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On the zeros of a class of arithmetical entire functions

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Introduction. Consider the class of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$ for which the *Fourier coefficients*

$$\widehat{f}(\lambda) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) e^{-2\pi i\lambda n} \tag{0.1}$$

exist for all $\lambda \in \mathbb{R}$. Since $\widehat{f}(\lambda+1) = \widehat{f}(\lambda)$, it follows that $\widehat{f}(\lambda)$ is completely determined by its values on $(0, 1]$. We are particularly interested in the case when $\widehat{f}(\cdot)$ is supported on the rationals; that is,

$$\widehat{f}(\lambda) = 0 \quad \text{ for } \lambda \notin \mathbb{Q}^+. \tag{0.2}$$

This occurs for many multiplicative functions (see for example [3]). We can then associate to f its *Fourier expansion*

$$f(n) \sim \sum_{0<q\le 1} \widehat{f}(q) e^{2\pi i q n}, \tag{0.3}$$

where q runs over the rationals in $(0, 1]$. (This series need not converge.) Many naturally occurring arithmetical functions have such an expansion. For example, the function $\sigma_{-\alpha}(n)$ (for $\alpha > 0$) has

$$\widehat{\sigma}_{-\alpha}(q) = \widehat{\sigma}_{-\alpha}\left(\frac{m}{n}\right) = \frac{\zeta(\alpha+1)}{n^{\alpha+1}} \quad \text{ for } m, n \text{ coprime},$$

where $\zeta(\cdot)$ is the Riemann zeta function.

These Fourier series and the closely related Ramanujan expansions (for which $\widehat{f}(m/n)$, with $(m, n) = 1$, depends only on n) have been studied in great detail (see for example [5], [6]).

An absolutely convergent Fourier series (0.3) extends naturally to an entire function of order 1. We study several aspects of these "arithmetical" entire functions, in particular the location of the zeros. In the case when the

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$\widehat{f}(q)$ are real and $\inf_{n\in\mathbb{Z}} f(n) > 0$, we prove that all the zeros of the entire extension are real and simple.

In the particular case of Ramanujan expansions, we obtain alternative series representations. Such representations can be used to extend functions whose Fourier series do not converge absolutely. For some particular examples, including an entire extension of the divisor function $d(n)$, we show that their zeros are again all real and simple, except for a conjugate pair of imaginary zeros.

Finally, we briefly discuss the asymptotic behaviour along the imaginary axis.

Some preliminaries. For a function $a : \mathbb{Q}^+ \to \mathbb{C}$ defined on the positive rationals, we define

$$\sum_{q\in\mathbb{Q}^+} a(q) = \lim_{N\to\infty} \sum_{\substack{m,n\le N\\(m,n)=1}} a\left(\frac{m}{n}\right), \qquad \text{whenever this limit exists.}$$

We shall sometimes abbreviate the left-hand sum by $\sum_q a(q)$ ([1]). For $q = m/n \in \mathbb{Q}^+$, with $(m,n) = 1$, we write $|q| = \max\{m,n\}$. Thus the above definition becomes

$$\sum_q a(q) = \lim_{N\to\infty} \sum_{|q|\le N} a(q).$$

As usual, we say $\sum_q a(q)$ converges *absolutely* if $\sum_q |a(q)|$ converges. In that case, we may sum the terms in any particular order.

We shall require Hurwitz's theorem concerning zeros of the uniform limit of holomorphic functions. There are various versions of this and we shall use it in two ways:

Hurwitz's theorem. *Let F_n be a sequence of holomorphic functions, and suppose $F_n \to F$ uniformly on a domain D, where F is not identically zero.*

(a) *If F_n has no zeros in D, then F has no zeros in D* ([1]).
(b) *If $F(z) \ne 0$ for z on a simple closed contour C lying entirely inside D, then for all n sufficiently large, F_n and F have the same number of zeros inside C* ([4]).

1. Fourier series and coefficients of arithmetical functions

Definition 1.1.

(i) Let $\mathcal{Q}$ denote the space of functions $f : \mathbb{N} \to \mathbb{C}$ for which (0.1) and (0.2) hold.

([1]) Throughout this article, a sum over q always denotes a sum over all the rationals q in the given range.

(ii) Let $\mathcal{Q}_1$ denote the subspace of $\mathcal{Q}$ consisting of absolutely convergent Fourier series; i.e. writing $\pi_q(n) = e^{2\pi iqn}$,

$$\mathcal{Q}_1 = \Big\{ \sum_{0<q\le 1} c(q)\pi_q : \sum_{0<q\le 1} |c(q)| < \infty \Big\}.$$

Note that if $f(n) = \sum_{0<q\le 1} c(q)e^{2\pi iqn} \in \mathcal{Q}_1$ then, by absolute convergence, for $\lambda \in (0,1]$ we have

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)e^{-2\pi i\lambda n} = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \sum_{0<q\le 1} c(q)e^{2\pi iqn}e^{-2\pi i\lambda n}$$

$$= \sum_{0<q\le 1} c(q) \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e^{2\pi i(q-\lambda)n} = \begin{cases} c(\lambda) & \text{if } \lambda \in \mathbb{Q}, \\ 0 & \text{if } \lambda \notin \mathbb{Q}, \end{cases}$$

i.e. the $c(q)$ are the Fourier coefficients of f. In fact, $\mathcal{Q}_1$ is a Banach algebra with norm $\|f\|_1 = \sum_{0<q\le 1} |\widehat{f}(q)|$.

A number of interesting examples of members of $\mathcal{Q}$ arise from a certain type of arithmetical function. For arithmetical functions f and g, denote by $f * g$ the Dirichlet convolution of f and g, i.e.

$$(f * g)(n) = \sum_{d|n} f(d)g\Big(\frac{n}{d}\Big).$$

THEOREM 1.1. *Let g be an arithmetical function for which $\sum_{n=1}^{\infty} |g(n)|/n$ converges and let $h \in \mathcal{Q}$ satisfy $\sum_{n\le x} |h(n)| = O(x)$. Then $f = g * h \in \mathcal{Q}$ with Fourier coefficients*

$$\widehat{f}(\lambda) = \sum_{n=1}^{\infty} \frac{g(n)}{n} \widehat{h}(\lambda n). \tag{1.1}$$

Furthermore, if $\sum_{n=1}^{\infty} |g(n)| < \infty$ and $h \in \mathcal{Q}_1$, then $f \in \mathcal{Q}_1$.

Proof. Let $\lambda \in \mathbb{R}$. We have

$$\frac{1}{x} \sum_{n\le x} f(n)e^{-2\pi i\lambda n} = \frac{1}{x} \sum_{n\le x} e^{-2\pi i\lambda n} \sum_{d|n} g(d)h\Big(\frac{n}{d}\Big)$$

$$= \frac{1}{x} \sum_{n\le x} g(n) \sum_{d\le x/n} h(d)e^{-2\pi i\lambda nd}.$$

Since $h \in \mathcal{Q}$, for fixed n we have

$$\frac{n}{x} \sum_{d\le x/n} h(d)e^{-2\pi i\lambda nd} \to \widehat{h}(\lambda n)$$

as $x \to \infty$. Furthermore, by assumption, the LHS above is bounded by an absolute constant A. Hence also $|\widehat{h}(\lambda n)| \le A$.

Let $\varepsilon > 0$. There exists N such that $\sum_{n>N} |g(n)|/n < \varepsilon$. So, for $x > N$,

$$\left|\frac{1}{x}\sum_{n\le x} f(n)e^{-2\pi i\lambda n} - \frac{1}{x}\sum_{n\le N} g(n)\sum_{d\le x/n} h(d)e^{-2\pi i\lambda nd}\right|$$
$$= \left|\sum_{N<n\le x}\frac{g(n)}{n}\frac{n}{x}\sum_{d\le x/n} h(d)e^{-2\pi i\lambda nd}\right| \le A\sum_{n>N}\frac{|g(n)|}{n} < A\varepsilon.$$

Let $x \to \infty$. Thus

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x} f(n)e^{-2\pi i\lambda n} - \sum_{n\le N}\frac{g(n)}{n}\widehat{h}(\lambda n)\right| \le A\varepsilon.$$

But $|\widehat{h}(\lambda n)| \le A$ so $|\sum_{n>N}(g(n)/n)\widehat{h}(\lambda n)| < A\varepsilon$. It follows that

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x} f(n)e^{-2\pi i\lambda n} - \sum_{n=1}^{\infty}\frac{g(n)}{n}\widehat{h}(\lambda n)\right| \le 2A\varepsilon.$$

This is true for all $\varepsilon > 0$, so the lim sup is in fact zero—proving that $f \in \mathcal{Q}$ and (1.1) holds.

Finally, if $\sum_{n=1}^{\infty} |g(n)| < \infty$ and $h \in \mathcal{Q}_1$, then we have

$$\sum_{0<q\le 1} |\widehat{f}(q)| \le \sum_{0<q\le 1}\sum_{n=1}^{\infty}\frac{|g(n)|}{n}|\widehat{h}(qn)| = \|h\|_1\sum_{n=1}^{\infty}|g(n)| < \infty,$$

since $\sum_{0<q\le 1}|\widehat{h}(qn)| = \sum_{0<q\le n}|\widehat{h}(q)| = n\|h\|_1$ by periodicity of $\widehat{h}(q)$, and so $f \in \mathcal{Q}_1$. ∎

The above result resembles, but is different from, Theorem 2.1 on p. 49 of [6], where h is assumed completely multiplicative. With $h \equiv 1$, we obtain the following corollary which contains Corollary 2.2 from [6].

Corollary 1.2. *Let $g : \mathbb{N} \to \mathbb{C}$ and let $f(n) = \sum_{d|n} g(d)$. If the series $\sum_{n=1}^{\infty} |g(n)|/n$ converges, then $f \in \mathcal{Q}$ with*

$$\widehat{f}(q) = \widehat{f}\left(\frac{m}{n}\right) = \sum_{k=1}^{\infty}\frac{g(kn)}{kn} \qquad \text{for } (m,n) = 1.$$

If we further assume that $\sum_{n=1}^{\infty} |g(n)|$ converges, then $f \in \mathcal{Q}_1$.

Examples 1.3.

(a) Let $g(n) = n^{-\alpha}$ in Corollary 1.2, so that $f(n) = \sum_{d|n} d^{-\alpha} = \sigma_{-\alpha}(n)$. Hence $\sigma_{-\alpha}(n) \in \mathcal{Q}$ for $\alpha > 0$ with

$$\widehat{\sigma}_{-\alpha}\left(\frac{m}{n}\right) = \sum_{k=1}^{\infty}\frac{1}{(kn)^{\alpha+1}} = \frac{\zeta(\alpha+1)}{n^{\alpha+1}}.$$

The Fourier series is therefore

$$\sigma_{-\alpha}(n) \sim \zeta(\alpha+1) \sum_{0<q\le 1} \frac{e^{2\pi i q n}}{|q|^{\alpha+1}}.$$

If $\alpha > 1$, then $\sigma_{-\alpha}(n) \in \mathcal{Q}_1$ and the above series converges absolutely to $\sigma_{-\alpha}(n)$.

(b) Let $g(n) = \mu(n)n^{-\alpha}$ in Corollary 1.2, so that $f(n) = \sum_{d|n} \mu(d)d^{-\alpha} = \varphi_\alpha(n)$ (2). For $\alpha > 0$ this lies in $\mathcal{Q}$ with

$$\widehat{f}\Big(\frac{m}{n}\Big) = \sum_{k=1}^{\infty} \frac{\mu(kn)}{(kn)^{\alpha+1}} = \frac{\mu(n)}{\zeta(\alpha+1)n^{\alpha+1}\varphi_{\alpha+1}(n)}.$$

(To see this, it is enough to consider n squarefree, in which case $\mu(kn)/\mu(n)$ is multiplicative as a function of k.) For $\alpha > 1$, $\varphi_\alpha(\cdot)$ is in $\mathcal{Q}_1$ and has the absolutely convergent Fourier series

$$\varphi_\alpha(n) = \frac{1}{\zeta(\alpha+1)} \sum_{0<q\le 1} \frac{\mu(|q|)}{|q|^{\alpha+1}\varphi_{\alpha+1}(|q|)}\, e^{2\pi i q n}.$$

2. Entire extensions. Every $f \in \mathcal{Q}_1$ can be extended, in a natural way, to an entire function. For $n \in \mathbb{N}$, we have

$$f(n) = \sum_{0<q\le 1} \widehat{f}(q) e^{2\pi i q n},$$

and we could define $f(z)$ ($z \in \mathbb{C}$) simply by replacing n by z in the above. However, we could equally well let q range over the rationals in $[0,1)$ (or indeed over any half-open interval between two integers). To avoid having to choose which endpoint will be included, we define instead

$$f(z) = \sum_{0\le q\le 1} w(q)\widehat{f}(q)e^{2\pi i q z} = {\sum_{0\le q\le 1}}' \widehat{f}(q)e^{2\pi i q z}, \tag{2.1}$$

where $w(q)$ is the following weight: $w(0) = w(1) = 1/2$, $w(q) = 1$ for $0 < q < 1$ rational (the $'$ indicating the $q = 0, 1$ terms are to be halved).

This series converges absolutely since $|e^{2\pi i q z}| \le e^{2\pi q|z|} \le e^{2\pi|z|}$, which is independent of q. In particular, this implies

$$|f(z)| \le e^{2\pi|z|} {\sum_{0<q\le 1}}' |\widehat{f}(q)| = \|f\|_1 e^{2\pi|z|}.$$

By standard theorems on holomorphic functions, $f(z)$ is entire with derivative

$$f'(z) = 2\pi i {\sum_{0\le q\le 1}}' q\widehat{f}(q)e^{2\pi i q z}.$$

Note that this is bounded on the real line.

(2) Here, we define $\varphi_\alpha(n) = \prod_{p|n}(1-p^{-\alpha})$. Thus $\varphi_1(n) = \phi(n)/n$.

The bound $|f(z)| \le \|f\|_1 e^{2\pi|z|}$ shows f is of finite order and the order is at most 1. Moreover, if $\widehat{f}(q) \ge 0$ for all q with at least one non-zero value, say $\widehat{f}(q') > 0$, then

$$f(-iy) = \sum_{0<q\le 1}{}' \widehat{f}(q)e^{2\pi qy} \ge \widehat{f}(q')e^{2\pi q'y},$$

showing that f has order 1 exactly.

There are of course infinitely many entire functions which interpolate a particular sequence, say $f(n)$. It is an interesting question to find the "smallest" such entire function. Here, "smallest" means with least growth. For example, if $f(n)$ is multiplicative but not of the form n^k ($k \in \mathbb{N}_0$), then the interpolating entire function must have order at least 1. For if not, let $F(z)$ be entire of order $\varrho < 1$ such that $F(n) = f(n)$. Then $G(z) := F(2z) - F(2)F(z)$ is entire of order at most ϱ, and not identically zero ([3]). Thus $G(z)$ has at most $O(r^{\varrho+\varepsilon})$ zeros for $|z| \le r$ for any $\varepsilon > 0$ (see for example [7]). But $G(k) = 0$ for every odd positive integer k, showing that there are at least $\frac{1}{2}r$ zeros here—a contradiction. In fact, a slight strengthening of this argument shows that $|G(z)| = O(e^{a|z|})$ is false for all $a > 0$ sufficiently small. Hence also for $F(z)$.

With the entire extension $f \in \mathcal{Q}_1$ defined by (2.1), a number of interesting questions present themselves. Amongst these are the location of any zeros, and the behaviour of $f(z)$ for large $|z|$, in particular for $z = \pm ix$ with x real. Also, if $f(n)$ is multiplicative, how does this manifest itself in the entire extension (2.1)?

2.1. *Zeros.* The entire extension of $f \in \mathcal{Q}_1$ as defined in (2.1) always has zeros, except in the case that the Fourier series of f is a single term. We are particularly interested in the case where $f(n)$ and $\widehat{f}(q)$ are real for all positive integers n and rational q, which we now assume. It follows easily that $\overline{f(z)} = f(-\overline{z})$, $\widehat{f}(1-q) = \widehat{f}(q)$, and

$$f(z) = e^{2\pi iz} f(-z).$$

In fact, we shall assume that $f(n)$ is positive for every $n \in \mathbb{N}_0$. As $f(-n) = f(n)$, this implies that $f(n) > 0$ for $n \in \mathbb{Z}$. Let g be the entire even function defined by

$$g(z) = e^{\pi z} f(iz) = \sum_{0\le q\le 1}{}' \widehat{f}(q)e^{(1-2q)\pi z}.$$

Then g is real on the imaginary axis (since $\overline{g(z)} = g(\overline{z})$) and $g(in) = (-1)^n f(n)$, which alternates in sign. Thus g has a zero on the imaginary axis between in and $i(n+1)$ for every $n \in \mathbb{Z}$.

([3]) If $F(2z) = F(2)F(z)$ for all z, then the Taylor coefficients a_n of $F(z)$ must satisfy $a_n(2^n - F(2)) = 0$. Hence $a_n = 0$ for all n except possibly one value. But $F(1) = f(1) = 1$, so $F(z) = z^k$ for some k.

Let $N(r)$ denote the number of zeros of g inside the disc $D(0,r)$. Thus from the above, $N(r) \geq 2[r]$. By Jensen's theorem ([7, p. 126]), we can obtain an upper bound. The bound $|f(-iz)| \leq \sum_{0<q\leq 1} |\widehat{f}(q)| e^{2\pi q \Re z} \leq \|f\|_1 e^{2\pi \Re z}$ whenever $\Re z \geq 0$ gives

$$|g(z)| \leq \|f\|_1 e^{\pi |\Re z|} \qquad (z \in \mathbb{C}).$$

Hence, by Jensen's theorem,

$$\begin{aligned} N_1(r) := \int_0^r \frac{N(t)}{t}\, dt &= \frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})|\, d\theta - \log |g(0)| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \pi r |\cos\theta|\, d\theta + \log \|f\|_1 - \log |g(0)| = 2r + A. \end{aligned}$$

But $N(r) \geq 2[r] > 2(r-1)$, so $N_1(r) \geq \int_1^r \frac{2(t-1)}{t}\, dt = 2r - 2\log r - 2$. Hence

$$N_1(r) = 2r + O(\log r).$$

Using the fact that $N(x)$ is increasing with x, for $x > y > 0$ we have

$$\frac{x-y}{y}\,(N_1(x) - N_1(x-y)) \leq N(x) \leq \frac{x+y}{y}\,(N_1(x+y) - N_1(x)).$$

Since $N_1(x+y) - N_1(x) = 2y + O(\log x)$ and $N_1(x) - N_1(x-y) = 2y + O(\log x)$, it follows that

$$N(x) = 2x\biggl(1 + O\biggl(\frac{y}{x}\biggr) + O\biggl(\frac{\log x}{y}\biggr)\biggr).$$

Choosing $y = \sqrt{x \log x}$ optimally gives

$$N(r) = 2r + O(\sqrt{r \log r}).$$

Since at least $2[r]$ of these zeros are on the imaginary axis, this suggests that they all lie on it. We prove this in Theorem 2.3.

Note that since $\widehat{f}(q) = \widehat{f}(1-q)$, we can write

$$\begin{aligned} g(z) &= \sum_{0\leq q\leq 1/2} w(q)\widehat{f}(q)(e^{(1-2q)\pi z} + e^{-(1-2q)\pi z}) - \widehat{f}\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\,\widehat{f}(1)(e^{\pi z} + e^{-\pi z}) + \widehat{f}\left(\frac{1}{2}\right) + \sum_{0<q<1/2} 2\widehat{f}(q) \cosh \pi(1-2q)z \\ &= \sum_{0\leq q\leq 1} \alpha_q \cosh \pi q z \end{aligned} \tag{2.2}$$

for some α_q real. (Indeed, $\alpha_0 = \widehat{f}(1/2)$, $\alpha_1 = \widehat{f}(1)$, and $\alpha_q = 2\widehat{f}((1-q)/2)$.)

LEMMA 2.1. *Let $\lambda \in (-1,1)$. All the zeros of the function $\cosh z - \lambda$ are imaginary and simple. More precisely, they are $i(\pm\theta + 2\pi k)$ $(k \in \mathbb{Z})$, where $\cos\theta = \lambda$ and $\theta \in (0,\pi)$.*

Proof. The equation $\cosh z = \lambda$ involves

$$e^{2z} - 2\lambda e^z + 1 = 0,$$

i.e. $(e^z - \lambda)^2 = \lambda^2 - 1$. Thus $e^z = \lambda \pm i\sqrt{1-\lambda^2}$ (since $\lambda \in (-1,1)$). But $|\lambda \pm i\sqrt{1-\lambda^2}| = 1$, hence $|e^z| = 1$ and z must be purely imaginary.

Writing $\lambda + i\sqrt{1-\lambda^2} = e^{i\theta}$, so that $\cos\theta = \lambda$ and $\theta \in (0,\pi)$, we have $e^z = e^{\pm i\theta}$. Thus the zeros are

$$z = \pm i\theta + 2\pi i k \quad (k \in \mathbb{Z}),$$

in particular they are all simple. ∎

THEOREM 2.2. *Let $N \in \mathbb{N}$ and let $\{\lambda_q\}_{0\le q\le 1}$ be real numbers for q rational and $|q| \le N$ (here $|0| = 1$). Let $A(z)$ be defined by*

$$A(z) = \sum_{\substack{0\le q\le 1\\ |q|\le N}} \lambda_q \cosh \pi q z$$

and suppose that $(-1)^n A(in) > 0$ for all $n \in \mathbb{Z}$. Then all the zeros of $A(z)$ are imaginary and simple. Furthermore, denoting the zeros by $\pm i\gamma_n$, where $\gamma_{n+1} > \gamma_n > 0$ $(n \in \mathbb{N})$, we have $n - 1 < \gamma_n < n$ for $n \ge 1$.

Proof. Let $M = \mathrm{lcm}\{1,\ldots,N\}$. Then for $q \in [0,1]$ rational such that $|q| \le N$, $Mq \in \mathbb{N}_0$. Hence

$$\cosh q\pi z = \cosh\biggl(Mq\,\frac{\pi z}{M}\biggr) = T_{Mq}(w),$$

where $w = \cosh(\pi z/M)$, and $T_n(\cdot)$ is the nth Chebyshev polynomial ([4]). Thus

$$A(z) = \sum_{\substack{0\le q\le 1\\ |q|\le N}} \lambda_q T_{Mq}(w) = P_M(w)$$

say, where $P_M(\cdot)$ is a polynomial of degree M with real coefficients. We shall see that all the roots are simple and lie in the open interval $(-1,1)$.

Let $\mu_n = \cos(\pi n/M)$ for $n \in \mathbb{Z}$. By supposition,

$$(-1)^n P_M(\mu_n) = (-1)^n P_M(\cosh(\pi i n/M)) = (-1)^n A(in) > 0.$$

But for $0 \le n \le M$, μ_n is a strictly decreasing sequence in $[-1,1]$, i.e.

$$-1 = \mu_M < \cdots < \mu_1 < \mu_0 = 1.$$

So there is a zero of $P_M(\cdot)$ between μ_n and μ_{n+1} for every $n = 0,1,\ldots,M-1$. This gives M simple zeros in $(-1,1)$, and as $P_M(\cdot)$ has degree M, they are all the zeros. Denote these zeros by $w_1,\ldots,w_M$ with $w_1 > \cdots > w_M$.

([4]) The nth Chebyshev polynomial is defined by $\cos n\theta = T_n(\cos\theta)$ $(n \in \mathbb{N}_0)$ and has degree n.

Now $A(z)=0$ if and only if $P_M(w)=0$, where $w=\cosh(\pi z/M)$. This gives the M equations

$$\cosh\frac{\pi z}{M}=w_r \quad (r=1,\dots,M).$$

By Lemma 2.1, each of them has all its roots on the imaginary axis. Apart from the simplicity of the roots, this proves the first part of the theorem.

For the second part, note that $\pi z/M=i(\pm\theta_r+2\pi k)$, where $\cos\theta_r=w_r$, $\theta_r\in(0,\pi)$, and $k\in\mathbb{Z}$. But $\mu_r<w_r<\mu_{r-1}$, so

$$\cos\frac{r\pi}{M}<\cos\theta_r<\cos\frac{(r-1)\pi}{M}.$$

As cos is a decreasing function on $[0,\pi]$, $(r-1)\pi/M<\theta_r<r\pi/M$. Thus

$$z=i(\pm\psi_r+2Mk) \quad \text{for } r=1,\dots,M \text{ and } k\in\mathbb{Z},$$

for some $\psi_r\in(r-1,r)$. Hence every interval $(in,i(n+1))$ contains exactly one zero. ∎

THEOREM 2.3. *Let $f\in\mathcal{Q}_1$ with $\widehat{f}(q)$ real for all q and such that $\inf_{n\in\mathbb{N}_0}f(n)>0$. Then the entire extension of f as defined by* (2.1) *has all its zeros real and simple*, *one in each interval $(n,n+1)$ for every $n\in\mathbb{Z}$.*

Proof. We can apply Hurwitz's theorem to our case, by taking

$$g_N(z)=\sum_{\substack{0\le q\le 1\\ |q|\le N}}\alpha_q\cosh\pi qz$$

with α_q as in (2.2). Then $g_N\to g$ uniformly on compact subsets of $\mathbb{C}$ and, crucially, $g_N\to g$ uniformly on the imaginary axis since

$$|g_N(ix)-g(ix)|\le\sum_{|q|>N}|\alpha_q|\to 0$$

independently of x. For all N sufficiently large, g_N satisfies the conditions of Theorem 2.2: the coefficients are real, and $(-1)^n g_N(in)$ is real and positive for all n (by uniform convergence on the imaginary axis and the assumption that $\inf_{n\in\mathbb{N}_0}f(n)>0$). Theorem 2.2 implies that g_N has all its zeros on the imaginary axis, one in each interval $i(n,n+1)$. By part (a) of Hurwitz's theorem, the zeros of g are also all on the imaginary axis (take D to be the half-plane $\{z\in\mathbb{C}:\Re z>0\}$ and $\{z\in\mathbb{C}:\Re z<0\}$ in turn).

Now fix $k\in\mathbb{Z}$ and let C be the circle of centre $i(k+1/2)$ and radius $1/2$. Then $g(z)\ne 0$ on C. By part (b) of Hurwitz's theorem, g has the same number of zeros inside C as g_n for all n sufficiently large, i.e. one. ∎

REMARK. The zeros are necessarily bounded away from the integers (and hence from each other). For we have $f(n+x)=f(n)+xf'(y)$ for some y with $0 < |y| < |x|$, so that

$$|f(n+x)| \geq f(n) - |x|\,|f'(y)| \geq \inf_{n\in\mathbb{N}_0} f(n) - |x| \sup_{y\in\mathbb{R}} |f'(y)| > 0$$

for all x sufficiently small, independently of n.

2.2. *Entire extensions associated to Ramanujan expansions.* In the special case when f has a Ramanujan expansion we can obtain a different representation of the entire extension. More specifically, as in Corollary 1.2, let $f(n) = \sum_{d|n} a_d$ where $\sum_n |a_n|$ converges, so that $f \in \mathcal{Q}_1$. Define the entire extension of f by (2.1). In particular we have (using $\widehat{f}(0) = \widehat{f}(1)$ and $\widehat{f}(m/n) = \widehat{f}(1/n)$ for m, n coprime)

$$f(0) = \sum_{n=1}^{\infty} \phi(n)\widehat{f}\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{k=1}^{\infty} \frac{a_{nk}}{k} = \sum_{m=1}^{\infty} \frac{a_m}{m} \sum_{d|m} \phi(m) = \sum_{m=1}^{\infty} a_m.$$

This can be interpreted by saying the divisors of 0 are all the positive integers, so that $f(0) = \sum_{d|0} a_d = \sum_{d=1}^{\infty} a_d$.

Proposition 2.4. *Let $f(n) = \sum_{d|n} a_d$, where $\sum_{n=1}^{\infty} |a_n|$ converges. Let f also denote the extension to $\mathbb{C}$ as defined in* (2.1). *Then for $z \in \mathbb{C}\setminus\mathbb{Z}$, we have*

$$f(z) = \left(\frac{e^{2\pi iz} - 1}{2\pi iz}\right)\left(f(0) - 2z^2 \sum_{n=1}^{\infty} \frac{f(n)}{n^2 - z^2}\right).$$

Proof. Let

$$S(n) = \sum_{m=1}^{n} e^{2\pi izm/n} = \frac{e^{2\pi iz} - 1}{1 - e^{-2\pi iz/n}} \quad \text{and} \quad T(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} e^{2\pi izm/n}.$$

Then $S(n) = \sum_{d|n} T(d)$ (see for example [2]). Using the fact that $\widehat{f}(0) = \widehat{f}(1) = \sum_{m=1}^{\infty} a_m/m$ we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \widehat{f}\left(\frac{1}{n}\right) T(n) - \frac{\widehat{f}(1)}{2}(e^{2\pi iz} - 1) \\ &= \sum_{n=1}^{\infty} \frac{T(n)}{n} \sum_{k=1}^{\infty} \frac{a_{kn}}{k} - \frac{\widehat{f}(1)}{2}(e^{2\pi iz} - 1) \\ &= \sum_{m=1}^{\infty} \frac{a_m}{m} \sum_{d|m} T(d) - \frac{1}{2}(e^{2\pi iz} - 1) \sum_{m=1}^{\infty} \frac{a_m}{m} \end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{a_m}{m}\left(S(m) - \frac{1}{2}(e^{2\pi iz} - 1)\right)$$
$$= (e^{2\pi iz} - 1)\sum_{n=1}^{\infty} \frac{a_n}{n}\left(\frac{1}{1 - e^{-2\pi iz/n}} - \frac{1}{2}\right).$$

Now for $z \notin \mathbb{Z}$,

$$\frac{1}{1 - e^{-2\pi iz/n}} - \frac{1}{2} = \frac{n}{2\pi iz} + \frac{inz}{\pi}\sum_{k=1}^{\infty} \frac{1}{n^2k^2 - z^2}.$$

Multiplying by $(e^{2\pi iz} - 1)a_n/n$ and summing over n gives

$$f(z) = (e^{2\pi iz} - 1)\left\{\frac{1}{2\pi iz}\sum_{n=1}^{\infty} a_n + \frac{iz}{\pi}\sum_{n=1}^{\infty} a_n \sum_{k=1}^{\infty} \frac{1}{n^2k^2 - z^2}\right\}$$
$$= \frac{e^{2\pi iz} - 1}{2\pi iz}\left\{f(0) - 2z^2\sum_{m=1}^{\infty} \frac{f(m)}{m^2 - z^2}\right\},$$

using $f(m) = \sum_{n|m} a_n$. ■

2.3. *Zeros again.* We have seen that under various conditions the zeros of f are all real. In the case that f is given as in Proposition 2.4, we find that we can relax these to the condition $f(n) > 0$ for all $n \in \mathbb{N}_0$. (This of course implies that a_n is real for all $n \in \mathbb{N}$ and hence that $\widehat{f}(q)$ is real.) We start with a lemma:

LEMMA 2.5. *Let a_n and b_n $(n = 0, \ldots, N)$ be real numbers such that $a_n > 0$ and $0 \le b_0 < b_1 < \cdots < b_N$. Let $f : \mathbb{C} \setminus \{-b_0, \ldots, -b_N\} \to \mathbb{C}$ be defined by*

$$f(z) = \sum_{n=0}^{N} \frac{a_n}{b_n + z}.$$

Then all zeros of f are real and negative, one in each interval $(-b_k, -b_{k-1})$.

Proof. We can write

$$f(z) = \frac{P_N(z)}{(b_0 + z)\ldots(b_N + z)},$$

where P_N is a polynomial of degree N. Hence f has N zeros (counting multiplicities).

Since the a_n and b_n are real, $f(z)$ is real for real z. Consider $f(x)$ for x real in an interval $(-b_k, -b_{k-1})$. We have $f(x) \to -\infty$ as $x \to -b_{k-1}$ and $f(x) \to \infty$ as $x \to -b_k$. It follows that $f(x)$ has a zero in the interval $(-b_k, -b_{k-1})$. This is true for each $k = 1, \ldots, N$, giving N zeros of f in the required intervals. ■

THEOREM 2.6. *Let a_n and b_n ($n \in \mathbb{N}_0$) be sequences of real numbers such that $a_n > 0$, $b_{n+1} > b_n \geq 0$, and $\sum_{n=1}^{\infty} a_n/b_n$ converges. Let $f : \mathbb{C} \setminus \{-b_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ be defined by*

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{b_n + z}.$$

Then all zeros of f are real and negative, one in each interval $(-b_k, -b_{k-1})$ ($k \in \mathbb{N}$).

Proof. Apply part (a) of Hurwitz's theorem to our case, by taking

$$f_N(z) = \sum_{n=0}^{N} \frac{a_n}{b_n + z}.$$

Then $f_N \to f$ uniformly on compact subsets of $\mathbb{C} \setminus \{-b_n : n \in \mathbb{N}_0\}$. By Lemma 2.5, f_N has all its zeros real and negative, one in each interval $(-b_k, -b_{k-1})$. By Hurwitz's theorem, the zeros of f are also all real and negative (take D to be the cut plane $\mathbb{C}_{\text{cut}} := \mathbb{C} \setminus (-\infty, 0]$).

For x real and not equal to $-b_k$ for any k,

$$f'(x) = -\sum_{n=0}^{\infty} \frac{a_n}{(b_n + x)^2} < 0.$$

Thus f is strictly decreasing on each interval $(-b_k, -b_{k-1})$. As before, $f(-b_k + 0) = \infty$ and $f(-b_{k-1} - 0) = -\infty$. Hence f has exactly one zero in each interval $(-b_k, -b_{k-1})$. ∎

COROLLARY 2.7. *Let f be as in Proposition 2.4 with entire extension as defined by (2.1), and suppose $f(n) > 0$ for all $n \in \mathbb{N}_0$. Then the zeros of f are real and simple, and are of the form $\pm\mu_n$ ($n \in \mathbb{N}$) with $n - 1 < \mu_n < n$.*

Proof. We have

$$f(z) = -2z^2 \left(\frac{e^{2\pi iz} - 1}{2\pi iz} \right) \left(\sum_{n=0}^{\infty} \frac{f_1(n)}{n^2 - z^2} \right),$$

where $f_1(0) = \frac{1}{2} f(0)$ and $f_1(n) = f(n)$ for $n \neq 0$. Thus $f_1(n) > 0$ and Theorem 2.6 can be applied. The zeros of f therefore occur when z^2 is real and positive, one in each interval $(n^2, (n+1)^2)$ ($n \in \mathbb{N}_0$); i.e. the zeros are all real and simple, one in each interval $(n, n+1)$ ($n \in \mathbb{Z}$). Since $e^{-\pi iz} f(z)$ is even, the zeros come in pairs, and are of the form stated. ∎

3. Some special and unbounded examples. From Examples 1.3, $\sigma_{-\alpha}(n)$ and $\varphi_\alpha(n)$ are in $\mathcal{Q}_1$ for $\alpha > 1$. Also, both $\sigma_{-\alpha}(n)$ and $\varphi_\alpha(n)$ have Ramanujan expansions and the results from 2.2 can be applied here. Their values at 0 are $\zeta(\alpha)$ and $1/\zeta(\alpha)$ respectively, and by Proposition 2.4 we have

$$\sigma_{-\alpha}(z) = \left(\frac{e^{2\pi iz}-1}{2\pi iz}\right)\left(\zeta(\alpha) - 2z^2\sum_{n=1}^{\infty}\frac{\sigma_{-\alpha}(n)}{n^2-z^2}\right), \tag{3.1}$$

$$\varphi_\alpha(z) = \left(\frac{e^{2\pi iz}-1}{2\pi iz}\right)\left(\frac{1}{\zeta(\alpha)} - 2z^2\sum_{n=1}^{\infty}\frac{\varphi_\alpha(n)}{n^2-z^2}\right). \tag{3.2}$$

If $\alpha \le 1$, then $\sigma_{-\alpha}$ and φ_α are not in $\mathcal{Q}_1$ any longer, and their Fourier series will not converge absolutely, so we cannot use (2.1) to define an entire extension. However, for $\alpha > -1$, we can still define them via the above. Note that in (3.1) there is no obvious choice for $\sigma_{-1}(\cdot)$ as the term $\zeta(\alpha)$ becomes infinite, so we avoid $\alpha = 1$ here. For $n \in \mathbb{N}$, $\sigma_{-\alpha}(n)$ and $\varphi_\alpha(n)$ are as usual, while $\sigma_{-\alpha}(-n) = \sigma_{-\alpha}(n)$, $\varphi_\alpha(-n) = \varphi_\alpha(n)$, and $\sigma_{-\alpha}(0) = \zeta(\alpha)$, $\varphi_\alpha(0) = 1/\zeta(\alpha)$.

For $\alpha > 1$, $\sigma_{-\alpha}(\cdot)$ and $\varphi_\alpha(\cdot)$ have all their zeros real and simple. We can ask whether the same holds true if $\alpha \le 1$. This turns out to be almost the case in that, except for a couple of imaginary zeros, all the zeros are real and simple.

We deduce the result by proving an adjusted version of Lemma 2.5. (In fact we only require the case $r = 0$.)

Lemma 3.1. *Let a_n and b_n $(n = 0,\dots,N)$ be real numbers such that $a_0,\dots,a_r < 0 < a_{r+1},\dots,a_N$ and $b_0 < b_1 < \dots < b_N$ (some $r \ge 0$). Further suppose that $\sum_{n=0}^{N} a_n > 0$. Let f be defined by*

$$f(z) = \sum_{n=0}^{N}\frac{a_n}{b_n+z} \qquad (z \ne -b_n).$$

Then all zeros of f are real and simple, one in each interval $(-b_k, -b_{k-1})$, $k = 1,\dots,N$, $k \ne r+1$, and one in $(-b_0,\infty)$.

Proof. As in Lemma 2.5, f has N zeros. Again by considering the behaviour of f near each $-b_n$, we find that f has a zero in $(-b_k, -b_{k-1})$ for $k = 1,\dots,N$ *except* when $k = r+1$. This gives $N-1$ zeros. The remaining zero occurs in $(-b_0,\infty)$ since $f(-b_0^+) = -\infty$ and

$$f(x) \sim \frac{1}{x}\sum_{n=0}^{N} a_n \qquad \text{as } x \to \infty,$$

so that $f(x) > 0$ for x sufficiently large. This gives N zeros and there are no others. ■

Theorem 3.2. *Let (a_n) and (b_n) $(n \in \mathbb{N}_0)$ be sequences of real numbers such that $a_0,\dots,a_r < 0 < a_n$ for all $n > r$ (some $r \ge 0$) and b_n is strictly increasing. Further suppose that $\sum_{n=0}^{N} a_n$ is eventually positive. Let*

$f : \mathbb{C} \setminus \{-b_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ be defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{b_n + z}.$$

Then the zeros of f are real and simple, one in each interval $(-b_k, -b_{k-1})$ $(k \neq r+1)$ and one in $(-b_0, \infty)$.

Proof. Apply Hurwitz's theorem to the partial sums f_N of f, which tend to f uniformly on compact subsets of $\mathbb{C} \setminus \{-b_n : n \in \mathbb{N}_0\}$. For all N sufficiently large, $\sum_{n=0}^{N} a_n > 0$, so f_N satisfies the conditions of Lemma 3.1. Hence (for such N) f_N has its zeros real, one in each interval $(-b_k, -b_{k-1})$ $(k \neq r+1)$ and one in $(-b_0, \infty)$. By Hurwitz's theorem, the zeros of f are also all real.

For $k \le r$,

$$f(x) \to \begin{cases} \infty & \text{if } x \to -b_k^-, \\ -\infty & \text{if } x \to -b_k^+. \end{cases}$$

Hence there exist $a, b \in (-b_k, -b_{k-1})$ such that $f < 0$ on $(-b_k, a]$ and $f > 0$ on $[b, -b_{k-1})$. Let C be a circular contour intersecting the real axis at a and b. Now $f_N \to f$ uniformly on C and its interior, and $f \neq 0$ on C. Thus by part (b) of Hurwitz's theorem, f and f_N have the same number of zeros inside C. But f has at least one zero here while f_N has at most one zero here. Hence f has exactly one zero in each interval $(-b_k, -b_{k-1})$. An identical argument shows this holds for $k \ge r+2$ as well.

A similar argument applies to the interval $(-b_0, \infty)$. Since $f(-b_0^+) = -\infty$ and f is eventually positive, there exists $c, d \in (-b_0, \infty)$ such that $f < 0$ on $(-b_0, c)$ and $f > 0$ on (d, ∞). Again f and f_N have the same number of zeros inside the circular contour intersecting the real axis at c and d for N sufficiently large, and this must again be 1. ∎

Corollary 3.3.

(i) *Let $-1 < \alpha < 1$. Then the zeros of $\sigma_{-\alpha}(\cdot)$ are of the form $\pm\mu_n$ $(n \in \mathbb{N})$, where $n < \mu_n < n+1$, and $\pm i\lambda$ for some $\lambda > 0$.*

(ii) *Let $0 < \alpha \le 1$. Then the zeros of $\varphi_\alpha(\cdot)$ are of the form $\pm\nu_n$ $(n \in \mathbb{N})$, where $n < \nu_n < n+1$, and $\pm i\kappa$ for some $\kappa \ge 0$. Further, $\kappa > 0$ if $\alpha \in (0,1)$ and $\kappa = 0$ if $\alpha = 1$.*

Proof. Apply Theorem 3.2 to the function

$$k(w) = \frac{1}{2w}\left(\zeta(\alpha) + 2w \sum_{n=1}^{\infty} \frac{\sigma_{-\alpha}(n)}{n^2 + w}\right) = \sum_{n=0}^{\infty} \frac{c_n}{n^2 + w},$$

where $c_0 = \frac{1}{2}\zeta(\alpha) < 0$ and $c_n = \sigma_{-\alpha}(n) > 0$ for $n \ge 1$. Thus the zeros of $k(w)$ are real and simple, one in each interval $(-(n+1)^2, -n^2)$ for $n \ge 1$, and one in $(0,\infty)$. Denote them by $-\mu_n^2$ and λ^2 respectively, where $n < \mu_n < n+1$ and $\lambda > 0$. Hence the zeros of $k(-z^2)$ are $\pm\mu_n$ and $\pm i\lambda$. Since $\sigma_{-\alpha}(z) = (i/\pi)k(-z^2)(e^{2\pi iz}-1)$ and $\sigma_{-\alpha}(\cdot)$ is non-zero on the integers, the result follows.

Similarly for $\varphi_\alpha(\cdot)$, but this time we cannot apply Theorem 3.2 for $\alpha \le 0$ since then $\varphi_\alpha(n) \not> 0$. For $\alpha \in (0,1)$, $\varphi_\alpha(0) < 0 < \varphi_\alpha(n)$ and identical arguments as above gives the result; namely, the zeros are $\pm\nu_n$ ($n \in \mathbb{N}$) and $\pm i\kappa$, where $\kappa > 0$.

Finally, if $\alpha = 1$, then $\varphi_1(0) = 0 < \varphi_1(n)$ and we must have $\kappa = 0$. ■

Remark. It would be interesting to study in more detail the distribution of the zeros of, say, $d(z)$ $(=\sigma_0(z))$. For example, a simple heuristic argument applied to the real function $e^{-i\pi x}d(x)$ suggests ([5]) the approximation

$$\mu_n \approx n + \frac{d(n)}{d(n)+d(n+1)}.$$

4. Behaviour of $f(ix)$ for large x. In general, we can say little about the behaviour of f, as defined by (2.1), along the imaginary axis. If $\sum_{0<q<1} a(q)$ is an absolutely convergent series over all the rationals between 0 and 1, then

$$\sum_{0<q<1} a(q)e^{-qx} \to 0$$

as $x \to \infty$. It follows immediately that $f(ix) = \frac{1}{2}\widehat{f}(1) + o(1)$.

If f has a Ramanujan expansion and is as in Proposition 2.4, we can say more regarding the $o(1)$ term.

Let $A(s)$ denote the Dirichlet series defined for $\Re s \ge 0$ by

$$A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Then we have ([6])

$$\begin{aligned} 2x^2 \sum_{n=1}^{\infty} \frac{f(n)}{n^2+x^2} &= \frac{1}{2\pi i} \int_{(c)} \frac{\pi}{\sin(\pi s/2)} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}\, x^s\, ds \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\pi}{\sin(\pi s/2)}\, \zeta(s)A(s)x^s\, ds, \end{aligned}$$

which is valid for $x > 0$ and any $1 < c < 2$. (Here $\int_{(c)}$ means $\lim_{T\to\infty} \int_{c-iT}^{c+iT}$.) By moving the contour to the left we pass the simple zero of $\zeta(s)$ at $s = 1$

([5]) Joining the points $(n, (-1)^n d(n))$ and $(n+1, (-1)^{n+1} d(n+1))$ with a straight line.

([6]) This is based on the identity $\frac{x^2}{1+x^2} = \frac{1}{2\pi i}\int_{(c)} \frac{\pi}{\sin(\pi s/2)} x^s\, ds$ for $0 < c < 2$.

picking up a residue $\pi A(1)x = \pi\widehat{f}(1)x$. We can go as far as $\Re s = \delta$ (any $\delta > 0$) so that

$$f(ix) = \frac{1}{2}\,\widehat{f}(1) + O(x^{\delta-1}).$$

With more knowledge of the analytic character of $A(s)$ we can push the contour further to the left to obtain better approximations.

In the special example of $\sigma_{-\alpha}(\cdot)$, we have $A(s) = \zeta(s)\zeta(s+\alpha)$, so that

$$2x^2\sum_{n=1}^{\infty}\frac{\sigma_{-\alpha}(n)}{n^2+x^2} = \frac{1}{2\pi i}\int_{(c)}\frac{\pi}{\sin(\pi s/2)}\,\zeta(s)\zeta(s+\alpha)x^s\,ds.$$

This is valid for $x > 0$ and any c such that $\max\{1, 1-\alpha\} < c < 2$.

Moving the line of integration to the left, we pick up the residues at the poles of the integrand. This is justifiable since $(\sin(\pi s/2))^{-1}$ is exponentially small on vertical lines. The residues are at $s = 1$, $s = 1-\alpha$ and $s = 0$ (the poles of $1/\sin(\pi s/2)$ at $s = -2n$ ($n \in \mathbb{N}$) are cancelled by the trivial zeros of $\zeta(s)$), and the residues are, respectively,

$$\pi\zeta(\alpha+1)x, \quad \frac{\pi\zeta(1-\alpha)}{\sin(\pi/2)(1-\alpha)}\,x^{1-\alpha}, \quad 2\zeta(0)\zeta(\alpha).$$

Since $\zeta(0) = -\frac{1}{2}$, we have $\sin(\pi/2)(1-\alpha) = \cos(\pi\alpha/2)$, and using the functional equation for ζ leads to

$$\sigma_{-\alpha}(ix) - \frac{1}{2}\,\zeta(\alpha+1) = \frac{\Gamma(\alpha)\zeta(\alpha)}{(2\pi x)^{\alpha}} + O\biggl(\frac{1}{x^A}\biggr)$$

for every A.

Similarly, for $\varphi_\alpha(\cdot)$ we obtain (after moving the line of integration past 0)

$$2x^2\sum_{n=1}^{\infty}\frac{\varphi_\alpha(n)}{n^2+x^2} = \frac{\pi}{\zeta(\alpha+1)}\,x - \frac{1}{\zeta(\alpha)} + \frac{1}{2\pi i}\int_{(c)}\frac{\pi}{\sin(\pi s/2)}\,\frac{\zeta(s)}{\zeta(s+\alpha)}\,x^s\,ds,$$

for $1-\alpha < c < 0$ (if $\alpha > 1$). Moving further to the left we encounter the poles of $1/\zeta(s+\alpha)$, i.e. the zeros of $\zeta(s+\alpha)$. It easily follows from this that the Riemann Hypothesis holds if and only if

$$\varphi_\alpha(ix) - \frac{1}{2\zeta(\alpha+1)} = O(x^{\varepsilon-\alpha-1/2})$$

for every $\varepsilon > 0$.

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