ACTA ARITHMETICA
C.2 (2001)

Certain classes of rapidly convergent series representations for $L(2n, \chi)$ and $L(2n+1, \chi)$

by

H. M. Srivastava (Victoria, BC) and Hirofumi Tsumura (Tokyo)

1. Introduction. For a non-trivial primitive Dirichlet character χ of modulus q, let $L(s, \chi)$ denote the Dirichlet L-function defined (for $\Re(s) > 1$) by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\Re(s) > 1), \tag{1.1}$$

and (for $\Re(s) \le 1$) by its analytic continuations (see, e.g., [5, Chapter 4]). Then, in terms of the familiar generalized Bernoulli numbers $B_{n,\chi}$ defined by means of the generating function

$$\sum_{k=1}^{q} \frac{\chi(k) t e^{kt}}{e^{qt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} \quad (|t| < 2\pi/q), \tag{1.2}$$

it is fairly well known that

$$L(2n+1,\chi) = \frac{(-1)^n i \tau(\chi)}{2\cdot(2n+1)!} \left(\frac{2\pi}{q}\right)^{2n+1} B_{2n+1,\overline{\chi}} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1,2,3,\dots\}) \tag{1.3}$$

and

$$L(2n,\chi) = \frac{(-1)^{n-1} \tau(\chi)}{2\cdot(2n)!} \left(\frac{2\pi}{q}\right)^{2n} B_{2n,\overline{\chi}} \quad (n \in \mathbb{N}) \tag{1.4}$$

for χ with $\chi(-1) = -1$ and $\chi(-1) = 1$, respectively; here $i := \sqrt{-1}$ and $\tau(\chi)$ is the Gauss sum defined by

$$\tau(\chi) := \sum_{k=1}^{q} \chi(k) \exp\left(\frac{2k\pi i}{q}\right).$$

2000 *Mathematics Subject Classification*: Primary 11M06; Secondary 11M38, 11B68.
Key words and phrases: Dirichlet L-functions, series representations, generating functions, Gauss sum, Mellin transformation, holomorphic function.

But no such simple (and useful) representations exist for $L(2n,\chi)$ and $L(2n+1,\chi)$ for χ with $\chi(-1)=-1$ and $\chi(-1)=1$, respectively. Recently, by making use of the Mellin transformation technique, Katsurada [1] proved the following series representations (see [1, p. 82, Theorem 3]): Let $u\in\mathbb{R}$ with $|u|\le 1$. If $\chi(-1)=1$ and $\chi\neq 1$, then

$$\begin{aligned}(1.5)\quad & nL(2n+1,\chi)-n\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+1}}\cos\left(\frac{2l\pi u}{q}\right)-\frac{\pi u}{q}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n}}\sin\left(\frac{2l\pi u}{q}\right)\\ &=(-1)^n\left(\frac{2\pi u}{q}\right)^{2n}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\cdot k}{(2n-2k)!}\cdot\frac{L(2k+1,\chi)}{(2\pi u/q)^{2k}}\right.\\ &\qquad\left.+\frac{\tau(\chi)}{q}\sum_{k=1}^{\infty}\frac{(2k)!}{(2n+2k)!}L(2k,\overline{\chi})u^{2k}\right]\quad(n\in\mathbb{N}).\end{aligned}$$

Furthermore, if $\chi(-1)=-1$, then

$$\begin{aligned}(1.6)\quad & L(2n,\chi)-\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n}}\cos\left(\frac{2l\pi u}{q}\right)\\ &=(-1)^n\left(\frac{2\pi u}{q}\right)^{2n-1}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{(2n-2k)!}\cdot\frac{L(2k,\chi)}{(2\pi u/q)^{2k-1}}\right.\\ &\qquad\left.+\frac{2\tau(\chi)i}{q}\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}L(2k+1,\overline{\chi})u^{2k+1}\right]\quad(n\in\mathbb{N}).\end{aligned}$$

The main object of this paper is to derive two (presumably new) members of the class of the series representations (1.5) and (1.6) (see [3, Section 3]), by using the same methods as those in our earlier work [2]. The infinite series occurring in these two members (see (3.1) and (3.2) below) converge remarkably faster than those in (1.5) and (1.6).

2. A set of lemmas. We make use of χ-analogues of the notations introduced already in our earlier paper [2]. Throughout this paper, we assume that χ is a non-trivial primitive Dirichlet character of modulus q. We define the sequence $\{\beta_{n,\chi}(x)\}_{n=0}^{\infty}$ by means of the generating function (cf. (1.2))

$$(2.1)\qquad F(x,t,\chi):=\sum_{k=1}^{q}\frac{\chi(k)tx^{q-k}e^{kt}}{e^{qt}-x^q}=\sum_{n=0}^{\infty}\beta_{n,\chi}(x)\frac{t^n}{n!}\quad(|t|<2\pi/q)$$
$$(1\le x\le 1+c;\ c>0),$$

so that, clearly,

$$(2.2)\qquad \beta_{n,\chi}(1)=B_{n,\chi}\quad(n\in\mathbb{N}_0).$$

We note that the numbers $\beta_{n,\chi}(x)$ are essentially the same as the generalized Euler numbers which were considered elsewhere by Tsumura [4, p. 282, (5)]. Since $\sum_{k=1}^{q}\chi(k)=0$ and since the zeros of $e^{qt}-x^q$ are given by

$$t=\frac{2n\pi i}{q}+\log x \qquad (n\in\mathbb{Z}), \tag{2.3}$$

the radius of convergence of the series in (2.1) is at least $2\pi/q$. Hence, by the Cauchy–Hadamard theorem for absolute convergence (cf., e.g., [6, p. 30]), we have

LEMMA 1. *Let the sequence* $\{\beta_{n,\chi}(x)\}_{n=0}^{\infty}$ *be defined by* (2.1). *Then there exists some non-negative real number* κ *such that*

$$\liminf_{n\to\infty}\left(\frac{|\beta_{n,\chi}(x)|}{n!}\right)^{-1/n}=\frac{2\pi}{q}+\kappa \qquad (\kappa\ge 0). \tag{2.4}$$

We now consider the following Dirichlet series (cf. (1.1)):

$$\omega(s,x,\chi):=\sum_{n=1}^{\infty}\frac{x^{-n}\chi(n)}{n^s} \qquad (1\le x\le 1+c;\ c>0), \tag{2.5}$$

so that, clearly,

$$\omega(s,1,\chi)=L(s,\chi). \tag{2.6}$$

In case $1<x\le 1+c$ $(c>0)$, we can see that the function $\omega(s,x,\chi)$ is holomorphic on the whole complex s-plane.

LEMMA 2. *Let* $\beta_{n,\chi}(x)$ *and* $\omega(s,x,\chi)$ *be defined by* (2.1) *and* (2.5), *respectively. Then*

$$\omega(1-n,x,\chi)=-\frac{\beta_{n,\chi}(x)}{n} \qquad (n\in\mathbb{N}). \tag{2.7}$$

Proof. The relationship (2.7) is well known when $x=1$. So we assume that $1<x\le 1+c$ $(c>0)$. For $t\in\mathbb{C}$ with $|t|<\log x$, it is easily seen that

$$\begin{aligned}\frac{1}{z^q-1}\sum_{k=1}^{q}\chi(k)z^k &= -\sum_{k=1}^{q}\chi(k)\sum_{j=1}^{\infty}z^{k+q(j-1)}\\ &= -\sum_{j=1}^{\infty}\sum_{k=1+q(j-1)}^{qj}\chi(k-qj+q)z^k\\ &= -\sum_{j=1}^{\infty}\chi(j)z^j \qquad (z:=e^t/x),\end{aligned}$$

since $|z|<1$. Thus, for $t\in\mathbb{C}$ and $|t|<\log x$, the generating function (2.1) readily yields

$$F(x,t,\chi) = -t\sum_{j=1}^{\infty} x^{-j}\chi(j)e^{jt} = -\sum_{j=1}^{\infty} x^{-j}\chi(j)\sum_{n=0}^{\infty}\frac{j^n t^{n+1}}{n!} \tag{2.8}$$
$$= -\sum_{n=0}^{\infty}\Big(\sum_{j=1}^{\infty} x^{-j}\chi(j)j^n\Big)\frac{t^{n+1}}{n!}$$
$$= -\sum_{n=1}^{\infty} n\Big(\sum_{j=1}^{\infty} x^{-j}\chi(j)j^{n-1}\Big)\frac{t^n}{n!},$$

where the inversion of the order of summation can be justified by absolute convergence of the series involved. The assertion (2.7) of Lemma 2 would now follow from (2.8) if we apply the definition (2.5) and compare the coefficients of t^n in (2.1) and (2.8).

Lemma 3. *Let* $n \in \mathbb{N}$ *and* $|\theta| < 2\pi/q$ $(\theta \in \mathbb{R})$.

(1) *If* $\chi(-1) = 1$ *and* $\chi \neq 1$, *then*

$$\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n+2}}\sin(k\theta) = \sum_{k=0}^{n}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\omega(2n-2k+1,x,\chi) - \sum_{k=n+1}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\cdot\frac{\beta_{2k-2n,\chi}(x)}{2k-2n} \quad (1 < x \leq 1+c;\ c>0). \tag{2.9}$$

(2) *If* $\chi(-1) = -1$, *then*

$$\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n+1}}\sin(k\theta) = \sum_{k=0}^{n-1}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\omega(2n-2k,x,\chi) - \sum_{k=n}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\cdot\frac{\beta_{2k-2n+1,\chi}(x)}{2k-2n+1} \quad (1 < x \leq 1+c;\ c>0). \tag{2.10}$$

Proof. Denote, for convenience, the left-hand side of (2.9) by $\Omega_n(x,\chi,\theta)$. Then it is easily seen that

$$\Omega_n(x,\chi,\theta) = \sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n+2}}\sum_{j=0}^{\infty}\frac{(-1)^j(k\theta)^{2j+1}}{(2j+1)!} \tag{2.11}$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n-2j+1}},$$

where the various interchanges of the order of summation are justified by absolute convergence of the series involved under the conditions stated already in Lemma 3. Upon separating the j-sum in (2.11) into two parts, we have

$$\begin{aligned}\Omega_n(x,\chi,\theta) &= \sum_{j=0}^{n}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n-2j+1}} \\ &\quad + \sum_{j=n+1}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\sum_{k=1}^{\infty}\frac{x^{-k}\chi(k)}{k^{2n-2j+1}} \\ &= \sum_{j=0}^{n}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\omega(2n-2j+1,x,\chi) \\ &\quad - \sum_{j=n+1}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\cdot\frac{\beta_{2j-2n,\chi}(x)}{2j-2n},\end{aligned}\tag{2.12}$$

by the definition (2.5) and Lemma 2, which evidently completes the proof of Lemma 3(1). By employing the same techniques as above, we can give the proof of Lemma 3(2).

3. The main series representations. By applying Lemma 3, we next prove the following

THEOREM. *Let $u\in\mathbb{R}$ and $|u|\le 1$.*

(1) *If $\chi(-1)=1$ and $\chi\ne 1$, then*

$$\begin{aligned}L(2n+1,\chi) &- \frac{q}{2\pi u}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+2}}\sin\left(\frac{2l\pi u}{q}\right) \\ &= (-1)^{n+1}\left(\frac{2\pi u}{q}\right)^{2n}\Bigg[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\cdot\frac{L(2k+1,\chi)}{(2\pi u/q)^{2k}} \\ &\quad + \frac{2\tau(\chi)}{q}\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}L(2k,\overline{\chi})u^{2k}\Bigg] \qquad (n\in\mathbb{N}).\end{aligned}\tag{3.1}$$

(2) *If $\chi(-1)=-1$, then*

$$\begin{aligned}L(2n,\chi) &- \frac{q}{2\pi u}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+1}}\sin\left(\frac{2l\pi u}{q}\right) \\ &= (-1)^{n+1}\left(\frac{2\pi u}{q}\right)^{2n-1}\Bigg[\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n+2k+1)!}\cdot\frac{L(2k,\chi)}{(2\pi u/q)^{2k-1}} \\ &\quad - \frac{2\tau(\chi)i}{q}\sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}L(2k+1,\overline{\chi})u^{2k+1}\Bigg] \qquad (n\in\mathbb{N}).\end{aligned}\tag{3.2}$$

Proof. Let $\theta = 2\pi u/q$. Then it is easily observed that the series on the left-hand sides of (2.9) and (2.10) are uniformly convergent with respect to x on the closed interval $[1, 1+c]$ $(c > 0)$. On the other hand, it follows from Lemma 1 that the series on the right-hand sides of (2.9) and (2.10) are also uniformly convergent with respect to x on $[1, 1+c]$ $(c > 0)$. Hence, by letting $x \to 1+$ in Lemma 3, we have, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k^{2n+2}} \sin\left(\frac{2k\pi u}{q}\right) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\left(\frac{2k\pi u}{q}\right)^{2k+1} L(2n-2k+1, \chi) - \sum_{k=n+1}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{2k\pi u}{q}\right)^{2k+1} \frac{B_{2k-2n,\chi}}{2k-2n} \tag{3.3}$$

and

$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k^{2n+1}} \sin\left(\frac{2k\pi u}{q}\right) = \sum_{k=1}^{n-1} \frac{(-1)^k}{(2k+1)!}\left(\frac{2k\pi u}{q}\right)^{2k+1} L(2n-2k, \chi) - \sum_{k=n}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{2k\pi u}{q}\right)^{2k+1} \frac{B_{2k-2n+1,\chi}}{2k-2n+1} \tag{3.4}$$

for $\chi \neq 1$ with $\chi(-1) = 1$ and $\chi(-1) = -1$, respectively. By using the relationships (1.3) and (1.4), we readily obtain the assertions (3.1) and (3.2) of the Theorem.

The infinite series occurring on the right-hand sides of (3.1) and (3.2) *obviously* converge more rapidly than the corresponding ones in (1.5) and (1.6), respectively.

Acknowledgements. The present investigation was initiated during the first-named author's visit to several universities and research institutes in Japan in May 2000. This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

[1] M. Katsurada, *Rapidly convergent series representations for $\zeta(2n+1)$ and their χ-analogue*, Acta Arith. 90 (1999), 79–89.

[2] H. M. Srivastava and H. Tsumura, *A certain class of rapidly convergent series representations for $\zeta(2n+1)$*, J. Comput. Appl. Math. 118 (2000), 323–335.

[3] —, —, *New rapidly convergent series representations for $\zeta(2n+1)$, $L(2n,\chi)$ and $L(2n+1,\chi)$*, Math. Sci. Res. Hot-Line 4 (2000), no. 7, 17–24.

[4] H. Tsumura, *On a p-adic interpolation of the generalized Euler numbers and its applications*, Tokyo J. Math. 10 (1987), 281–293.

[5] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Springer, New York, 1997.

[6] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, 4th ed., Cambridge Univ. Press, Cambridge, 1927.

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
E-mail: harimsri@math.uvic.ca

Department of Management
Tokyo Metropolitan College
Akishima, Tokyo 196-8540, Japan
E-mail: tsumura@tmca.ac.jp

Received on 18.8.2000
and in revised form on 20.11.2000 (3872)