and 0, respectively. Furthermore, for any positive integer $h \leq i$, we have

\begin{equation}
\sum_{j=0}^{i-h} \frac{a_j Q_j}{N} < 2^{1-h}.
\end{equation}

**Proof.** We begin by dividing the interval $0 \leq n < N$ into the subintervals $0 \leq n < a_i Q_i$ and $a_i Q_i \leq n < N$, to obtain

\begin{equation}
\frac{1}{N} \sum_{n=0}^{N-1} g(n) = \frac{1}{N} \sum_{0 \leq n < a_i Q_i} g(n) + \frac{1}{N} \sum_{a_i Q_i \leq n < N} g(n).
\end{equation}

We have, by Lemma 2,

\begin{equation}
\frac{1}{N} \sum_{0 \leq n < a_i Q_i} g(n) = \frac{a_i Q_i}{N} \left( \frac{1}{a_i} \sum_{n=0}^{a_i-1} g_i(n) \right) \left( \frac{1}{Q_i} \sum_{n=0}^{Q_i-1} g(n) \right) = \frac{a_i Q_i}{N} \left( \frac{1}{a_i} \sum_{n=0}^{a_i-1} g_i(n) \right) \left( \prod_{k=0}^{i-1} \mu_k(g) \right).
\end{equation}

Furthermore, by the $Q$-multiplicativity of $g$, we also have, for all $n$ with $a_i Q_i \leq n < N$, $g(n) = g(n - a_i Q_i) g_i(a_i)$. Thus,

\begin{equation}
\frac{1}{N} \sum_{a_i Q_i \leq n < N} g(n) = \frac{1}{N} \sum_{a_i Q_i \leq n < N} g(n - a_i Q_i) g_i(a_i)
\end{equation}

\begin{equation}
= g_i(a_i) \frac{1}{N} \sum_{a_i Q_i \leq n < N} g(n - a_i Q_i)
\end{equation}

\begin{equation}
= g_i(a_i) \frac{1}{N} \sum_{0 \leq n < N - a_i Q_i} g(n)
\end{equation}

\begin{equation}
= g_i(a_i) \frac{N - a_i Q_i}{N} \left( \frac{1}{N - a_i Q_i} \sum_{0 \leq n < N - a_i Q_i} g(n) \right).
\end{equation}
It follows that
\[
\frac{1}{N} \sum_{n=0}^{N-1} g(n) = \frac{a_i Q_i}{N} \left( \frac{1}{a_i} \sum_{n=0}^{a_i-1} g_i(n) \right) \left( \prod_{k=0}^{i-1} \mu_k(g) \right) \\
+ g_i(a_i) \frac{N - a_i Q_i}{N} \left( \frac{1}{N - a_i Q_i} \sum_{0 \leq n < N - a_i Q_i} g(n) \right).
\]

Iterating the last expression \(i-1\) times gives (2.3). Inequality (2.4) follows from the chain of inequalities
\[
\frac{1}{N} \sum_{j=0}^{i-h} a_j Q_j \leq \frac{1}{Q_i} \sum_{j=0}^{i-h} a_j Q_j \leq \frac{Q_i - h + 1}{Q_i} = \prod_{j=i-h+1}^{i-1} \frac{1}{q_j} \leq 2^{1-h}.
\]

3. Mean value theorems for \(Q\)-multiplicative functions. Throughout this section, we let \(Q\) be a mixed radix system with factors \(\{q_j\}_{j \geq 0}\). For a given \(Q\)-multiplicative function \(g\) with component functions \(g_j\), we define the mean value of \(g\) by
\[
M(g) = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n),
\]
provided this limit exists. We set
\[
\sigma_j(g) = \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} (1 - \text{Re}(g_j(m))),
\]
and recall the notation
\[
\mu_j(g) = \frac{1}{q_j} \sum_{0 \leq n < q_j} g_j(n)
\]
introduced in the previous section.

The following theorem, due to Coquet [Co1, Lemma 1], gives a characterization of \(Q\)-multiplicative functions of modulus at most 1 which have mean value 0. We present a proof here for completeness.

**Theorem 2.** Let \(g\) be a \(Q\)-multiplicative function satisfying \(|g| \leq 1\). The mean value \(M(g)\) exists and is equal to 0 if and only if at least one of the following two conditions holds:

(i) For some \(j \geq 0\), \(\mu_j(g) = 0\).

(ii) The series \(\sum_{j \geq 0} (1 - |\mu_j(g)|)\) diverges.

**Proof.** Assume first that \(M(g) = 0\). Then, by (2.2) of Lemma 2,
\[
\prod_{i=0}^{\infty} \mu_i(g) = \lim_{j \to \infty} \prod_{i=0}^{j} \mu_i(g) = \lim_{j \to \infty} \frac{1}{Q_{j+1}} \sum_{n=0}^{Q_{j+1}-1} g(n) = 0.
\]
By Lemma 3, this implies that at least one of conditions (i) or (ii) holds.
Conversely, assume that at least one of conditions (i) or (ii) holds. Applying Lemma 3 again, we conclude that $\prod_{j=0}^{i} \mu_j(g) = 0$. We now show that $M(g)$ exists and is equal to 0. Let $N$ be a positive integer with base-$Q$ representation $\sum_{j=0}^{i} a_j Q_j$, where $a_i > 0$. Applying Lemma 4 with $h = \lfloor i/2 \rfloor$, we obtain

$$\left| \frac{1}{N} \sum_{0 \leq n < N} g(n) \right| < \sum_{j=0}^{i} \frac{a_j Q_j}{N} \left| \prod_{k=0}^{j-1} \mu_k(g) \right| < 2^{1-[i/2]} + \sum_{j=\lfloor i/2 \rfloor + 1}^{i} \frac{a_j Q_j}{N} \left| \prod_{k=0}^{j-1} \mu_k(g) \right|$$

$$\leq 2^{2-i/2} + \sum_{j=\lfloor i/2 \rfloor}^{i} \frac{a_j Q_j}{N} \left| \prod_{k=0}^{j-1} \mu_k(g) \right|.$$

Since $i$ tends to infinity as $N$ tends to infinity and $\sum_{j=\lfloor i/2 \rfloor}^{i} a_j Q_j \leq N$, the right-hand side tends to 0 as $N$ tends to infinity. Thus, $M(g) = 0$. This completes the proof of Theorem 2.

We now characterize those $Q$-multiplicative functions of modulus at most 1 having a non-zero mean value, a case that was not considered by Coquet. This characterization is the content of the following theorem which represents an analog of the well known mean value theorem of Delange [De1], and generalizes a result of Delange [De3] for the case of ordinary base-$q$ expansions.

**Theorem 3.** Let $g$ be a $Q$-multiplicative function satisfying $|g| \leq 1$. The mean value $M(g)$ exists and is non-zero if and only if the following three conditions all hold:

(i) For each $j \geq 0$, $\mu_j(g) \neq 0$.

(ii) $\sum_{j \geq 0} (1 - \mu_j(g))$ converges.

(iii) $\lim_{j \to \infty} \sigma_j(g) = 0$.

**Proof.** For simplicity of notation, we will write $\mu_j = \mu_j(g)$ and $\sigma_j = \sigma_j(g)$ throughout the proof.

Assume first that $M(g) = L$ for some number $L \neq 0$. Then, in particular, we have

$$\lim_{j \to \infty} \frac{1}{Q_j} \sum_{0 \leq n < Q_j} g(n) = L.$$

By (2.2) of Lemma 2, this implies that $\prod_{j=0}^{\infty} \mu_j = L$. By Lemma 3, the convergence of the product $\prod_{j=0}^{\infty} \mu_j$ to a non-zero value implies that conditions (i) and (ii) of the theorem hold.
It remains to show that condition (iii) also holds, i.e., we wish to show that the quantity
\[ \sigma_j = \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} (1 - \text{Re}(g_j(m))) \]
tends to zero as \( j \) tends to infinity. Let \( \{n_j\}_{j=0}^{\infty} \) be a sequence of integers such that the maximum in the definition of \( \sigma_j \) is attained at \( n = n_j \), so that
\[ \sigma_j = 1 - \text{Re} \left( \frac{1}{n_j} \sum_{0 \leq m < n_j} g_j(m) \right). \]
Applying (2.1) of Lemma 2 with \( r = n_j \), we obtain
\[ \lim_{j \to \infty} \frac{1}{n_j} \sum_{0 \leq m < n_j} g_j(m) = \lim_{j \to \infty} \frac{(1/n_j Q_j) \sum_{0 \leq n < n_j} g(n)}{(1/Q_j) \sum_{0 \leq n < Q_j} g(n)} = \frac{L}{L} = 1. \]
Therefore, \( \sigma_j \) tends to 0 as \( j \) tends to infinity, which proves condition (iii).

Conversely, assume that conditions (i), (ii), and (iii) all hold. The first two conditions imply that the infinite product \( \prod_{j=0}^{\infty} \mu_j \) converges to a non-zero value, by Lemma 3. Let \( L \) denote this value. We will show that \( M(g) \) exists and is equal to \( L \).

We first note that, by the bound \( |g_j| \leq 1 \) and the general inequality
\[ |1 - z|^2 = 1 + |z|^2 - 2 \text{Re} z \leq 2(1 - \text{Re} z) \quad (|z| \leq 1), \]
condition (iii) is equivalent to
\[ \text{(iii)'} \quad \lim_{j \to \infty} \max_{0 < n \leq q_j} \left| \frac{1}{n} \sum_{0 \leq m < n} g_j(m) \right| = 0. \]
Furthermore, (iii)’ implies that
\[ (3.1) \quad \lim_{j \to \infty} g_j(m) = 1 \]
for any fixed \( m \).

Let \( \varepsilon > 0 \) be given and choose \( h \) and \( i_0 \) such that \( 2^{1-h} < \varepsilon \), and for \( i \geq i_0 \) we have the following three conditions:

(a) \( |\prod_{j=0}^{i-1} \mu_j - L| < \varepsilon \).
(b) \( |(1/n) \sum_{0 \leq m < n} g_i(m) - 1| < \varepsilon \) \quad (0 < n \leq q_i).
(c) \( |\prod_{k < j \leq i} g_j(m_j) - 1| < \varepsilon \) \quad (i - h \leq k \leq i, \ 0 \leq m_j < 1/\varepsilon).

Condition (a) is possible since \( \prod_{j=0}^{\infty} \mu_j = L \), while conditions (b) and (c) can be met in view of condition (iii)’ and (3.1).
Let $N$ be a positive integer with base-$Q$ representation $N = \sum_{j=0}^{i} a_j Q_j$ where $a_i > 0$, and suppose that $N$ is sufficiently large and $i > i_0 + h$. Applying Lemma 4, we have

$$\left| \frac{1}{N} \sum_{0 \leq n < N} g(n) - L \right|$$

$$= \left| \sum_{j=0}^{i} \frac{a_j Q_j}{N} \left( \prod_{m=j+1}^{i} g_m(a_m) \right) \left( \frac{1}{a_j} \sum_{0 \leq n < a_j} g_j(n) \right) \left( \prod_{k=0}^{j-1} \mu_k \right) - L \right|$$

$$\leq \left| \sum_{j=i-h+1}^{i} \frac{a_j Q_j}{N} \left( \prod_{m=j+1}^{i} g_m(a_m) \right) \left( \frac{1}{a_j} \sum_{0 \leq n < a_j} g_j(n) \right) \left( \prod_{k=0}^{j-1} \mu_k \right) - L \right|$$

$$+ 2 \sum_{j=0}^{i-h} \frac{a_j Q_j}{N},$$

where in the last step we have used the fact that $g_m, g, \mu_k$, and $L$ are at most $1$ in modulus. By inequality (2.4) of Lemma 4, the second term on the right hand side is at most $2(2^{1-h}) < 2\varepsilon$. Moreover, by the triangle inequality, the first term is bounded by

$$\sum_{j=i-h+1}^{i} \frac{a_j Q_j}{N} \left( \prod_{m=j+1}^{i} g_m(a_m) \right) - 1 \cdot \left( \frac{1}{a_j} \sum_{0 \leq n < a_j} g_j(n) \right) \left( \prod_{k=0}^{j-1} \mu_k \right)$$

$$+ \sum_{j=i-h+1}^{i} \frac{a_j Q_j}{N} \left( \left| \frac{1}{a_j} \sum_{0 \leq n < a_j} g_j(n) - 1 \right| \prod_{k=0}^{j-1} \mu_k \right) + \sum_{j=0}^{i-h} \frac{a_j Q_j}{N} \left( \prod_{k=0}^{j-1} \mu_k - L \right)$$

$$= \Sigma_1 + \Sigma_2,$$

say. Since $i - h > i_0$, we have, by assumptions (a) and (b) above,

$$\Sigma_2 < 2\varepsilon \sum_{j=i-h+1}^{i} \frac{a_j Q_j}{N} \leq 2\varepsilon.$$

To estimate $\Sigma_1$, we distinguish two cases. If $a_j < 1/\varepsilon$ for all $j$ with $i - h < j \leq i$, then by assumption (c), $|\prod_{m=j+1}^{i} g_m(a_m) - 1| < \varepsilon$ for all $j$ and therefore $\Sigma_1 < \varepsilon$. Otherwise, let $j_0$ be the largest value of $j$ in the range $i - h < j \leq i$ for which $a_{j_0} \geq 1/\varepsilon$. The contribution of terms with $j_0 \leq j \leq i$ to $\Sigma_1$ is, as before, at most $\varepsilon$. Thus,

$$\Sigma_1 < \varepsilon + \sum_{j=i-h+1}^{j_0-1} \frac{a_j Q_j}{N} \left( \prod_{m=j+1}^{i} g_m(a_m) \right) - 1 \cdot \left( \frac{1}{a_j} \sum_{0 \leq n < a_j} g(n) \right) \left( \prod_{k=0}^{j-1} \mu_k \right)$$
\[ \leq \varepsilon + 2 \sum_{j=0}^{j_0-1} \frac{a_j Q_j}{N} \leq \varepsilon + 2 \sum_{j=0}^{j_0-1} \frac{(q_j - 1)Q_j}{N} = \varepsilon + 2 \sum_{j=0}^{j_0-1} \frac{Q_{j+1} - Q_j}{N} \]

\[ < \varepsilon + \frac{2Q_{j_0}}{N} \leq \varepsilon + \frac{2Q_{j_0}}{a_{j_0}Q_{j_0}} \leq 3\varepsilon. \]

In either case, we have

\[ \left| \frac{1}{N} \sum_{0 \leq n < N} g(n) - L \right| < 7\varepsilon. \]

Since \( \varepsilon \) was arbitrary, we have shown that \( M(g) = L \). This completes the proof of the theorem.

The following result is an immediate consequence of Theorems 2 and 3.

**Theorem 4.** Let \( g \) be a \( Q \)-multiplicative function satisfying \( |g| \leq 1. \) The mean value \( M(g) \) exists if and only if at least one of the following three conditions holds:

(i) For some \( j \geq 0, \mu_j(g) = 0. \)

(ii) The series \( \sum_{j \geq 0} (1 - |\mu_j(g)|) \) diverges.

(iii) \( \sum_{j \geq 0} (1 - \mu_j(g)) \) converges, and \( \lim_{j \to \infty} \sigma_j(g) = 0. \)

The mean value is zero if either condition (i) or (ii) holds.

4. **Proof of Theorem 1.** Let \( Q = \{Q_j\}_{j \geq 0} \) be a mixed radix system, with factors \( q_j = Q_{j+1}/Q_j \), and let \( f \) be a real-valued \( Q \)-additive function with component functions \( f_j \).

For each integer \( k \neq 0 \), we set \( g^{(k)}(n) = e(kf(n)) \). Then each function \( g^{(k)} \) is a \( Q \)-multiplicative function with component functions \( g_j^{(k)}(n) = e(kf_j(n)) \). We write

\[ \mu_j^{(k)} = \mu_j(g^{(k)}) = \frac{1}{q_j} \sum_{0 \leq n < q_j} g_j^{(k)}(n), \]

and

\[ \sigma_j^{(k)} = \sigma_j(g^{(k)}) = \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} (1 - \text{Re}(g_j^{(k)}(n))), \]

and denote the mean value of \( g^{(k)} \) by \( M_k \), whenever this mean value exists.

By Lemma 1, \( f \) has a limit distribution modulo 1 if and only if, for each integer \( k \neq 0 \), the mean value \( M_k \) exists, and the distribution is uniform if and only if, for each integer \( k \neq 0 \), \( M_k = 0 \). By Theorem 4, for each \( k \neq 0 \), \( M_k \) exists if and only if at least one of the following three conditions holds:

(i) For some \( j \geq 0, \mu_j^{(k)} = 0. \)

(ii) The series \( \sum_{j \geq 0} (1 - |\mu_j^{(k)}|) \) diverges.
(iii) \( k \sum_{j \geq 0} (1 - \mu_j^{(k)}) \) converges, and \( \lim_{j \to \infty} \sigma_j^{(k)} = 0. \)

Further, \( M_k = 0 \) if and only if either condition (i)\( k \) or (ii)\( k \) holds. Therefore, it remains only to show that, for each integer \( k \neq 0 \), conditions (i)\( k \), (ii)\( k \), and (iii)\( k \) are equivalent to conditions (i), (ii), and (iii) of Theorem 1, respectively.

To prove this, we fix an integer \( k \neq 0 \). Conditions (i)\( k \) and (ii)\( k \) are identical to conditions (i) and (ii) of Theorem 1, respectively, by the definition of \( \mu_j^{(k)} \). The equivalence between condition (iii)\( k \) and condition (iii) of Theorem 1 follows from the definition of \( \mu_j^{(k)} \) and the relation

\[
\sigma_j^{(k)} = \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} (1 - \text{Re}(g_j^{(k)}(n))) \geq \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} \|k f_j(n)\|^2,
\]

which holds since

\[
1 - \text{Re}e(x) = 1 - \cos(2\pi x) \simeq \|x\|^2
\]

for any real number \( x \). This completes the proof of Theorem 1.

5. Proof of the corollaries

**Proof of Corollary 1.** Fix an integer \( k \neq 0 \). For each \( j \geq 0 \), let \( n_j \) be such that \( \max_{0 < n < q_j} \|k f_j(n)\|^2 \) is attained at \( n = n_j \). First we note that by the elementary inequality

\[
|1 + e(x)| \leq 2 - 2\|x\|^2 \quad (x \in \mathbb{R}),
\]

we have, for all \( j \),

\[
\frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(k f_j(n)) \right| \leq \frac{1}{q_j} \left| \sum_{1 \leq n < q_j} e(k f_j(n)) \right| + \frac{1}{q_j} \left| 1 + e(k f_j(n_j)) \right|
\]

\[
\leq \frac{1}{q_j} (|q_j - 2| + \left| 1 + e(k f_j(n_j)) \right|)
\]

\[
\leq \frac{1}{q_j} ((q_j - 2) + (2 - 2\|k f_j(n_j)\|^2))
\]

\[
= \frac{1}{q_j} (q_j - 2\|k f_j(n_j)\|^2)
\]

and thus

\[
1 - \frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(k f_j(n)) \right| \geq \frac{2\|k f_j(n_j)\|^2}{q_j} \geq \frac{2}{q_j^2} \sum_{0 \leq n < q_j} \|k f_j(n_j)\|^2.
\]

Since, by assumption, the factors \( q_j \) are bounded, the divergence of the series

\[
\sum_{j \geq 0} \sum_{0 \leq n < q_j} \|k f_j(n)\|^2
\]

(5.1)
implies that condition (ii) of Theorem 1 holds. Hence, if for all \( k \neq 0 \) either the series in (5.1) diverges or condition (i) of Theorem 1 holds, then Theorem 1 implies that \( f \) has a uniform limit distribution modulo 1.

Conversely, assume that \( f \) is uniformly distributed modulo 1. Then, for each \( k \neq 0 \), either condition (i) or condition (ii) of Theorem 1 holds. We will show that if condition (ii) holds for some \( k \neq 0 \) then the series in (5.1) diverges. Fix \( k \neq 0 \). Since, for all real \( x \),

\[
1 - \Re e(x) = 1 - \cos(2\pi x) \leq 2\pi^2 \|x\|^2,
\]

we have, for all \( j \),

\[
1 - \frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(kf_j(n)) \right| \leq 1 - \frac{1}{q_j} \Re \sum_{0 \leq n < q_j} e(kf_j(n))
\]

\[
= \frac{1}{q_j} \sum_{0 \leq n < q_j} (1 - \Re e(kf_j(n)))
\]

\[
\leq \frac{1}{q_j} \sum_{0 \leq n < q_j} 2\pi^2 \|kf_j(n)\|^2
\]

\[
\leq \pi^2 \sum_{0 \leq n < q_j} \|kf_j(n)\|^2.
\]

Thus, condition (ii) of Theorem 1 implies the divergence of the series in (5.1), as claimed.

**Proof of Corollary 2.** Assume first that \( \alpha \) is irrational. If the factors \( q_j \) are bounded, then, since \( f_j(1) = \alpha \) for all \( j \), it follows from Corollary 1 that \( f \) is uniformly distributed modulo 1. It remains to deal with the case when the factors \( q_j \) are unbounded.

Fix \( k \neq 0 \). Then we have, for all \( j \geq 0 \),

\[
\frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(kf_j(n)) \right| = \frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(k\alpha n) \right|
\]

\[
= \frac{1}{q_j} \left| 1 - e(k\alpha q_j) \right| \leq \frac{2}{q_j(1 - e(\alpha k))}.
\]

Since the factors \( q_j \) are unbounded and \( \alpha \) is irrational, this quantity is \( \leq 1/2 \) for infinitely many \( j \), and so condition (ii) of Theorem 1 is satisfied. Therefore, \( f \) has a uniform limit distribution modulo 1.

On the other hand, if \( \alpha \) is rational, then \( f \) takes on only finitely many values modulo 1, and thus \( f \) cannot be uniformly distributed modulo 1.

**Proof of Corollary 3.** Let \( F = \alpha f \). Then \( F \) is completely \( Q \)-additive with component functions \( F_j = \alpha f_j \). As in Corollary 2, if \( \alpha \) is rational then \( F \) cannot be uniformly distributed modulo 1. Assume therefore that \( \alpha \) is
irrational. We will show that condition (ii) of Theorem 1 is satisfied (with \( F \) in place of \( f \)) for all \( k \neq 0 \). By Theorem 1 it then follows that \( F \) is uniformly distributed modulo 1. Fix an integer \( k \neq 0 \) and let \( a \) be as in the statement of the corollary, so that \( f_0(a) > 0 \) and \( a < q_j \) for all \( j \). As in the proof of Corollary 1, we have, for all \( j \),

\[
1 - \frac{1}{q_j} \left| \sum_{0 \leq n < q_j} e(kF_j(n)) \right| \geq \frac{2\|kF_j(a)\|}{q_j} = \frac{2\|kaF_j(a)\|}{q_j} = \frac{2\|kaF_0(a)\|}{q_j}.
\]

Since \( \|kaF_0(a)\| \neq 0 \) by our assumptions that \( \alpha \) is irrational, \( k \neq 0 \), and \( f_0(a) \neq 0 \), and since, by the hypothesis of Corollary 2, the series \( \sum_{j \geq 0} 1/q_j \) diverges, condition (ii) of Theorem 1 is satisfied as claimed.

**Proof of Corollary 4.** We note first that the component functions \( f_j(n) \) of \( f(n) = \alpha M(n) \) are given by

\[
f_j(n) = \begin{cases} \alpha, & n = q_j - 1, \\ 0, & 0 \leq n < q_j - 1. \end{cases}
\]

Thus we have, for any integer \( k \neq 0 \),

\[
\sum_{0 \leq n < q_j} e(kf_j(n)) = q_j - 1 + e(k\alpha).
\]

It follows that condition (i) of Theorem 1 is satisfied if and only if, for some \( j \), \( q_j = 2 \) and \( \|k\alpha\| = 0 \). Since \( \alpha \) is irrational, this is impossible unless \( k = 0 \). Therefore, condition (i) of Theorem 1 does not hold for any \( k \neq 0 \).

We next show that condition (ii) of Theorem 1 is equivalent to the divergence of \( \sum_{j \geq 0} 1/q_j \). In view of (5.2), condition (ii) of Theorem 1 is equivalent to

\[
\sum_{j \geq 0} \left( 1 - \frac{1}{q_j} \left| q_j - 1 + e(k\alpha) \right| \right) = \infty.
\]

To show the equivalence between (5.3) and the divergence of \( \sum_{j \geq 0} 1/q_j \), we will establish the inequalities

\[
\frac{4\|k\alpha\|^2}{q} \leq 1 - \frac{1}{q} \left| q - 1 + e(k\alpha) \right| \leq \frac{2}{q}
\]

for any integer \( q \geq 2 \) and any real number \( \alpha \).

The upper bound in (5.4) is trivial. To prove the lower bound, we note that

\[
\left( \frac{1}{q} \left| q - 1 + e(k\alpha) \right| \right)^2 \leq \frac{1}{q^2}((q - 1)^2 + 1 + 2(q - 1)\cos(2\pi k\alpha))
\]

\[
= \frac{1}{q^2}(q^2 - 2(q - 1)(1 - \cos(2\pi k\alpha))).
\]
\[ = 1 - \frac{2(q-1)}{q^2} (1 - \cos(2\pi k\alpha)) \]
\[ \leq \left( 1 - \frac{q-1}{q^2} (1 - \cos(2\pi k\alpha)) \right)^2 \]
\[ \leq \left( 1 - \frac{(q-1)(8\|k\alpha\|^2)}{q^2} \right)^2 \]
\[ \leq \left( 1 - \frac{4\|k\alpha\|^2}{q} \right)^2 . \]

It follows that
\[ \frac{1}{q} |q - 1 + e(k\alpha)| \leq 1 - \frac{4\|k\alpha\|^2}{q} , \]
which implies the lower bound in (5.4). Since \( \alpha \) is irrational, we have \( \|k\alpha\| \neq 0 \) for all non-zero integers \( k \). Thus condition (ii) of Theorem 1 holds for all \( k \neq 0 \) if and only if \( \sum_{j \geq 0} 1/q_j \) diverges. From the theorem it therefore follows that \( f \) is uniformly distributed modulo 1 if and only if \( \sum_{j \geq 0} 1/q_j \) diverges.

It remains to show that \( f \) has a non-uniform limit distribution modulo 1 if and only if the series \( \sum_{j \geq 0} 1/q_j \) converges. To this end we note that, by (5.2), the first part of condition (iii) of Theorem 1 is equivalent to the convergence of
\[ \sum_{j \geq 0} \frac{1 - e(k\alpha)}{q_j} , \]
which in turn is equivalent to the convergence of \( \sum_{j \geq 0} 1/q_j \), since \( \alpha \) is irrational. Therefore it remains only to show that if \( \sum_{j \geq 0} 1/q_j \) converges, then the second part of condition (iii) of Theorem 2.1 holds for all \( k \neq 0 \). This follows immediately from the observation that, for all \( k \neq 0 \),
\[ \max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} \|kf_j(m)\|^2 = \frac{\|k\alpha\|^2}{q_j} \to 0 , \]
as \( j \) tends to infinity, since the convergence of \( \sum_{j \geq 0} 1/q_j \) implies that \( 1/q_j \) tends to 0.

Proof of Corollary 5. We note first that, for all \( j \) with \( q_j > a \),
\[ f_j(n) = \begin{cases} \alpha, & n = a, \\ 0, & \text{otherwise}. \end{cases} \]
(5.5)

Thus, we have
\[ \sum_{0 \leq n < q_j} e(kf_j(n)) = \begin{cases} q_j - 1 + e(k\alpha), & q_j > a, \\ q_j, & q_j \leq a. \end{cases} \]
(5.6)
As in the proof of Corollary 4, this implies that condition (i) of Theorem 1 does not hold for any \( k \neq 0 \). Moreover, using (5.4), we see that condition (ii) of Theorem 1 is satisfied for all \( k \neq 0 \) if and only if \( \sum q_j > a \) diverges. Therefore, \( f \) is uniformly distributed modulo 1 if and only if \( \sum q_j \) diverges. This proves the first assertion of the corollary.

To prove the second assertion of the corollary, we note that by (5.5), we have, for all \( k \neq 0 \),
\[
\max_{0 < n \leq q_j} \frac{1}{n} \sum_{0 \leq m < n} \|k f_j(m)\|^2 = \begin{cases} \|k \alpha\|^2/a, & q_j > a, \\ 0, & q_j \leq a. \end{cases}
\]
Therefore, the limit in condition (iii) of Theorem 1 is 0 for all \( k \neq 0 \) if and only if \( q_j \leq a \) for all but at most finitely many \( j \). It remains only to show that under the same condition, the series in condition (iii) of Theorem 1 converges for all \( k \neq 0 \). This follows immediately, since, by (5.6),
\[
\sum_{j \geq 0} \left( 1 - \frac{1}{q_j} \sum_{0 \leq n < q_j} e(k f_j(n)) \right) = \sum_{q_j > a} \frac{1 - e(k \alpha)}{q_j}.
\]

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