On the Kummer conjecture

by

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1. Introduction. Let \( q \) be an odd prime, and let \( h_q \) and \( h_q^+ \) be the class numbers of \( \mathbb{Q}(\zeta_q) \) and \( \mathbb{Q}(\zeta_q + \zeta_q^{-1}) \) respectively (\( \zeta_q = e^{2\pi i/q} \)). It is well known that \( h_q^+ \mid h_q \). The famous Kummer conjecture states that

\[
h_q^- = h_q/h_q^+ \sim 2q \left( \frac{q}{4\pi^2} \right)^{(q-1)/4} \quad (\text{as } q \to \infty),
\]

is equivalent to

\[
\prod_{\chi(-1)=-1} L(1, \chi) \sim 1 \quad (\text{as } q \to \infty),
\]

where \( \chi \) runs through Dirichlet characters modulo \( q \). This conjecture has not been proved yet, but there are several works on the upper bound of \( h_q^- \); for example, by using elementary methods, Feng Keqin [1] proved that

\[
h_q^- < 2q \left( \frac{q - 1}{31.997158 \ldots} \right)^{(q-1)/4},
\]

and in [2] it is mentioned that the following results have been proved:

\[
\frac{2}{3} < \prod_{\chi(-1)=-1} L(1, \chi) < \frac{3}{2} \quad (\text{for } 5 \leq q \leq 523),
\]

\[
\frac{e^{-12.93}}{Lq^{1/2}(\log q)^4} < \prod_{\chi(-1)=-1} L(1, \chi) < e^{15.49}L(\log q)^5 \quad (\text{for any } q)
\]

with \( L = e^{4.66/\log q} \). In [4], M. Ram Murty and Yiannis N. Petridis proved the following weak Kummer conjecture: There exists a positive constant \( c \)

2000 Mathematics Subject Classification: Primary 11M20.

Key words and phrases: Kummer conjecture, Dirichlet \( L \)-function, bound estimate.

This work is supported by N.S.F. (No.10601039) of P.R. China.
such that for almost all primes \( q \),
\[
c^{-1} \leq \prod_{\chi(-1) = -1} L(1, \chi) \leq c.
\]

In this paper, we shall prove the following

**Theorem 1.** For every sufficiently large prime \( q \) with \( q \equiv 1 \pmod{4} \),
\[
\prod_{\chi(-1) = -1} L(1, \chi) \geq e^{-1.4 (\log q)^{-4/3} (\log \log q)^{-1}} \left( 1 + O \left( \frac{1}{\log \log q} \right) \right),
\]
\[
\prod_{\chi(-1) = -1} L(1, \chi) \leq e^{0.84 (\log q)^{7/6} (\log \log q)} \left( 1 + O \left( \frac{1}{\log \log q} \right) \right),
\]
where \( \chi \) runs through Dirichlet characters modulo \( q \).

Using the same method, we can prove the following conclusions:

**Theorem 2.** Let \( q \) be any sufficiently large prime. Assume that there is no exceptional zero for \( L(s, \chi) \), where \( \chi \) is any Dirichlet character modulo \( q \). Then we have the same estimates as in Theorem 1.

**Theorem 3.** Assuming the Generalized Riemann Hypothesis (GRH), for every sufficiently large prime \( q \) we have
\[
\prod_{\chi(-1) = -1} L(1, \chi) \geq e^{-2.1 (\log q)^{-4/3} (\log \log q)^{-1}} \left( 1 + O \left( \frac{1}{\log \log q} \right) \right),
\]
\[
\prod_{\chi(-1) = -1} L(1, \chi) \leq e^{1.53 (\log q)^{7/6} (\log \log q)} \left( 1 + O \left( \frac{1}{\log \log q} \right) \right).
\]

**Theorem 4.** For any fixed \( \varepsilon > 0 \), there is a positive number \( Q \), which depends only on \( \varepsilon \), such that if \( q \) is a prime greater than \( Q \), we have
\[
e^{-1.4 q^{-\varepsilon} (\log q)^{-1/3}} \leq \prod_{\chi(-1) = -1} L(1, \chi) \leq e^{0.84 q^\varepsilon (\log q)^{1/6}}.
\]

The following symbols will be used in the proof of these theorems. For \( q \geq 3 \), \((l, q) = 1, 1 \leq l < q \), we write
\[
\pi(x; q, l) = \sum_{p \leq x \atop p \equiv l \pmod{q}} 1, \quad \theta(x; q, l) = \sum_{p \leq x \atop p \equiv l \pmod{q}} \log p,
\]
\[
\psi(x; q, l) = \sum_{n \leq x \atop n \equiv l \pmod{q}} \Lambda(n), \quad \psi(x; \chi) = \sum_{n \leq x} \Lambda(n) \chi(n),
\]
\[
\psi(x) = \sum_{n \leq x} \Lambda(n), \quad x_1 = q^{\log \log q}, \quad x_2 = e^q,
\]
where \( \Lambda(n) \) is the von Mangoldt function.
2. Some lemmas

**Lemma 1.** For every prime $q \geq 3$,

$$\sum_{\chi(-1)=-1} \sum_{p>x_2} \frac{\chi(p)}{p} \ll q^2 e^{-c_1 \sqrt{q}}, \quad (6)$$

where $\chi$ runs through Dirichlet characters modulo $q$, and $c_1 > 0$ is a constant independent of $q$.

**Proof.** For an arbitrary number $y > x_2$, by using the identity

$$\sum_{\chi(-1)=1} \chi(n) = \begin{cases} \frac{1}{2}(q-1) & \text{if } n \equiv \pm 1 \pmod{q}, \\ 0 & \text{otherwise}, \end{cases} \quad (7)$$

we have

$$\sum_{\chi(-1)=-1} \sum_{x_2 < p \leq y} \frac{\chi(p)}{p} = \sum_{x_2 < p \leq y} \frac{1}{p} \sum_{\chi(-1)=-1} \chi(p)$$

$$= \frac{1}{2} \left( \sum_{x_2 < p \leq y, \atop p \equiv 1 \pmod{q}} \frac{1}{p} - \sum_{x_2 < p \leq y, \atop p \equiv -1 \pmod{q}} \frac{1}{p} \right)$$

$$= \frac{q-1}{2} \int_{x_2}^{y} \frac{1}{u} \{ \pi(u; q, 1) - \pi(u; q, -1) \}.$$  

The Siegel–Walfisz theorem yields

$$\sum_{\chi(-1)=-1} \sum_{x_2 < p \leq y} \frac{\chi(p)}{p} \ll q e^{-c_1 \sqrt{\log x_2}} + q \int_{x_2}^{y} e^{-c_1 \sqrt{\log u}} \frac{du}{u}$$

$$\ll q e^{-c_1 \sqrt{\log x_2}} + q e^{-c_1 \sqrt{\log x_2} \log x_2} \ll q^2 e^{-c_1 \sqrt{q}}.$$  

Since $y$ is arbitrary, we can easily get (6) by letting $y$ tend to infinity in the last formula.

**Lemma 2** (see [5, §17.1]). Assume $q \geq 3$ is any integer and $s = \sigma + it$. Then there is at most one character $\chi$ modulo $q$ such that the function $L(s, \chi)$ has a zero in the region

$$\sigma \geq 1 - \frac{c_2}{\log(q(|t| + 2))}, \quad (8)$$

where $c_2$ is a positive constant. If such an exceptional function exists, the corresponding character $\tilde{\chi}$ must be a nonprincipal real character modulo $q$, and $L(s, \tilde{\chi})$ has only one zero $\beta$ (this zero must be a real zero) in the above region.

**Lemma 3** (see [5, Theorem 33.3.1]). If the exceptional zero $\tilde{\beta}$ in Lemma 2 exists, then there are positive constants $c_3, c_4$ such that $\tilde{\beta}$ is the only zero
of $\prod_{\chi \mod q} L(s, \chi)$ in the region

$$
\begin{cases}
\sigma \geq 1 - \frac{c_3}{\log(q(|t| + 2))} \log \frac{c_4 \varepsilon}{\delta \log(q(|t| + 2))}, \\
\tilde{\delta} \log(q(|t| + 2)) \leq c_4,
\end{cases}
$$

where $\tilde{\delta} = 1 - \tilde{\beta}$.

**Lemma 4** (see [5, Theorem 33.2.8]). Assume $q \geq 3$ is any integer and $\chi$ is a Dirichlet character modulo $q$. Denote by $N(\alpha, T, \chi)$ the number of zeros of $L(s, \chi)$ in the region $\alpha \leq \sigma \leq 1$, $|t| \leq T$, and write $N(\alpha, T, q) = \sum_{\chi \mod q} N(\alpha, T, \chi)$. Then for any $T \geq 2$ and $1/2 \leq \alpha \leq 1$, we have

$$
N(\alpha, T, q) \ll (qT)^{3(1-\alpha)}.
$$

**Lemma 5** (see [5, Theorem 18.1.5]). Assume $x \geq 2$, $T \geq 2$, $q \geq 3$. Then for every nonprincipal character $\chi$ modulo $q$,

$$
\psi(x, \chi) = -\tilde{E} x^\tilde{\beta} - \sum' x^g \frac{\log(x)}{q} + O\left(\frac{x \log^2(xqT)}{T} + \log^2(xq) + \tilde{E} x^{1/4}\right),
$$

where

$$
\tilde{E} = \begin{cases} 
1, & \chi = \tilde{\chi} \\
0, & \chi \neq \tilde{\chi},
\end{cases}
$$

$\tilde{\chi}$ is the exceptional character that possibly exists, and $\sum'$ is the sum over all nontrivial zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ except the exceptional zeros $\tilde{\beta}$ and $1 - \tilde{\beta}$.

**Lemma 6.** Assume $x \geq 2$, $T \geq 2$, $q \geq 3$ is a prime number, and $l$ is a positive integer satisfying $1 \leq l < q$. Then

$$
\theta(x; q, l) = \psi(x) - \tilde{E}(q) \tilde{\chi}(l) \frac{x^\beta}{\beta} - \frac{1}{q - 1} \sum_{\chi \neq \chi^0} \tilde{\chi}(l) \sum' x^g \frac{\log(x)}{q}
$$

$$
+ O\left(\frac{x \log^2(xqT)}{T} + \log^2(xq) + x^{1/2}\right)
$$

where

$$
\tilde{E}(q) = \begin{cases} 
1 & \text{if the exceptional character } \tilde{\chi} \mod q \text{ exists}, \\
0 & \text{otherwise}.
\end{cases}
$$

**Proof.** We have

$$
\psi(x; q, l) = \sum_{n \leq x} \frac{\Lambda(n)}{\phi(q)} \sum_{\chi \mod q} \tilde{\chi}(l) \chi(n) = \frac{1}{q - 1} \sum_{\chi \mod q} \tilde{\chi}(l) \psi(x, \chi).
$$
Since $q$ is a prime number, we easily get
\[\psi(x, \chi^0) = \sum_{n \leq x} A(n) = \psi(x) - \sum_{n \leq x, q|n} A(n)\]
\[= \psi(x) - \sum_{q^m \leq x} \log q = \psi(x) + O(\log x).\]

Combining this with Lemma 5, we have
\[\psi(x; q, l) = \frac{\psi(x)}{q-1} + \frac{1}{q-1} \sum_{\chi \neq \chi^0} \chi(l) \psi(x, \chi) + O\left(\frac{1}{q} \log x\right)\]
\[= \frac{\psi(x)}{q-1} - \tilde{E}(q) \tilde{\chi}(l) \cdot \frac{x^{\tilde{\beta}}}{\tilde{\beta}} - \frac{1}{q-1} \sum_{\chi \neq \chi^0} \chi(l) \sum_{|\gamma| \leq T} \frac{x^\gamma}{\gamma} \]
\[+ O\left(\frac{x \log^2(xqT)}{T} + \log^2(qx) + \tilde{E}(q) \frac{x^{1/4}}{q}\right),\]
which proves the lemma by using $\theta(x; q, l) = \psi(x; q, l) + O(x^{1/2})$.

**Lemma 7.** Let $A = \min(c_2, c_3, c_4)$, where $c_2, c_3, c_4$ are defined in Lemmas 2 and 3. Then for every sufficiently large integer $q$, we have
\[
\sum_{\chi \mod q} \sum' u^{\beta-1} \ll \left(\frac{u}{(qT)^3}\right)^{-A/\log(qT)} + u^{-1/2}qT \log(qT) \quad (12)
\]
for $u \geq x_1 = q^{\log \log q}$ and $T = q^4$; $\sum'$ and $\beta$ are defined in Lemma 5.

**Proof.** From Lemmas 2 and 3, we have:

(i) If the exceptional zero $\tilde{\beta}$ exists and satisfies $\tilde{\delta} \log(qT) \leq A$, then
\[\prod_{\chi \mod q} L(s, \chi)\]
does not vanish in the region
\[\sigma \geq 1 - \frac{A}{\log(qT)} \log \frac{Ae}{\delta \log(qT)}, \quad |t| \leq T\]
except at $s = \tilde{\beta}$.

(ii) If $\tilde{\delta} \log(qT) > A$ or the exceptional zero does not exist, then
\[\prod_{\chi \mod q} L(s, \chi)\]
does not vanish in the region $\sigma \geq 1 - A/\log(qT)$, $|t| \leq T$.

Hence if we choose
\[\eta_0 = A \log \frac{Ae}{\delta_0 \log(qT)},\]
where
\[\delta_0 = \begin{cases} \tilde{\delta}, & \delta \log(qT) \leq A, \\ A/\log(qT), & \delta \log(qT) > A \text{ or the exceptional zero does not exist,} \end{cases}\]
then $\eta_0 \geq A$, and $\prod_{\chi \mod q} L(s, \chi) \neq 0$ in the region $\sigma \geq 1 - \eta_0 / \log(qT)$, $|t| \leq T$ except at one point $s = \tilde{\beta}$. Hence

$$\sum_{\chi \mod q} \sum'_{|\gamma| \leq T} u^{\beta - 1} \ll \sum_{\chi \mod q} \sum'_{|\gamma| \leq T} u^{\beta - 1} = \int_{1/2} \sum_{\chi \mod q} 1^{1 - \eta_0 / \log(qT)} u^{\alpha - 1} d_\alpha N(\alpha, T, q).$$

Making use of $N(1/2, T, q) \ll qT \log(qT)$ and Lemma 4, we obtain

$$\sum_{\chi \mod q} \sum'_{|\gamma| \leq T} u^{\beta - 1} \ll \int_{1/2} N(\alpha, T, q) u^{\alpha - 1} \log u d\alpha + u^{-1/2} qT \log(qT)$$

$$\ll (\log u) \int_{1/2} \left( \frac{u}{(qT)^3} \right)^{\alpha - 1} d\alpha + u^{-1/2} qT \log(qT)$$

$$\ll \frac{\log u}{\log u - 3 \log(qT)} \left( \frac{u}{(qT)^3} \right)^{-\eta_0 / \log(qT)} + u^{-1/2} qT \log(qT),$$

and (12) follows at once from the choice of $u$, $T$ and $\eta_0 \geq A$.

**Lemma 8.** For every sufficiently large prime $q \equiv 1 (\mod 4)$, we have

$$(13) \quad \sum_{\chi \mod q} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} \ll (\log q)^{-A/5} (\log \log q)^{-1},$$

where $A$ is defined in Lemma 7.

**Proof.** Making use of (7) and Lemma 6, we have

$$\sum_{\chi \mod q} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} \quad \chi(-1) = -1$$

$$= \frac{q - 1}{2} \left( \sum_{x_1 < p \leq x_2} \frac{1}{p} - \sum_{x_1 < p \leq x_2} \frac{1}{p} \quad \chi(-1) = -1 \right.$$

$$= \frac{q - 1}{2} \int_{x_1}^{x_2} \frac{1}{u \log u} d\{\theta(u; q, 1) - \theta(u; q, -1)\}$$

$$= \int_{x_1}^{x_2} \frac{1}{u \log u} d\left\{ \frac{\tilde{E}(q)}{2} (\tilde{\chi}(-1) - 1) \frac{u^\tilde{\beta}}{\tilde{\beta}} - \frac{1}{2} \sum_{\chi \notin \chi^0} (1 - \tilde{\chi}(-1)) \sum_{|\gamma| \leq T} \frac{u^\gamma}{\gamma q} \right\}$$

$$+ O(\frac{q \log^2(x_2 q T)}{T \log x_2} + \frac{q}{\sqrt{x_1 \log x_1}} + \int_{x_1}^{x_2} \left( \frac{uq \log^2(uqT)}{T} + u^{-1/2} q \right) \log u + \frac{1}{u^2 \log^2 u} du),$$
Notice that \( q \) is a prime and \( \tilde{\chi} \) is nonprincipal real character, so that
\[
\tilde{\chi}(-1) = \left( \frac{-1}{q} \right) = (-1)^{(q-1)/2} = 1,
\]
where \( \left( \frac{n}{q} \right) \) is the Legendre symbol modulo \( q \). Hence
\[
\sum_{\chi \mod q} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} = -\sum_{x_1} \left( \sum_{\chi(-1)=-1} \sum' u^{q-1} \right) \frac{du}{u \log u}
\]
\[
+ O\left( \frac{q \log^2(x_2 q T)}{T \log x_2} + \frac{q}{\sqrt{x_1 \log x_1}} + \int \left( \frac{u q \log^2(u q T)}{T} + u^{1/2} q \right) \frac{\log u + 1}{u^2 \log^2 u} \right)
\]
\[
= -\sum_{x_1} \left( \sum_{\chi(-1)=-1} \sum' u^{q-1} \right) \frac{du}{u \log u} + O\left( \frac{q \log^2(q T) \log^2 x_2}{T} + \frac{q}{\sqrt{x_1 \log x_1}} \right).
\]
If we choose \( T = q^4 \) and make use of Lemma 7 as well as the definition of \( x_1, x_2 \), we obtain
\[
\sum_{\chi \mod q} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} \ll \sum_{x_1} \left( \sum_{\chi \mod q | \gamma| \leq T} \sum' u^{q-1} \right) \frac{du}{u \log u} + \frac{\log^2 q}{q}
\]
\[
\ll \int_{x_1}^{x_2} \left\{ \left( \frac{u}{q^{15}} \right)^{-A/5 \log q} + u^{-1/2} q^{5 \log q} \right\} \frac{du}{u \log u} + \frac{\log^2 q}{q}
\]
\[
\ll \int_{x_1}^{x_2} u^{-1-A/5 \log q} \frac{du}{\log u} + \frac{\log^2 q}{q}
\]
\[
\ll \frac{1}{\log x_1} \cdot \frac{5 \log q}{A} x_1^{-A/5 \log q} + \frac{\log^2 q}{q}
\]
\[
\ll (\log q)^{-A/5} (\log \log q)^{-1}.
\]
This completes the proof of the lemma.

**Lemma 9** (see [5, Theorem 28.6.1]). Assume \( (q, l) = 1 \) and \( 1 \leq l < q < y \leq x \). Then
\[
\pi(x; q, l) - \pi(x - y; q, l) < \frac{2}{\phi(q)} \cdot \frac{y}{\log(y/q)}.
\]

**Lemma 10.** Let \( q \geq 3 \) be a prime number, \( 1 \leq l < q \), \( l \) is not a prime, and \( x \geq q^2 \). Then
\[
\sum_{p \leq x \atop p \equiv l (\mod q)} \frac{1}{p} \leq \frac{2}{q - 1} \left( \log \log \frac{x}{q} + \frac{1}{2} + \frac{1}{\log q} \right).
\]
Proof. Using Abel’s identity, we obtain

\[ \sum_{\substack{2q < p \leq x \mod q \equiv l \mod q}} \frac{1}{p} = \frac{1}{x} \pi(x; q, l) - \frac{1}{2q} \pi(2q; q, l) + \int \frac{1}{2q} \pi(u; q, l) \, du. \]  

If we choose \( x = y > q \) in Lemma 9, we have

\[ \pi(x; q, l) < \frac{2}{\phi(q)} \cdot \frac{x}{\log(x/q)} \quad (x > q). \]

Combining this with (16), we find that

\[ \sum_{\substack{2q < p \leq x \mod q \equiv l \mod q}} \frac{1}{p} \leq \frac{1}{x} \cdot \frac{2}{q-1} \cdot \frac{x}{\log(x/q)} + \int \frac{1}{2q} \cdot \frac{u}{q-1} \cdot \frac{u}{\log(u/q)} \, du \]

\[ \leq \frac{2}{q-1} \log \log \frac{x}{q} + \frac{2}{q-1} \cdot \frac{1}{\log q}; \]

we have used \( x \geq q^2 \) in the last inequality. Since \( l \) is not a prime, we have

\[ \sum_{\substack{p \leq 2q \mod q \equiv l \mod q}} \frac{1}{p} \leq \frac{1}{q}, \]

and we easily deduce (15) from the discussion above.

**Lemma 11.** Assume \( q \) is a sufficiently large prime with \( q \equiv 1 \mod 4 \), and define

\[ \sum_1 = \sum_{\chi \mod q} \sum_{p \leq x_1} \frac{\chi(p)}{p}. \]

Then

\[ \left| \sum_1 \right| \leq \log \log q + \log \log \log q + \frac{1}{2} + O \left( \frac{1}{\log \log q} \right). \]

**Proof.** From (7) we get

\[ \sum_{\chi(-1) = -1} \sum_{p \leq x_1} \frac{\chi(p)}{p} = \sum_{p \leq x_1} \frac{1}{p} \sum_{\chi(-1) = -1} \chi(p) \]

\[ = \frac{q - 1}{2} \left( \sum_{p \equiv 1 \mod q} \frac{1}{p} - \sum_{p \equiv -1 \mod q} \frac{1}{p} \right). \]

Thus

\[ -\frac{q - 1}{2} \sum_{\substack{p \leq x_1 \mod q \equiv -1 \mod q}} \frac{1}{p} \leq \sum_{\chi(-1) = -1} \sum_{p \leq x_1} \frac{\chi(p)}{p} \leq \frac{q - 1}{2} \sum_{\substack{p \leq x_1 \mod q \equiv 1 \mod q}} \frac{1}{p}. \]

If we choose \( x = x_1, l = 1 \) and \( x = x_1, l = q - 1 \) respectively in Lemma 10,
and put the results into (18), we obtain
\[
\left| \sum_{\chi(-1)=-1} \sum_{p \leq x_1} \frac{\chi(p)}{p} \right| \leq \log \log \frac{x_1}{q} + \frac{1}{2} + \frac{1}{\log q}
\leq \log \log q + \log \log \log q + \frac{1}{2} + \frac{1}{\log q}.
\]

Applying Lemmas 1 and 8 yields (17).

**Lemma 12.** Let \( q \) be a sufficiently large prime, and define
\[
\sum_2 = \sum_{\chi \mod q} \sum_p \sum_{j=2}^{\infty} \frac{\chi(p^j)}{jp^j}.
\]
Then
\[
\sum_2 \geq -\frac{1}{3} \log \log q - \frac{1}{6} \left( 2 + \log \frac{8}{\log^2 2} + \gamma \right) + O\left( \frac{1}{\log q} \right),
\]
(19)
\[
\sum_2 \leq \frac{1}{6} \log \log q + \frac{1}{12} \left( 2 + \log \frac{2}{\log^2 2} + \gamma \right) + O\left( \frac{1}{\log q} \right),
\]
(20)
where \( \gamma \) is the Euler constant.

**Proof.** We have
\[
\sum_2 = \sum_p \sum_{j=2}^{\infty} \frac{1}{jp^j} \sum_{\chi(-1)=-1} \chi(p^j)
\]
\[
= \frac{q-1}{2} \left\{ \sum_{p \mod q \equiv 1} \sum_{j=2}^{\infty} \frac{1}{jp^j} - \sum_{p \mod q \equiv -1} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right\}.
\]
Since
\[
\sum_{p \mod q \equiv 1} \sum_{j=2}^{\infty} \frac{1}{jp^j} = \sum_{p \mod q \equiv -1} \sum_{j=2}^{\infty} \frac{1}{jp^j} + O\left( \sum_{p \mod q \equiv 1} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right)
\]
\[
= \sum_{p \mod q \equiv 1} \sum_{j=2}^{\infty} \frac{1}{jp^j} + O\left( \sum_{p \mod q \equiv -1} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right)
\]
\[
= \sum_{p \mod q \equiv 1} \sum_{j=2}^{\infty} \frac{1}{jp^j} + O\left( \frac{1}{q \log q} \right),
\]
and in the same way
\[
\sum_{p \mod q \equiv -1} \sum_{j=2}^{\infty} \frac{1}{jp^j} = \sum_{p \mod q \equiv -1} \sum_{j=2}^{\infty} \frac{1}{jp^j} + O\left( \frac{1}{q \log q} \right),
\]
we deduce from (21) that
\[
\sum_2 = \frac{q-1}{2} \left( \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} - \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right) + O\left( \frac{1}{\log q} \right).
\]

Let \( g \) be a primitive root modulo \( q \), \( \delta(p) \) be the exponent order of \( p \) modulo \( q \) for \( p < q \) (that is to say, \( \delta(p) = \min\{d > 0 : p^d \equiv 1 \pmod{q}\} \)), and let \( k(p) \) (\( 1 \leq k(p) \leq q-1 \)) denote the integer satisfying \( p \equiv g^{k(p)} \pmod{q} \) for \( p < q \).

Then
\[
\delta(p) = \frac{q-1}{(k(p), q-1)},
\]
and thus
\[
\sum_2 = \frac{q-1}{2} \left( \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} - \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right) + O\left( \frac{1}{\log q} \right)
\]
\[
= \frac{q-1}{2} \left( \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} - \sum_{p<q} \sum_{j=2}^{\infty} \frac{1}{jp^j} \right) + O\left( \frac{1}{\log q} \right).
\]

Since \( \delta(p) \geq 3 \) when \( p < q \) because \( q \) is a sufficiently large prime, we have
\[
(22) \quad \sum_2 = \frac{q-1}{2} \left( \sum_{p<q} \sum_{j=1}^{\infty} \frac{1}{j\delta(p)p^j\delta(p)} - \sum_{p<q} \sum_{j=1}^{\infty} \frac{1}{(j-1/2)\delta(p)p^{(j-1/2)\delta(p)}} \right) + O\left( \frac{1}{\log q} \right)
\]
\[
= \frac{q-1}{2} \left( - \sum_{p<q} \frac{1}{\delta(p)} \log \left( 1 - \frac{1}{p^{\delta(p)}} \right) - \sum_{p<q} \frac{1}{\delta(p)} \log \frac{1 + p^{-\delta(p)/2}}{1 - p^{-\delta(p)/2}} \right) + O\left( \frac{1}{\log q} \right)
\]
\[
= \frac{q-1}{2} \left\{ \sum_{p<q} \frac{1}{\delta(p)} \log \left( 1 + \frac{1}{p^{\delta(p)} - 1} \right) - 2 \sum_{p<q} \frac{1}{\delta(p)} \log \left( 1 + \frac{1}{p^{\delta(p)/2}} \right) \right\} + O\left( \frac{1}{\log q} \right)
\]
and this implies that
\[ \sum_2 \geq -(q - 1) \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{\delta(p)} \log \left( 1 + \frac{1}{p^{\delta(p)/2}} \right) + O \left( \frac{1}{\log q} \right) \]
\[ \geq -(q - 1) \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)/2}} + O \left( \frac{1}{\log q} \right). \]

Notice that \(2 \mid \delta(p)\) implies \(p^{\delta(p)/2} \equiv -1 \pmod{q}\), thus \(p^{\delta(p)/2} \geq q - 1\) and
\[ \frac{1}{p^{\delta(p)/2}} \leq \frac{1}{p^{\delta(p)/2} + 1} \cdot \frac{q}{q - 1}, \]
so that
\[ \sum_2 \geq -q \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)/2} + 1} + O \left( \frac{1}{\log q} \right). \]

On the other hand, we can also deduce from (22) that
\[ \sum_2 \leq \frac{q - 1}{2} \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)/2} - 1} + O \left( \frac{1}{\log q} \right). \]

Since \(\delta(p) \geq 3\) for \(p < q\), we have
\[ \sum_2 \geq -q \left( \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{p^{\delta(p)/2} + 1} + O \left( \frac{1}{\log q} \right) \right), \]
\[ \sum_2 \leq q \left( \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{p^{\delta(p)/2} - 1} + O \left( \frac{1}{\log q} \right) \right). \]

We write \(p^{\delta(p)} = l(p)q + 1\). Because \(2 \mid \delta(p)\) implies \(p^{\delta(p)/2} \equiv -1 \pmod{q}\), we can write \(p^{\delta(p)/2} = h(p)q - 1\) if \(2 \mid \delta(p)\). Then
\[ \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{p^{\delta(p)/2} - 1} = \sum_{p < q \atop 2 \nmid \delta(p)} \frac{1}{l(p)q} \]
\[ = \frac{1}{q} \left( \sum_{p < q \atop 2 \nmid \delta(p), l(p) < q} \frac{1}{l(p)} + \sum_{p < q \atop 2 \mid \delta(p), l(p) \geq q} \frac{1}{l(p)} \right) \]
\[ = \frac{1}{q} \sum_{p < q \atop 2 \mid \delta(p), l(p) < q} \frac{1}{l(p)} + O \left( \frac{1}{q \log q} \right). \]
Using the same method, we have

\[ \sum_{p/q} \frac{1}{p^\delta(p)/2 + 1} = \frac{1}{q} \sum_{p/q} \frac{1}{h(p)} + O\left(\frac{1}{q \log q}\right). \]

Since \( l(p) < q \Rightarrow \delta(p) \leq n_1 = [(2 \log q)/\log 2] \) and \( h(p) < q \Rightarrow \delta(p) \leq n_2 = [(4 \log q)/\log 2] \), we deduce from (25), (26) that

\[ \sum_{p/q} \frac{1}{p^\delta(p)/2 + 1} + 1 = \frac{1}{q} \sum_{p/q} \frac{1}{h(p)} + O\left(\frac{1}{q \log q}\right). \]

It is well known that for every \( d \mid q - 1 \), there are exactly \( \phi(d) \) integers in the reduced residue class modulo \( q \) with \( d \) as their exponent order modulo \( q \), so the number of terms of the sum on the right hand side of (27) is less than

\[ \sum_{n \leq n_1} \phi(n) \leq n_1 \sum_{n \leq n_1} \frac{1}{q} \sum_{n \mid q-1, 2n} 1 \leq \frac{1}{2} n_1^2 \leq \frac{2}{\log^2 2} \log^2 q. \]

It is obvious that \( l(p) \) is different for each \( p \), and

\[ l(p) \begin{cases} 
\text{is even} & \text{if } p > 2, \\
\geq 1 & \text{if } p = 2,
\end{cases} \]

so we infer from (27) that

\[ \sum_{p/q} \frac{1}{p^\delta(p) - 1} \leq \frac{1}{q} \left(1 + \sum_{n \leq n_1} \frac{1}{2n} \sum_{n \mid q-1, 2n} 1 \right) + O\left(\frac{1}{q \log q}\right) \]

\[ = \frac{1}{q} \left\{ 1 + \frac{1}{2} \left( \log \left( \frac{2}{\log^2 2} \log^2 q \right) + \gamma + O\left(\frac{1}{\log^2 q}\right) \right) \right\} \]

\[ + O\left(\frac{1}{q \log q}\right) \]

\[ = \frac{1}{q} \left( \log \log q + 1 + \frac{1}{2} \log \frac{2}{\log^2 2} + \frac{\gamma}{2} \right) + O\left(\frac{1}{q \log q}\right). \]

Also, the number of terms of the sum on the right hand side of (28) is less than

\[ \sum_{n \leq n_2} \phi(n) \leq \frac{1}{2} n_2^2 \leq \frac{8}{\log^2 2} \log^2 q. \]
Because \( h(p) \) is different for each \( p \), and
\[
  h(p) \begin{cases} \text{is even} & \text{if } p > 2, \\ \geq 1 & \text{if } p = 2, \end{cases}
\]
we deduce from (28) that
\[
  \sum_{\substack{p < q \\ 2 \mid \delta(p)}} \frac{1}{p^{\delta(p)/2} + 1} \leq \frac{1}{q} \left( 1 + \sum_{n \leq \frac{8}{\log^2 q}} \frac{1}{2n} \right) + O\left( \frac{1}{q \log q} \right)
\]
\[
= \frac{1}{q} \left( \log \log q + 1 + \frac{1}{2} \log \frac{8}{\log^2 q} + \frac{\gamma}{2} \right) + O\left( \frac{1}{q \log q} \right).
\]
The lemma now follows at once from (23), (24), (29), (30).

### 3. Proof of the theorems

**Proof of Theorem 1.** It is easy to verify that \( \prod_{\chi(-1)=-1} L(1, \chi) \) is a positive real number. Making use of the identity
\[
L(1, \chi) = \prod_p \left( 1 - \frac{\chi(p)}{p} \right)^{-1} \quad (\chi \neq \chi^0),
\]
we have
\[
\log \left( \prod_{\chi(-1)=-1} L(1, \chi) \right) = \log \left( \prod_{\chi(-1)=-1} \prod_p \left( 1 - \frac{\chi(p)}{p} \right)^{-1} \right)
\]
\[
= - \sum_{\chi(-1)=-1} \sum_p \log \left( 1 - \frac{\chi(p)}{p} \right)
\]
\[
= \sum_{\chi(-1)=-1} \sum_p \sum_{j=1}^{\infty} \frac{\chi(p^j)}{jp^j} = \sum_1 + \sum_2,
\]
so that Lemmas 11 and 12 imply
\[
\log \left( \prod_{\chi(-1)=-1} L(1, \chi) \right) \geq -\frac{4}{3} \log \log q - \log \log \log q + \log A_1 + O\left( \frac{1}{\log \log q} \right),
\]
\[
\log \left( \prod_{\chi(-1)=-1} L(1, \chi) \right) \leq \frac{7}{6} \log \log q + \log \log \log q + \log A_2 + O\left( \frac{1}{\log \log q} \right),
\]
with
\[
A_1 = \exp \left( -\frac{1}{6} \left( 5 + \log \frac{8}{\log^2 2} + \gamma \right) \right),
\]
\[
A_2 = \exp \left( \frac{1}{12} \left( 8 + \log \frac{2}{\log^2 2} + \gamma \right) \right).
\]
A simple calculation shows that
\begin{equation}
A_1 > e^{-1.4}, \quad A_2 < e^{0.84},
\end{equation}
and Theorem 1 follows at once.

Proof of Theorem 2. Inspecting the proof of Lemma 8 shows that if the exceptional zero does not exist, then the assertion of Lemma 8 holds for every sufficiently large prime \( q \), and therefore so does the assertion of Lemma 11. If we now invoke Lemma 12, the result follows at once.

Proof of Theorem 3. We separate \( \log(\prod_{\chi(-1)=-1} L(1, \chi)) \) into two parts as in (31). Because we are assuming GRH, we have
\begin{equation}
\theta(x; q, l) = \frac{\psi(x)}{q-1} + O(x^{1/2} \log^2 x)
\end{equation}
for \( x \geq 2, q \geq 3, (q, l) = 1 \). This implies
\begin{equation}
\sum_{\chi(-1)=-1} \sum_{p > q^3} \frac{\chi(p)}{p} \ll \frac{\log q}{\sqrt{q}}.
\end{equation}
Choosing \( x = q^3 \) in Lemma 10 yields
\begin{equation}
\sum_{p \equiv l \pmod{q}} \frac{1}{p} \leq \frac{2}{q-1} \left( \log \log q + \frac{1}{2} + \log 2 + \frac{1}{\log q} \right)
\end{equation}
for \( l = \pm 1 \). Similar to Lemma 11, from (34), (35) we deduce that assuming GRH, for every sufficiently large prime \( q \),
\begin{equation}
\left| \sum_{1} \right| \leq \log \log q + \frac{1}{2} + \log 2 + O \left( \frac{1}{\log q} \right).
\end{equation}
By using (36) and Lemma 12, we have
\begin{align*}
\log \left( \prod_{\chi(-1)=-1} L(1, \chi) \right) &\geq -\frac{4}{3} \log \log q + \log A_3 + O \left( \frac{1}{\log q} \right), \\
\log \left( \prod_{\chi(-1)=-1} L(1, \chi) \right) &\leq \frac{7}{6} \log \log q + \log A_4 + O \left( \frac{1}{\log q} \right),
\end{align*}
with
\begin{align*}
A_3 &= \exp \left( -\frac{1}{6} (5 + 9 \log 2 - 2 \log \log 2 + \gamma) \right), \\
A_4 &= \exp \left( \frac{1}{12} (8 + 13 \log 2 - 2 \log \log 2 + \gamma) \right).
\end{align*}
A simple calculation shows that
\[ A_3 > e^{-2.1}, \quad A_4 < e^{1.53}, \]
which completes the proof of Theorem 3.

**Proof of Theorem 4.** The procedure is similar to the proof of Theorem 3, but now we have to use the Siegel–Walfisz theorem instead of (33). Hence, for every \( \varepsilon > 0 \), write \( x_3 = \exp(q^\varepsilon) \). We have the following formula corresponding to (34):

\[
\sum_{\chi(-1) = -1} \sum_{p > x_3} \frac{\chi(p)}{p} \ll q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}),
\]

where \( c_5 \) is a constant dependent on \( \varepsilon \). Choosing \( x = x_3 \) in Lemma 10 yields

\[
\sum_{p \leq x_3} \frac{1}{p} \leq \frac{2}{q-1} \left( \varepsilon \log q + \frac{1}{2} + \frac{1}{q} \right)
\]

for \( l = \pm 1 \). Similar to Lemma 11, from (37), (38) we infer that for every sufficiently large prime \( q \) and all \( \varepsilon > 0 \),

\[
\left| \sum_1 \right| \leq \varepsilon \log q + \frac{1}{2} + O\left( q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q} \right).
\]

Combining this with Lemma 12 and (31), we conclude that for all \( \varepsilon > 0 \) and every sufficiently large prime \( q \),

\[
\log \left( \prod_{\chi(-1) = -1} L(1, \chi) \right) \geq -\varepsilon \log q - \frac{1}{3} \log \log q + \log A_1
\]
\[ + O\left( q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q} \right), \]

\[
\log \left( \prod_{\chi(-1) = -1} L(1, \chi) \right) \leq \varepsilon \log q + \frac{1}{6} \log \log q + \log A_2
\]
\[ + O\left( q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q} \right). \]

Together with (32) this completes the proof of Theorem 4.

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Received on 24.5.2007

(5448)