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On the Barban–Davenport–Halberstam theorem: XV

by

C. Hooley (Cardiff)

1. Introduction. We revert to the subject of the thirteenth article XIII [5] of this series (throughout we refer to such previous articles by the Roman numerals corresponding to their position in the series, full biographical details of those cited being given at the end), in which we discussed and augmented Liu's [6] lower bounds for the Barban–Davenport–Halberstam moments

$$S(x,Q)=\sum_{k\le Q}G(x,k)=\sum_{k\le Q}\sum_{\substack{0<a\le k\\(a,k)=1}}E^2(x;a,k)=\sum_{k\le Q}\sum_{\substack{0<a\le k\\(a,k)=1}}\left(\theta(x;a,k)-\frac{x}{\phi(k)}\right)^2 \tag{1}$$

that involve the prime number counting functions

$$\theta(x;a,k)=\sum_{\substack{p\le x\\p\equiv a,\bmod k}}\log p.$$

First, to put our previous and present work in an appropriate context, we must mention that we were somewhat remiss in XIII in merely reporting Liu's statement about $S(x,Q)$ in his Theorem 2 when a more careful reading of his Introduction would have revealed he had overlooked the fact that a stronger result was an immediate corollary of his Theorem 1. Associated more fundamentally in a statistical sense with $\theta(x;a,k)$ than $S(x,Q)$, the subject of the latter theorem was the parallel sum

$$S^*(x,Q)=\sum_{k\le Q}G^*(x,k)=\sum_{k\le Q}\sum_{\substack{0<a\le k\\(a,k)=1}}E^{*2}(x;a,k) \tag{2}$$

in which the differences

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$$(3)\qquad E^*(x;a,k) = \theta(x;a,k) - \frac{1}{\phi(k)} \sum_{\substack{p\le x\\ p\nmid k}} \log p = \theta(x;a,k) - \frac{\theta_k(x)}{\phi(k)}, \quad \text{say},$$

appear instead of the approximate $E(x;a,k)$. Consequently, being a true dispersion because the sum of $\theta(x;a,k)$ over a complete set of reduced residues a, mod k, is $\theta_k(x)$, the sum $G^*(x,k)$ appearing here is a minorant for sums such as $G(x,k)$. However, rather than appealing to this simple principle that provides a lower estimate for $S(x,Q)$ at least as good as one for $S^*(x,Q)$, Liu effected the transition from the latter sum to the former by using the estimate

$$\theta(x) - x = O(xe^{-A(\log x)^{3/5-\varepsilon}})$$

and thereby unnecessarily reduced the range of validity of his estimate for $S(x,Q)$ to

$$(4)\qquad Q > xe^{-A(\log x)^{3/5-\varepsilon}}$$

from the wider

$$(5)\qquad Q > xe^{-o(\log x/\log\log x)}$$

that was valid for $S^*(x,Q)$. Nevertheless, it is clear that in any fair comparison of [6] with other work both of the inequalities

$$(6)\qquad S^*(x,Q),\ S(x,Q) > \left(\frac{1}{4} - \varepsilon\right)Qx\log x \qquad (x > x_0(\varepsilon))$$

for the range (5) should essentially be attributed to Liu.

Perelli [8] in 1995 improved Liu's results by shewing that the inequalities (6) were true in any longer range of the type

$$(7)\qquad Q > x^{1-o(1)}$$

and that actually

$$(8)\qquad S^*(x,Q) > (A_1(\delta) - \varepsilon)Qx\log x$$

for $Q = x^{1-\delta}$ and a function $A_1(\delta)$ tending to $1/4$ as $\delta \to 0$, although he like us afterwards cited Liu's second inequality for the unnecessarily attenuated range (4). His paper, of whose existence we were unfortunately ignorant when XIII was written, sketches a refinement in Liu's method that relates primarily to any exceptional zeros of Dirichlet's L-functions formed with real characters, taking account of the fact neglected by Liu that, if χ be a character, mod k, associated with a primitive character, mod q, then $\tau(\chi) = 0$ unless $(k/q,q) = 1$.

In a similar direction, the procedures of XIII in its unconditional aspect were tantamount to gaining the lower bound

$$(9)\qquad S^*(x,Q) > (1-\varepsilon)Qx\log x$$

for the range (4) and the deduction of a like bound for $S(x,Q)$ through the above stated minimal property of $S^*(x,Q)$. Thus our results were more accurate than Liu's and Perelli's within the range (4) but had a narrower sphere of applicability. To these, we must then add the bounds

$$S^*(x,Q),\ S(x,Q) > \left(2 - \frac{\log x}{\log Q} - \varepsilon\right) Qx \log Q \tag{10}$$

that were shewn in XIII to hold on the assumption that the Riemann zeta-function $\zeta(s)$ have no zeros with real part exceeding $3/4$ and that therefore provide a conditional lower bound of expected magnitude for values of Q down to about $x^{1/2}$. But a re-examination of our treatment reveals that we too were guilty of an oversight because, although we were mindful of the basic inequality between $S(x,Q)$ and $S^*(x,Q)$, we failed to avail ourselves of the actual positive value of their difference. The first purpose of this paper is therefore to remedy the resulting underperformance and to shew that the bound (10) for $S(x,Q)$ is true without qualification; we thus advance far ahead of (8) for $S(x,Q)$ and produce the first unconditional bound for a Barban–Davenport–Halberstam sum that is valid for all values of Q above about $x^{1/2}$ (compare with the asymptotic formulae for $S(x,Q)$ that were obtained in I and II on the assumption of the extended Riemann hypothesis).

Yet, though the latter of the two sums $S^*(x,Q), S(x,Q)$ may be the more natural to study from the viewpoint of the prime number theorem for arithmetical progressions, it is clear that it is the former that is of greater basic interest because of its interpretation as a sum of dispersions of $\theta(x;a,k)$, a realization that is strengthened by our appreciation of the way $\theta(x;a,k)$ can be expressed in terms of $\theta_k(x)/\phi(k)$ and sums $\theta(x,\chi)$ affected by non-principal characters, mod k. Here, putting aside the conditional result on $S^*(x,Q)$ in (10), we are left with the comparison between the unconditional (8) and (9) for $S^*(x,Q)$ and the need to improve their joint effect.

It is to the last requirement that we denote the second and principal part of this paper, proving the better inequality

$$S^*(x,Q) > \left(\frac{\pi^2}{12} - \varepsilon\right) Qx \log x \qquad (x > x_0(\varepsilon))$$

in the range (7) and in fact that $A_1(\delta) \to \pi^2/12$ in (8) as $\delta \to 0$. Being almost $5/6$, the multiplier here of $Qx\log x$ does not fall far short of the value 1 in (9), while the range of applicability is much wider (down to $Q = x^{1-\delta_1}$) than the former range in which the latter value had been shewn to be appropriate. Two main refinements in the methods of Liu and Perelli are the source of the improvement and are so incorporated in the treatment that it is desirable to rework with more finesse those aspects of the method that depend on the ideas of Gallagher [2].

Two other matters should be mentioned. The first is that alongside the sums $S(x,Q), S^*(x,Q)$ there is a third one $S^{**}(x,Q)$ that emerges from the replacement of $\theta_k(x)$ in $E^*(x;a,k)$ by $\theta(x)$ and that in some ways is the easiest to work with. Since

$$\theta_k(x) = \theta(x) - \sum_{\substack{p\le x\\ p\mid k}} \log p = \theta(x) + O(\log k), \tag{11}$$

$S^*(x,Q)$ and $S^{**}(x,Q)$ are virtually indistinguishable and are subject in most instances to comparable bounds. The other is that we refer the reader to the end for a discussion about the sharpness of our estimations and how the method could be varied to provide either a shorter treatment or a more accurate one.

2. Notation and conventions. The symbol ε denotes an arbitrarily small positive number while A is a positive absolute constant; in accord with normal practice, neither of these necessarily remains the same on all occasions or, indeed, within given equations and inequalities. The symbol x is a positive variable to be regarded as tending to infinity so that every stated relation is valid for sufficiently large x; in particular, those involving ε are true for values of x exceeding some number that depends on ε but that is independent of the parameter δ. The constants implied by the O-notation are of type A.

3. The improved lower bound for $S(x,Q)$. As implied in the Introduction, we advance to a better unconditional lower bound for $S(x,Q)$ by replacing an inequality by a stronger equality in the preliminary §4 of XIII, in which all relevant definitions and conditions are to be retained. Letting $G^*(x,Q)$ denote the exact dispersion

$$\sum_{\substack{0<a\le k\\ (a,k)=1}} (\theta(x;a,k) - \theta_k(x)/\phi(k))^2,$$

we now begin with the two relations

$$G^*(x,Q) = \sum_{\substack{0<a\le k\\ (a,k)=1}} \theta^2(x;a,k) - \frac{\theta_k^2(k)}{\phi(k)} \tag{12}$$

and

$$G(x,Q) = G^*(x,Q) + \frac{\{x-\theta_k(x)\}^2}{\phi(k)}$$

that imply that

$$G(x,Q) = \sum_{\substack{0<a\le k\\ (a,k)=1}} \theta^2(x;a,k) - \frac{\theta_k^2(x)}{\phi(k)} + \frac{\{x-\theta_k(x)\}^2}{\phi(k)}$$

$$= \sum_{\substack{0<a\le k\\(a,k)=1}} \theta^2(x;a,k) - \frac{\theta^2(x)}{\phi(k)} + \frac{\{x-\theta(x)\}^2}{\phi(k)} + O\left(\frac{x\log k}{\phi(k)}\right)$$

in virtue of (11). This ousts the first inequality for $G(x,k)$ in XIII and therefore equations (58) and then (64) therein remain true if the additional term

$$\frac{\zeta(2)\zeta(3)}{\zeta(6)}\{x-\theta(x)\}^2\log\frac{Q_2}{Q_1} \tag{13}$$

be added to their final right-hand sides.

To benefit from this adjustment it is preferable to use the structure of the second form of treatment originated in §6 of XIII. Having arrived at equation (88) therein, we add the term (13) to its first two constituents and therefore annihilate the effect of $(x-\theta(x))^2$ on the proceedings. Since it was only here that the conditional estimate (83) was used in the work, we have therefore elevated the proof to an unconditional status. Consequently, we can now state

THEOREM 1. *Let* $E(x;a,k)$ *be defined as in* (1). *Then, if* $Q \le x$, *we have*

$$\sum_{k\le Q}\sum_{\substack{0<a\le k\\(a,k)=1}} E^2(x;a,k) > \left(2 - \frac{\log x}{\log Q} - \varepsilon\right)Qx\log Q$$

for $x > x_0(\varepsilon)$.

It was stated in the Introduction that the initial inequality for $G(x,k)$ in XIII was founded on a comparable one for $G^*(x,k)$. Since this is easily confirmed, we see that we may replace $E(x;a,k)$ by $E^*(x;a,k)$ in Theorem 1 here or in Theorem 2 of XIII provided that we still assume that $\zeta(s)$ has no zeros with real part exceeding 3/4.

4. The exponential sum $f(\theta)$ for rational values of θ and its relation to $G^*(x,Q)$. Since the exponential sum

$$f(\theta) = \sum_{p\le x}\log p\, e^{2\pi ip\theta}$$

plays an indispensable rôle in our work, we first develop some of its properties for rational values of θ and, in particular, its relationships with the entities $G^*(x,k)$ and $E^*(x;a,k)$ in (2) and (3).

First, the sum of $f(b/q)$ taken over a complete set of reduced residues b, mod q, is

$$\sum_{\substack{0<b\le q\\(b,q)=1}} f\left(\frac{b}{q}\right) = \sum_{\substack{0<b\le q\\(b,q)=1}}\sum_{p\le x}\log p\, e^{2\pi ipb/q} = \sum_{p\le x} c_q(p)\log p,$$

in which $c_q(p)$ is a sum that equals $\mu(q)$ or $\mu(q)-\mu(q/p)p$ according as $p\nmid q$ or $p\,|\,q$. Hence, for $q\le x$,

$$\sum_{\substack{0<b\le q\\(b,q)=1}} f\left(\frac{b}{q}\right)=\mu(q)\sum_{p\le x}\log p-\sum_{p|q}\mu\left(\frac{q}{p}\right)p\log p$$
$$=\mu(q)\theta(x)+O\Big(q\sum_{p|q}\log p\Big)=\mu(q)\theta(x)+O(q\log q)$$

and the dispersion $D(x,q)$ of $f(b/q)$ is accordingly determined by

$$(14)\qquad \phi(q)D(x,q)=\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2-\frac{1}{\phi(q)}(\mu(q)\theta(x)+O(q\log 2q))^2$$
$$=\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2-\frac{\mu^2(q)\theta^2(x)}{\phi(q)}+O\left(\frac{xq\log^2 2q}{\phi(q)}\right)$$

in this case.

Secondly, in somewhat similar vein, we deduce from equations (12) and (11) the preparatory equation

$$(15)\qquad G^*(x,k)=\sum_{\substack{0<a\le k\\(a,k)=1}}\theta^2(x;a,k)-\frac{\theta^2(x)}{\phi(k)}+O\left(\frac{x\log 2k}{\phi(k)}\right)\qquad (k\le x)$$

that participates in another analysis of $f(b/q)$. This begins with the equation

$$\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2=\sum_{0<b\le q}\left|f\left(\frac{b}{q}\right)\right|^2\sum_{d|b,\,d|q}\mu(d)=\sum_{d|q}\mu(d)\sum_{0<b'\le q/d}\left|f\left(\frac{b'}{q/d}\right)\right|^2$$
$$=\sum_{k|q}\mu\left(\frac{q}{k}\right)\sum_{0<b'\le k}\left|f\left(\frac{b'}{k}\right)\right|^2,$$

wherein the last inner sum equals

$$\sum_{0<b'\le k}\sum_{p,p'\le x}\log p\,\log p'\,e^{2\pi i(p-p')b'/k}=\sum_{p,p'\le x}\log p\,\log p'\sum_{0<b'\le k}e^{2\pi i(p-p')b'/k}$$
$$=k\sum_{\substack{p,p'\le x\\p-p'\equiv 0,\,\mathrm{mod}\,k}}\log p\,\log p'=k\sum_{0<a\le k}\theta^2(x;a,k)$$
$$=k\sum_{\substack{0<a\le k\\(a,k)=1}}\theta^2(x;a,k)+k\sum_{p|k}\log^2 p=k\sum_{\substack{0<a\le k\\(a,k)=1}}\theta^2(x;a,k)+O(k\log^2 2k)$$

when $k \le x$. Taken together and then considered with (15), these two equations imply that

$$(16) \quad \sum_{\substack{0<b\le q\\(b,q)=1}} \left| f\left(\frac{b}{q}\right)\right|^2$$
$$= \sum_{k|q} \mu\left(\frac{q}{k}\right) k \sum_{\substack{0<a\le k\\(a,k)=1}} \theta^2(x;a,k) + O\Big(\sum_{k|q} k\log^2 2k\Big)$$
$$= \sum_{k|q} \mu\left(\frac{q}{k}\right) k \left\{ G^*(x,k) + \frac{\theta^2(x)}{\phi(k)} + O\left(\frac{x\log 2k}{\phi(k)}\right)\right\}$$
$$+ O(\sigma(q)\log^2 2q)$$
$$= \sum_{k|q} \mu\left(\frac{q}{k}\right) kG^*(x,k) + \theta^2(x)\sum_{k|q}\mu\left(\frac{q}{k}\right)\frac{k}{\phi(k)}$$
$$+ O(xd_3(q)\log 2q) + O(\sigma(q)\log^2 2q)$$
$$= \sum_{k|q} \mu\left(\frac{q}{k}\right) kG^*(x,k) + \frac{\mu^2(q)\theta^2(x)}{\phi(q)} + O(xd_3(q)\log^2 2q),$$

which evaluation is the source of two useful conclusions. To obtain the first, we substitute in (14) to get

$$\phi(q)D(x,q) = \sum_{k|q}\mu\left(\frac{q}{k}\right)kG^*(x,k) + O(xd_3(q)\log^2 2q)$$

with the implication that

$$(17) \qquad \sum_{k|q}\mu\left(\frac{q}{k}\right)kG^*(x,k) > -Axd_3(q)\log^2 x$$

for $q \le x$. Hence, on writing

$$(18) \quad H(x,q) = \frac{1}{q}\sum_{k|q}\mu\left(\frac{q}{k}\right)kG^*(x,k) \quad \text{and} \quad L(x,U) = \sum_{q\le U} H(x,q),$$

we deduce that

$$(19) \qquad L(x,U) > -Ax\log^2 x\sum_{q\le U}\frac{d_3(q)}{q} > -Ax\log^5 x$$

for $U \le x$.

The virtues of this inequality rest on its universality and the fairly good lower bound it provides. Accompanied by the positive bound for $L(x,U)$ the main method produces for values of U larger than $x^{1-\delta}$, it allows us to proceed to a satisfactory final result in a way that would otherwise be denied us. Very possibly the left side of (19) is positive for much smaller values of U but such a property we neither need nor are able to prove.

The second inference from (16), which forms the first step in the main treatment of $L(x,U)$, is that

$$(20)\qquad L(x,U_2)-L(x,U_1)=\sum_{U_1<q\le U_2}H(x,q)$$
$$=\sum_{U_1<q\le U_2}\frac{1}{q}\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2-\theta^2(x)\sum_{U_1<q\le U_2}\frac{\mu^2(q)}{q\phi(q)}$$
$$+O\left(x\log^2 x\sum_{q\le U_2}\frac{d_3(q)}{q}\right)$$
$$=\sum_{U_1<q\le U_2}\frac{1}{q}\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2+O\left(\frac{x^2}{U_1}\right)+O(x\log^5 x).$$

Finally we should note that (16) is a much more accurate reflection of the relationship between $G^*(x,q)$ and $\sum_b |f(b/q)|^2$ than the corresponding inequality

$$G(x,q)\ge\frac{1}{q}\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2$$

found in [6] and [8] by the use of character sums. But to take advantage of this superiority we need (17), which had no counterpart in the earlier works.

5. Preliminary treatment of the second moment of $f(\theta)$. We shall now follow with appropriate modifications the mainstream of Liu's argument until its confluence in §7 with the tributary material of §4. First, letting c be a positive number exceeding 1 whose definition must await the introduction of the later equation (31), we let

$$(21)\qquad \delta_1<\frac{1}{4c}<\frac{1}{4}$$

be a suitable positive constant and bring in the primary number Q that we express as $x^{1-\delta}$, where it is supposed that $0\le\delta\le\delta_1$ and where δ will be seen to be a parameter with respect to which all limiting processes pertaining to the passage of x to infinity are uniform. In terms of Q, we then further introduce the quantities

$$(22)\qquad Q_1=Q/\log x,\qquad Q_0=(x/Q)\log x=x^\delta\log x$$

even though, as we shall later see, we could dispense with Q_1 and only work with a larger

$$Q_0 = (x^3/Q^3)\log x = x^{3\delta}\log x \tag{23}$$

provided that we were willing to accept a smaller useful value of δ.

Thus equipped, we bound the integral

$$\int_0^1 |f(\theta)|^2\,d\theta = \sum_{p\le x}\log^2 p = x\log x + O(x) \tag{24}$$

from above by

$$\sum_{q\le Q}\ \sum_{\substack{0<b\le q\\(b,q)=1}}\ \int_{-1/Qq}^{1/Qq}\left|f\left(\frac{b}{q}+\phi\right)\right|^2 d\phi = \sum_{q\le Q_0} + \sum_{Q_0<q\le Q_1} + \sum_{Q_1<q\le Q} \tag{25}$$

$$= \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3, \quad \text{say},$$

because the unit interval, mod 1, is covered by all intervals of the type $|\phi - b/q| \le 1/Qq$ that answer to the above conditions of summation. But, for interest and for a later ephemeral requirement, we should mention the companion inequality

$$\mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3 \le 2\int_0^1 |f(\theta)|^2\,d\theta, \tag{26}$$

which is verified by noting that, if $b_1/q_1 < b_2/q_2$ be two adjacent fractions in the Farey series of order $[Q]$, then their distance apart is $(b_2q_1 - b_1q_2)/q_1q_2 = 1/q_1q_2 > 1/Qq_1, 1/Qq_2$ so that no point in the unit interval, mod 1, is covered more than twice by the intervals $b/q \pm 1/Qq$. In the meanwhile, having deduced that

$$\mathrm{S}_3 \ge x\log x - \mathrm{S}_1 - \mathrm{S}_2 + O(x), \tag{27}$$

we need upper bounds for S_1 and S_2 to find a lower bound for S_3.

As in [6] and [8], an upper bound for S_2 is easily found through the large sieve inequality although here we prefer to avoid using a dissection of the range of q. This we do by changing the order in which the summation and integration are performed, thereby shewing that

$$\mathrm{S}_2 = \int_{-1/QQ_0}^{1/QQ_0}\ \sum_{Q_0<q\le\min(1/Q|\phi|,Q_1)}\ \sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}+\phi\right)\right|^2 d\phi \tag{28}$$

$$= O\Big(\int_0^{1/QQ_0}\{x^2\log x + \min^2(1/Q\phi, Q_1)x\log x\}\,d\phi\Big)$$

$$= O\left(\frac{x^2\log x}{QQ_0}\right) + O\left(x^2Q_1^2\log x\int_0^{1/QQ_1} d\phi\right) + O\left(\frac{x\log x}{Q^2}\int_{1/QQ_1}^{\infty}\frac{d\phi}{\phi^2}\right)$$

$$= O\left(\frac{x^2\log x}{QQ_0}\right) + O\left(\frac{xQ_1\log x}{Q}\right) = O(x)$$

owing to (22).

Before we embark on the estimation of S_1, which represents the hardest part of the work, it is desirable to summarize some definitions and theorems relating to characters and Dirichlet's L-functions that are specifically stated to suit the present context (Davenport's monograph [1] is a useful reference). Here χ_q denotes a character, mod q, induced by a primitive character $\chi^*_{q^*}$, where of course $q^* \mid q$ and the trivial character, mod 1, is regarded as being primitive. Then the sum

$$\tau(\chi_q) = \sum_{0<h\le q}\chi_q(h)e^{2\pi ih/q}$$

has the properties that

(29) $$|\tau(\chi^*_{q^*})| = q^{*1/2}$$

and

(30) $$\tau(\chi_q) = \begin{cases} \mu(r)\chi^*_{q^*}(r)\tau(\chi^*_{q^*}) & \text{if } q = q^*r \text{ and } (q^*, r) = 1,\\ 0 & \text{otherwise,}\end{cases}$$

so that $|\tau(\chi_q)|$ never exceeds $q^{1/2}$. Next, if $N(\alpha, T, \chi^*_{q^*})$ denote the number of zeros of $L(s, \chi^*_{q^*})$ in the region

$$\sigma > \alpha,\ |t| \le T \quad (\alpha \le 1/2 - 0; T > Q' \ge 2)$$

then

(31) $$\sum_{q^*\le Q'}\sum_{\chi^*_{q^*}} N(\alpha, T, \chi^*_{q^*}) = O(T^{c(1-\alpha)}),$$

as was shewn by Gallagher [2] for a suitable positive constant $c > 1$; this is the number used in (21) in anticipation of this statement. Also, if now

(32) $$T = Q_0^2\log^3 x$$

be chosen in terms of the Q_0 defined by (22) and if d be a suitably small positive absolute constant, then for $q^* \le Q_0$ no zeros $\beta + i\gamma$ of the functions $L(s, \chi^*_{q^*})$ lie in the region

(33) $$\sigma > 1 - \frac{d}{\log T} = 1 - T_1, \quad \text{say,} \quad |t| \le T$$

with the possible exception of one real zero $\beta_0 = \beta_0(Q_0)$ of one such function formed with a non-principal real character $\chi^*_{q_0^*}$ mod q_0^*.

6. Estimations of S_1 and S_3. For any positive integer not exceeding x,

$$f(\theta) = \sum_{\substack{p\le x\\ p\nmid q}} \log p\, e^{2\pi i p\theta} + O\Big(\sum_{p\mid q}\log p\Big)$$

$$= \sum_{\substack{n\le x\\ (n,q)=1}} \Lambda(n)e^{2\pi i n\theta} + O(\log x) + O(x^{1/2})$$

$$= f_q(\theta) + O(x^{1/2}), \quad \text{say,}$$

and thus

$$|f(\theta)|^2 = |f_q(\theta)|^2 + O(x^{3/2}).$$

Also, at $\theta = b/q + \phi$, we have in the usual way that

$$f_q(\theta) = \sum_{\substack{0<h\le q\\ (h,q)=1}} e^{2\pi i hb/q} \sum_{\substack{n\le x\\ n\equiv h,\ \mathrm{mod}\, q}} \Lambda(n)e^{2\pi i n\phi}$$

$$= \frac{1}{\phi(q)} \sum_{\substack{0<h\le q\\ (h,q)=1}} e^{2\pi i hb/q} \sum_{\chi_q} \overline{\chi}_q(h) \sum_{n\le x} \chi_q(n)\Lambda(n)e^{2\pi i n\phi}$$

$$= \frac{1}{\phi(q)} \sum_{\chi_q} \chi_q(b)\tau(\overline{\chi}_q) \sum_{n\le x} \chi_q(n)\Lambda(n)e^{2\pi i n\phi}$$

$$= \frac{1}{\phi(q)} \sum_{\chi_q} \chi_q(b)\tau(\overline{\chi}_q)F(\phi,\chi_q), \quad \text{say.}$$

From these two equations we see that the inner sum in S_1 as defined by (25) is given by

$$\sum_{\substack{0<b\le q\\ (b,q)=1}} \int_{-1/Qq}^{1/Qq} \left| f_q\Big(\frac{b}{q}+\phi\Big)\right|^2 d\phi + O\Big(\sum_{0<b\le q} x^{3/2} \int_{-1/Qq}^{1/Qq} d\phi\Big)$$

$$= \int_{-1/Qq}^{1/Qq} \sum_{\substack{0<b\le q\\ (b,q)=1}} \left| f_q\Big(\frac{b}{q}+\phi\Big)\right|^2 d\phi + O\Big(\frac{x^{3/2}}{Q}\Big),$$

the integrand in which is

$$\frac{1}{\phi^2(q)} \sum_{\chi_q,\chi'_q} \tau(\overline{\chi}_q)\overline{\tau}(\overline{\chi}_q)F(\phi,\chi_q)F(-\phi,\overline{\chi}_q) \sum_{\substack{0<b\le q\\ (b,q)=1}} \chi_q\overline{\chi}'_q(b)$$

$$= \frac{1}{\phi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 |F(\phi,\chi_q)|^2.$$

Hence, on summing over q, we infer that

$$(34)\qquad \mathrm{S}_1 = \sum_{q\le Q_0} \frac{1}{\phi(q)} \int_{-1/Qq}^{1/Qq} \sum_{\chi_q} |\tau(\chi_q)|^2 F(\phi,\chi_q)|^2\, d\phi + O\biggl(\frac{x^{3/2}Q_0}{Q}\biggr)$$
$$= \mathrm{S}_1' + O(x), \quad \text{say},$$

on account of (22) and (21).

If a typical character χ_q appearing in the definition of S_1' be induced by the primitive character $\chi_{q^*}^*$ where $q^* \mid q$, then

$$F(\phi,\chi_q) = F(\phi,\chi_{q^*}^*) + O(\log^2 x)$$

as usual so that we have the estimate

$$|F(\phi,\chi_q)|^2 = |F(\phi,\chi_{q^*}^*)|^2 + O(x\log^2 x),$$

which with (34), (30), and (29) implies that

$$(35)\qquad \mathrm{S}_1 \le \sum_{q^*\le Q_0} \frac{q^*}{\phi(q^*)} \int_{-1/Qq^*}^{1/Qq^*} \sum_{\chi_{q^*}^*} |F(\phi,\chi_q^*)|^2\, d\phi \sum_{\substack{r\le Q_0/q^*\\ (r,q^*)=1}} \frac{\mu^2(r)}{\phi(r)}$$
$$+ O\biggl(\frac{x\log^2 x}{Q}\sum_{q\le Q_0} 1\biggr) + O(x)$$
$$\le (1+\varepsilon)\log Q_0 \sum_{q^*\le Q_0} \sum_{\chi_{q^*}^*} \int_{-1/Qq^*}^{1/Qq^*} |F(\phi,\chi_{q^*}^*)|^2\, d\phi + O(x)$$
$$= (1+\varepsilon)\log Q_0 \sum_{q^*\le Q_0} \sum_{\chi_{q^*}^*} G(\chi_{q^*}^*) + O(x)$$
$$= (1+\varepsilon)\log Q_0\ H(Q_0) + O(x), \quad \text{say},$$

by (22).

To treat $G(\chi_{q^*}^*)$ we use a theorem of Gallagher's that is described in Lemma 1.9 of Montgomery's tract [7]. Here, taking their T and δ to be $1/Qq^*$ and εQq^* and considering the ensuing constants implicit in their calculation, we find that

$$\delta^2(1+\varepsilon)^{-1} \int_{-T}^{T} |F(\phi,\chi_{q^*}^*)|^2\, d\phi \le \delta^2 \frac{\sin^2 \pi\varepsilon}{\pi^2\varepsilon^2} \int_{-T}^{T} |F(\phi,\chi_{q^*}^*)|^2\, d\phi$$
$$\le \int_{-\infty}^{\infty} \biggl| \sum_{\substack{t-\varepsilon Qq^* < n\le t\\ 0<n\le x}} \Lambda(n)\chi_{q^*}^*(n)\biggr|^2 dt$$

and hence that

$$G(\chi^*_{q^*}) \le \frac{1+\varepsilon}{\varepsilon^2(Qq^*)^2} \times \int\limits_0^{x+\varepsilon Qq^*} |\psi\{\min(t,x),\chi^*_{q^*}\} - \psi\{\max(0,t-\varepsilon Qq^*),\chi^*_{q^*}\}|^2\,dt, \tag{36}$$

where the value of ε will not be allowed to change until the end of (40). But, for $\varepsilon Qq^* < t \le x + \varepsilon Qq^*$,

$$|\psi\{\min(t,x),\chi^*_{q^*}\} - \psi\{\max(0,t-\varepsilon Qq^*),\chi^*_{q^*}\}| \le \psi(t) - \psi(t-\varepsilon Qq^*) < (1+\varepsilon)\varepsilon Qq^*$$

because $Q > x^{3/4}$ and $Qq^* \le x\log x$, the inequality remaining true for $t \le \varepsilon Qq^*$ as then its left side does not exceed

$$\psi(t) < (1+\varepsilon)t \le (1+\varepsilon)\varepsilon Qq^*.$$

Thus we arrive at the inequality

$$G(\chi^*_{q^*}) \le \frac{(1+\varepsilon)^2}{\varepsilon Qq^*} \times \int\limits_0^{x+\varepsilon Qq^*} |\psi\{\min(t,x),\chi^*_{q^*}\} - \psi\{\max(0,t-\varepsilon Qq^*),\chi^*_{q^*}\}|\,dt \tag{37}$$

that lends itself to an application of the explicit formula ([1]) for $\psi(w,\chi^*_{q^*})$.

To facilitate our expressing $\psi(w,\chi^*_{q^*})$ in terms of the zeros $\varrho = \beta + i\gamma$ of $L(s,\chi^*_{q^*})$, we let $\upsilon(\chi^*_{q^*})$ be defined as 1 or 0 accordingly as $\chi^*_{q^*}$ is principal or otherwise with the consequence that

$$\psi(w,\chi^*_{q^*}) = \upsilon(\chi^*_{q^*})w - \sum_{\substack{|\gamma|\le T\\ \beta\ge 1/2}} \frac{w^\varrho}{\varrho} + O\Big(\frac{x\log^2 x}{T}\Big) \tag{38}$$

if $0 \le w \le x$ and T be given by (31), whence

$$|\psi(w_2,\chi^*_{q^*}) - \psi(w_1,\chi^*_{q^*})| \le \upsilon(\chi^*_{q^*})(w_2-w_1) + \sum_{\substack{|\gamma|\le T\\ \beta\ge 1/2}} \left|\frac{w_2^\varrho - w_1^\varrho}{\varrho}\right| + O\Big(\frac{x\log^2 x}{T}\Big) \tag{39}$$

([1]) The explicit formulae for $\psi(w,\chi)$ containing an infinite series summed over ϱ are false when $w < 1$; the same is not true of (38) as the result is seen to be trivial over the latter range because of the relationship between x and T.

$$= v(\chi^*_{q^*})(w_2 - w_1) + \sum_{\substack{|\gamma|\le T\\ \beta\ge 1/2}} \Big|\int_{w_1}^{w_2} u^{\varrho-1}\,du\Big| + O\Big(\frac{x\log^2 x}{T}\Big)$$

$$\le v(\chi^*_{q^*})(w_2 - w_1) + \sum_{\substack{|\gamma|\le T\\ \beta\ge 1/2}} \int_{w_1}^{w_2} u^{\beta-1}\,du + O\Big(\frac{x\log^2 x}{T}\Big)$$

for $0 \le w_1 \le w_2 \le x$. Then, by taking $w_1 = \max(0, t-\varepsilon Qq^*), w_2 = \min(t, x)$ to treat the integrand in (37), we infer that the impact of an individual integrand in the final line of (39) on $G(\chi^*_{q^*})$ does not exceed

$$(40)\qquad \frac{(1+\varepsilon)^2}{\varepsilon Qq^*}\int_0^{x+\varepsilon Qq^*} dt \int_{\max(0,t-\varepsilon Qq^*)}^{\min(t,x)} u^{\beta-1}\,du$$

$$= \frac{(1+\varepsilon)^2}{\varepsilon Qq^*}\int_0^{x} u^{\beta-1}\,du \int_u^{u+\varepsilon Qq^*} dt = \frac{(1+\varepsilon)^2x^\beta}{\beta} < \frac{(1+\varepsilon)x^\beta}{\beta},$$

on restoring our introductory convention regarding ε that has been held in abeyance since the formation of (36). In particular, if there be an exceptional zero $\beta_0 = \beta_0(Q_0)$ of the type described through (33) in §5, then for $\chi^*_{q^*}$ the upper bound supplied by (40) becomes

$$(41)\qquad \frac{(1+\varepsilon)x^{\beta_0}}{\beta_0} < \frac{(1+\varepsilon)x}{1-T_1} < (1+\varepsilon)x$$

when $\beta = \beta_0$; similarly, should $\chi^*_{q^*}$ be the principal character, then we must also account for the additional effect of not more than

$$(42)\qquad (1+\varepsilon)x$$

from the first term in (38).

The contribution of the special terms (40) and (41) to $H(Q_0)$ in (35) can in no circumstances exceed $(2+\varepsilon)x$, while that of the remainder term in (39) is

$$O\Big(\frac{x\log^2 x(x+\varepsilon Q_0Q)}{\varepsilon T}\sum_{q^*\le Q_0}\frac{q^*}{Qq^*}\Big) = O\Big(\frac{xQ_0\log^2 x}{T}\sum_{q\le Q_0} 1\Big)$$

$$= O\Big(\frac{xQ_0^2\log^2 x}{T}\Big) = O\Big(\frac{x}{\log x}\Big)$$

by (22), (37), and (32). Hence, in summation of what has been recently gained, we have

$$H(Q_0) < (2+\varepsilon)x + 2\sum_{q^*\le Q_0}\sum_{\chi^*_{q^*}}\sum_{\substack{|\gamma|\le T\\ 1/2\le\beta<1-T_1}} x^{\beta} \tag{43}$$
$$= (2+\varepsilon)x - 2\int_{1/2-0}^{1-T_1} x^{\sigma}\, d\Big(\sum_{q^*\le Q_0}\sum_{\chi^*_{q^*}} N(\sigma,T,\chi^*_{q^*})\Big),$$

to which we should append the inequalities

$$T = Q_0^2\log^3 x = x^{2\delta}\log^5 x < x^{3/4c}, \qquad T^c < x^{3/4}$$

that are based on (32), (22), and (21).

At this point we exploit Gallagher's bound (31) and partial integration to deduce that the above integral is

$$O(x^{1/2}T^{1/2c}) + O\bigg(x\log x\int_{T_1}^{1/2}\Big(\frac{x}{T^c}\Big)^{-\sigma'}d\sigma'\bigg)$$
$$= O(x^{7/8}) + O\bigg\{x\Big(\frac{x}{T^c}\Big)^{-T_1}\bigg\} = O\bigg\{x\Big(\frac{x}{T^c}\Big)^{-T_1}\bigg\},$$

in which, by (33),

$$T_1 = d/(2\delta+5X)\log x$$

with $X = (\log\log x)/\log x$. Thence

$$\begin{aligned} H(Q_0) &< (2+\varepsilon)x + O(x\cdot x^{-d\{1-c(2\delta+5X)\}/(2\delta+5X)\log x})\\ &= (2+\varepsilon)x + O(xe^{-d/(2\delta+5X)})\\ &= (2+\varepsilon)x + O(xe^{-d/2\delta}) + O\Big(\frac{x\log\log x}{\log x}\Big)\\ &\le (2+\varepsilon)x + Axe^{-d/2\delta}, \end{aligned}$$

the definition of $e^{-d/2\delta}$ at $\delta = 0$ being taken to be 0.

Assimilating this in (35), we then reach the inequality

$$\begin{aligned} \mathrm{S}_1 &< (1+\varepsilon)(2+Ae^{-d/2\delta}+\varepsilon)x\log Q_0 + O(x)\\ &< \{(2+Ae^{-d/2\delta})\delta+\varepsilon\}x\log x, \end{aligned}$$

which in combination with (27) and (28) yields the lower bound

$$\mathrm{S}_3 > \{1-\delta(2+Ae^{-d/2\delta})-\varepsilon\}x\log x = (A(\delta)-\varepsilon)x\log x, \quad \text{say,} \tag{44}$$

we sought. Note here that $A(\delta)$ is a decreasing function of δ that tends to 1 as $\delta \to 0$ and that its zero is less than 1/2. However, we cannot easily tell if this zero be greater than or less than the number δ_1 that essentially constrains the scope of the method on account of the rôle of c in the application of (31) to (45), although it can be seen that these two numbers have roughly the same order of size in terms of c.

7. Transition to $L(x,Q)$. We move from S_3 to $L(x,Q)$ by comparing the integrals in the former with their integrands at $\phi=0$. In fact

$$\left|f\left(\frac{b}{q}+\phi\right)-f\left(\frac{b}{q}\right)\right| = \left|\sum_{p\le x}(e^{2\pi i(b/q+\phi)p}-e^{2\pi ibp/q})\log p\right| \le |\phi|\sum_{p\le x}p\log p = O\left(\frac{x^2}{Qq}\right)$$

when $|\phi|\le 1/Qq$. So, with Q and Q_1 as in (22),

$$\frac{2}{Q}\sum_{Q_1<q\le Q}\frac{1}{q}\sum_{\substack{0<b\le q\\(b,q)=1}}\left|f\left(\frac{b}{q}\right)\right|^2 = \sum_{Q_1<q\le Q}\sum_{\substack{0<b\le q\\(b,q)=1}}\int_{-1/Qq}^{1/Qq}\left|f\left(\frac{b}{q}+\phi\right)+O\left(\frac{x^2}{Qq}\right)\right|^2 d\phi, \tag{45}$$

and this by a derivative of Minkowski's theorem is not less than

$$\begin{aligned}
&\sum_{Q_1\le q\le Q}\sum_{\substack{0<b\le q\\(b,q)=1}}\int_{-1/Qq}^{1/Qq}\left|f\left(\frac{b}{q}+\phi\right)\right|^2 d\phi\\
&\quad+O\left\{\frac{x^2}{Q}\left(\sum_{Q_1<q\le Q}\sum_{\substack{0<b\le q\\(b,q)=1}}\int_{-1/Qq}^{1/Qq}\left|f\left(\frac{b}{q}+\phi\right)\right|^2 d\phi\right)^{1/2}\right.\\
&\quad\times\left.\left(\sum_{Q_1<q\le Q}\sum_{0<b\le q}\frac{1}{q^2}\int_{-1/Qq}^{1/Qq}d\phi\right)^{1/2}\right\}\\
&\quad+O\left(\frac{x^4}{Q^2}\sum_{Q_1<q\le Q}\sum_{0<b\le q}\frac{1}{q^2}\int_{-1/Qq}^{1/Qq}d\phi\right)\\
&=\mathrm{S}_3+O\left\{\frac{x^2}{Q^{3/2}}\left(\sum_{q>Q_1}\frac{1}{q^2}\right)^{1/2}\mathrm{S}_3^{1/2}\right\}+O\left(\frac{x^4}{Q^3}\sum_{q>Q_1}\frac{1}{q^2}\right)\\
&=\mathrm{S}_3+O\left(\frac{x^2}{Q^{3/2}Q_1^{1/2}}\,\mathrm{S}_3^{1/2}\right)+O\left(\frac{x^4}{Q^3Q_1}\right)\\
&=\mathrm{S}_3+O(x^{1/2}\,\mathrm{S}_3^{1/2})+O(x).
\end{aligned}$$

From this we can easily navigate to the required inequality for the left-side of (45) by a variety of routes. For example, most obviously, we may liken the size of an expression $u^2 + au + b$ containing bounded values of a and b with u^2 as $u \to \infty$; alternatively, but unnecessarily, we may use in the second term of the final line above the inequality $\mathrm{S}_3 < 3x \log x$ that stems from (26) and (24). Consequently, whatever argument be used, we infer from (44) and (45) that

$$\sum_{Q_1<q\le Q} \frac{1}{q} \sum_{\substack{0<b\le q\\(b,q)=1}} \left| f\left(\frac{b}{q}\right)\right|^2 > \frac{1}{2}\{A(\delta)-\varepsilon\}Qx\log x$$

and observe in passing that Liu's procedure at this point would only have derived an inequality of this type with the factor 1/4 instead of 1/2 in the right-hand side.

Interpreted in the language of equation (20) of §4, this inequality gives

$$\begin{aligned} L(x,Q)-L(x,Q_1) &> \frac{1}{2}\{A(\delta)-\varepsilon\}Qx\log x + O\left(\frac{x^2}{Q_1}\right) + O(x\log^5 x)\\ &> \frac{1}{2}\{A(\delta)-\varepsilon\}Qx\log x, \end{aligned}$$

to which we add the estimate (19) for $L(x,Q_1)$ to produce

$$L(x,Q) > \frac{1}{2}\{A(\delta)-\varepsilon\}Qx\log x - Ax\log^5 x > \frac{1}{2}\{A(\delta)-\varepsilon\}Qx\log x. \tag{46}$$

8. The final theorem. To derive the final theorem we first aver that, whatever value of δ_1 be chosen in (21), the inequality (46) remains true when δ_1 is replaced by a larger fixed constant less than $1/4c$. Then, still assuming that $\delta < \delta_1$ and letting $\xi = \xi(x)$ be a sufficiently slowly increasing function of x that tends to infinity, we use the Möbius inversion formula to rewrite (18) as

$$G^*(x,k) = \sum_{lm=k} \frac{H(x,m)}{l}$$

so that, by (2),

$$\begin{aligned} S^*(x,Q) &= \sum_{lm\le Q} \frac{H(x,m)}{l} = \sum_{l\le Q}\frac{1}{l}\sum_{m\le Q/l} H(x,m) = \sum_{l\le Q}\frac{1}{l}\,L(x,Q/l)\\ &= \sum_{l\le \xi}\frac{1}{l}\,L(x,Q/l) + \sum_{\xi<l\le Q}\frac{1}{l}\,L(x,Q/l)\\ &= S_a^*(x,Q) + S_b^*(x,Q), \quad \text{say.} \end{aligned} \tag{47}$$

The exponent $\delta_{(l)}$ pertaining to the parameter $Q/l = x^{1-\delta_{(l)}}$ in the primary sum $S_a^*(x,Q)$ equals

$$\delta + \frac{\log l}{\log x} \le \delta_1 + \frac{\log \xi}{\log x}$$

with the consequence that here

$$L(x,Q/l) > \frac{1}{2}\{A(\delta_{(l)}) - \varepsilon\}(Q/l)x\log x$$

by (46) and that also $A(\delta_{(l)}) > A(\delta) - \varepsilon$. Therefore

$$S_a^*(x,Q) > \frac{1}{2}\{A(\delta) - \varepsilon\}Qx\log x \sum_{l\le\xi}\frac{1}{l^2} > \frac{\pi^2}{12}\{A(\delta) - \varepsilon\}Qx\log x. \tag{48}$$

As for $S_b^*(x,Q)$, equation (19) gives

$$S_b^*(x,Q) > -Ax\log^5 x \sum_{\xi<l\le Q}\frac{1}{l} = O(x\log^6 x),$$

which with (48) and (47) yields the final estimate

$$S^*(x,Q) > \frac{\pi^2}{12}\{A(\delta) - \varepsilon\}Qx\log x$$

that we embody in

THEOREM 2. *Let*

$$S^*(x,Q) = \sum_{k\le Q}\sum_{\substack{0<a\le k\\(a,k)=1}} E^{*2}(x;a,k),$$

where $E^*(x;a,k)$ *is defined in equation* (2) *of the Introduction. Suppose also that* Q *is a number not exceeding* x *that is expressed as* $x^{1-\delta}$, *where* δ *is less than some positive constant* δ_1. *Then there is a decreasing continuous function* ([2]) $A(\delta)$ *of* δ *equal to* 1 *at* $\delta = 0$ *with the property that*

$$S^*(x,Q) > \frac{\pi^2}{12}\{A(\delta) - \varepsilon\}Qx\log x \qquad (x > x_0(\varepsilon))$$

uniformly with respect to δ. *In particular*,

$$S^*(x,Q) > \left(\frac{\pi^2}{12} - \varepsilon\right)Qx\log x$$

if $\delta = o(1)$ *as* $x\to\infty$.

Improvements in this theorem of two types are desirable. Putting on one side the first in which we would like to increase the constant in the inequality to a value beyond $\pi^2/12$, we make some comments on finding good bounds on the values of Q for which $S^*(x,Q)$ is bounded below by some (possibly small) constant multiple of $Qx\log x$. Such a search, as explained at the end of §6, is governed both directly by the value of c in (31) and the form of the function $A(\delta)$ that can be produced for our theorem. As it is, without unnecessarily complicating the argument, we have laid out the exposition

([2]) We can naturally arrange for $A(\delta)$ to be always positive by adjusting the value of δ_1 if necessary.

so that an example of $A(\delta)$ could be explicitly constructed through the constants latent in (31) and (32) without gratuitous wastage. In fact, we have already advanced below the limits set in Liu's method by a more efficacious use of Gallagher's procedure in (36), which obviates the inefficiencies associated with the previous need to lift out certain items from $F(\phi, \chi_q)$ when χ_q is associated with a principal or exceptional character. As a consequence, the deficiency from 1 represented in $A(\delta)$ in (44) has been reduced to at most a third of what it would have been had the previous procedure been explicitly conducted through its appeal to the Cauchy–Schwarz inequality. So whether or not our improvement has reduced the range of Q for which there is no meaningful result, we have at least strengthened the bounds for $S^*(x, Q)$ for the larger values of Q.

In an opposite direction, if we wished for a shorter derivation at the cost of a weaker value of $A(\delta)$, we could begin by dispensing with the large sieve by setting $Q_1 = Q_0$ as in (23) so that the sum S_2 disappears (the presence of the $\log x$ in (23) is inessential but facilitates comparison between the two methods). The treatment of S_3 remains the same, whereas it quickly becomes clear that 3δ takes over the rôle previously played by δ in the estimation of S_1 although it is now wise to make a harmless adjustment to the value of δ_1. Thus, without taking into account any compensating features of the new circumstances, we would obtain

$$S^*(x, Q) > \frac{\pi^2}{6}\{A(3\delta) - \varepsilon\}Qx\log x$$

in place of Theorem 2.

But, if we sought to curtail the treatment yet further, we could avoid all explicit reference to the zero-free regions and exceptional zeros of Dirichlet's L-functions and rely only on Gallagher's estimate (31). In fact, had we abstained from using the properties and definitions associated with equation (33), we would still have had available the analogue

$$H(Q_0) < (1+\varepsilon)x + 2\sum_{q^* \le Q_0}\sum_{\chi^*_{q^*}}\sum_{\substack{|\gamma| \le T \\ 1/2 \le \beta < 1}} x^{\beta}$$

of (43), to which we could apply (31) to gain the estimate

$$H(Q_0) < (1+\varepsilon)x + O(x) < Ax$$

in place of the previous more accurate one. This then leads ultimately to the lower bound

$$S^*(x, Q) > \frac{\pi^2}{12}(1 - A\delta - \varepsilon)$$

that is obviously inferior to what was obtained before. However, since Gallagher's estimate does itself depend on the properties of exceptional zeros, this last diversion involves an illusory economy and represents an inefficient deployment of resources.

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School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4YH, U.K.

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