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On the Kummer conjecture

by

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1. Introduction. Let q be an odd prime, and let h_q and h_q^+ be the class numbers of $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q+\zeta_q^{-1})$ respectively ($\zeta_q=e^{2\pi i/q}$). It is well known that $h_q^+ \mid h_q$. The famous *Kummer conjecture* states that

$$h_q^- = h_q/h_q^+ \sim 2q\left(\frac{q}{4\pi^2}\right)^{(q-1)/4} \quad (\text{as } q\to\infty)$$

is equivalent to

$$\prod_{\chi(-1)=-1} L(1,\chi)\sim 1 \quad (\text{as } q\to\infty),$$

where χ runs through Dirichlet characters modulo q. This conjecture has not been proved yet, but there are several works on the upper bound of h_q^-; for example, by using elementary methods, Feng Keqin [1] proved that

$$h_q^- < 2q\left(\frac{q-1}{31.997158\ldots}\right)^{(q-1)/4},$$

and in [2] it is mentioned that the following results have been proved:

$$\frac{2}{3} < \prod_{\chi(-1)=-1} L(1,\chi) < \frac{3}{2} \quad (\text{for } 5\le q\le 523),$$

$$\frac{e^{-12.93}}{Lq^{1/2}(\log q)^4} < \prod_{\chi(-1)=-1} L(1,\chi) < e^{15.49}L(\log q)^5 \quad (\text{for any } q)$$

with $L=e^{4.66/\log q}$. In [4], M. Ram Murty and Yiannis N. Petridis proved the following *weak Kummer conjecture*: There exists a positive constant c

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such that for almost all primes q,

$$c^{-1} \le \prod_{\chi(-1)=-1} L(1,\chi) \le c.$$

In this paper, we shall prove the following

THEOREM 1. *For every sufficiently large prime q with $q \equiv 1 \pmod 4$,*

$$\prod_{\chi(-1)=-1} L(1,\chi) \ge e^{-1.4}(\log q)^{-4/3}(\log\log q)^{-1}\left(1+O\left(\frac{1}{\log\log q}\right)\right), \tag{1}$$

$$\prod_{\chi(-1)=-1} L(1,\chi) \le e^{0.84}(\log q)^{7/6}(\log\log q)\left(1+O\left(\frac{1}{\log\log q}\right)\right), \tag{2}$$

where χ runs through Dirichlet characters modulo q.

Using the same method, we can prove the following conclusions:

THEOREM 2. *Let q be any sufficiently large prime. Assume that there is no exceptional zero for $L(s,\chi)$, where χ is any Dirichlet character modulo q. Then we have the same estimates as in Theorem* 1.

THEOREM 3. *Assuming the Generalized Riemann Hypothesis* (*GRH*), *for every sufficiently large prime q we have*

$$\prod_{\chi(-1)=-1} L(1,\chi) \ge e^{-2.1}(\log q)^{-4/3}\left(1+O\left(\frac{1}{\log q}\right)\right), \tag{3}$$

$$\prod_{\chi(-1)=-1} L(1,\chi) \le e^{1.53}(\log q)^{7/6}\left(1+O\left(\frac{1}{\log q}\right)\right). \tag{4}$$

THEOREM 4. *For any fixed $\varepsilon > 0$, there is a positive number Q, which depends only on ε, such that if q is a prime greater than Q, we have*

$$e^{-1.4}q^{-\varepsilon}(\log q)^{-1/3} \le \prod_{\chi(-1)=-1} L(1,\chi) \le e^{0.84}q^{\varepsilon}(\log q)^{1/6}. \tag{5}$$

The following symbols will be used in the proof of these theorems. For $q \ge 3$, $(l,q)=1$, $1 \le l < q$, we write

$$\pi(x;q,l) = \sum_{\substack{p\le x\\ p\equiv l \,(\mathrm{mod}\, q)}} 1, \qquad \theta(x;q,l) = \sum_{\substack{p\le x\\ p\equiv l \,(\mathrm{mod}\, q)}} \log p,$$

$$\psi(x;q,l) = \sum_{\substack{n\le x\\ n\equiv l \,(\mathrm{mod}\, q)}} \Lambda(n), \qquad \psi(x;\chi) = \sum_{n\le x} \Lambda(n)\chi(n),$$

$$\psi(x) = \sum_{n\le x} \Lambda(n), \qquad x_1 = q^{\log\log q}, \qquad x_2 = e^q,$$

where $\Lambda(n)$ is the von Mangoldt function.

2. Some lemmas

LEMMA 1. *For every prime* $q \geq 3$,

$$\sum_{\chi(-1)=-1} \sum_{p>x_2} \frac{\chi(p)}{p} \ll q^2 e^{-c_1\sqrt{q}}, \tag{6}$$

where χ *runs through Dirichlet characters modulo* q, *and* $c_1 > 0$ *is a constant independent of* q.

Proof. For an arbitrary number $y > x_2$, by using the identity

$$\sum_{\chi(-1)=1} \chi(n) = \begin{cases} \frac{1}{2}(q-1) & \text{if } n \equiv \pm 1 \pmod{q}, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

we have

$$\begin{aligned} \sum_{\chi(-1)=-1} \sum_{x_2<p\leq y} \frac{\chi(p)}{p} &= \sum_{x_2<p\leq y} \frac{1}{p} \sum_{\chi(-1)=-1} \chi(p) \\ &= \frac{q-1}{2}\Bigg(\sum_{\substack{x_2<p\leq y \\ p\equiv 1 \pmod{q}}} \frac{1}{p} - \sum_{\substack{x_2<p\leq y \\ p\equiv -1 \pmod{q}}} \frac{1}{p}\Bigg) \\ &= \frac{q-1}{2} \int_{x_2}^{y} \frac{1}{u}\, d\{\pi(u;q,1) - \pi(u;q,-1)\}. \end{aligned}$$

The Siegel–Walfisz theorem yields

$$\begin{aligned} \sum_{\chi(-1)=-1} \sum_{x_2<p\leq y} \frac{\chi(p)}{p} &\ll qe^{-c_1\sqrt{\log x_2}} + q\int_{x_2}^{y} e^{-c_1\sqrt{\log u}} \frac{du}{u} \\ &\ll qe^{-c_1\sqrt{\log x_2}} + qe^{-c_1\sqrt{\log x_2}} \log x_2 \ll q^2 e^{-c_1\sqrt{q}}. \end{aligned}$$

Since y is arbitrary, we can easily get (6) by letting tend y to infinity in the last formula.

LEMMA 2 (see [5, §17.1]). *Assume* $q \geq 3$ *is any integer and* $s = \sigma + it$. *Then there is at most one character* χ *modulo* q *such that the function* $L(s,\chi)$ *has a zero in the region*

$$\sigma \geq 1 - \frac{c_2}{\log(q(|t|+2))}, \tag{8}$$

where c_2 *is a positive constant. If such an exceptional function exists, the corresponding character* $\widetilde{\chi}$ *must be a nonprincipal real character modulo* q, *and* $L(s,\widetilde{\chi})$ *has only one zero* $\widetilde{\beta}$ (*this zero must be a real zero*) *in the above region.*

LEMMA 3 (see [5, Theorem 33.3.1]). *If the exceptional zero* $\widetilde{\beta}$ *in Lemma* 2 *exists, then there are positive constants* c_3, c_4 *such that* $\widetilde{\beta}$ *is the only zero* *of* $\prod_{\chi \bmod q} L(s,\chi)$ *in the region*

$$\begin{cases} \sigma \geq 1 - \dfrac{c_3}{\log(q(|t|+2))} \log \dfrac{c_4 e}{\widetilde{\delta}\log(q(|t|+2))}, \\ \widetilde{\delta}\log(q(|t|+2)) \leq c_4, \end{cases} \tag{9}$$

where $\widetilde{\delta} = 1 - \widetilde{\beta}$.

LEMMA 4 (see [5, Theorem 33.2.8]). *Assume* $q \geq 3$ *is any integer and* χ *is a Dirichlet character modulo* q. *Denote by* $N(\alpha, T, \chi)$ *the number of zeros of* $L(s,\chi)$ *in the region* $\alpha \leq \sigma \leq 1$, $|t| \leq T$, *and write* $N(\alpha, T, q) = \sum_{\chi \bmod q} N(\alpha, T, \chi)$. *Then for any* $T \geq 2$ *and* $1/2 \leq \alpha \leq 1$, *we have*

$$N(\alpha, T, q) \ll (qT)^{3(1-\alpha)}. \tag{10}$$

LEMMA 5 (see [5, Theorem 18.1.5]). *Assume* $x \geq 2$, $T \geq 2$, $q \geq 3$. *Then for every nonprincipal character* χ *modulo* q,

$$\psi(x,\chi) = -\widetilde{E}\,\frac{x^{\widetilde{\beta}}}{\widetilde{\beta}} - \sideset{}{'}\sum_{|\gamma| \leq T} \frac{x^{\varrho}}{\varrho} + O\biggl(\frac{x\log^2(xqT)}{T} + \log^2(xq) + \widetilde{E}x^{1/4}\biggr), \tag{11}$$

where

$$\widetilde{E} = \begin{cases} 1, & \chi = \widetilde{\chi}, \\ 0, & \chi \neq \widetilde{\chi}, \end{cases}$$

$\widetilde{\chi}$ *is the exceptional character that possibly exists, and* $\sum'$ *is the sum over all nontrivial zeros* $\varrho = \beta + i\gamma$ *of* $L(s,\chi)$ *except the exceptional zeros* $\widetilde{\beta}$ *and* $1 - \widetilde{\beta}$.

LEMMA 6. *Assume* $x \geq 2$, $T \geq 2$, $q \geq 3$ *is a prime number, and* l *is a positive integer satisfying* $1 \leq l < q$. *Then*

$$\theta(x;q,l) = \frac{\psi(x)}{q-1} - \frac{\widetilde{E}(q)\widetilde{\chi}(l)}{q-1} \cdot \frac{x^{\widetilde{\beta}}}{\widetilde{\beta}} - \frac{1}{q-1} \sum_{\chi \neq \chi^0} \overline{\chi}(l) \sideset{}{'}\sum_{|\gamma| \leq T} \frac{x^{\varrho}}{\varrho}$$
$$+ O\biggl(\frac{x\log^2(xqT)}{T} + \log^2(xq) + x^{1/2}\biggr)$$

where

$$\widetilde{E}(q) = \begin{cases} 1 & \text{if the exceptional character } \widetilde{\chi} \bmod q \text{ exists}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We have

$$\psi(x;q,l) = \sum_{n \leq x} \frac{\Lambda(n)}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(l)\chi(n) = \frac{1}{q-1} \sum_{\chi \bmod q} \overline{\chi}(l)\psi(x,\chi).$$

Since q is a prime number, we easily get

$$\psi(x,\chi^0)=\sum_{\substack{n\le x\\(n,q)=1}}\Lambda(n)=\psi(x)-\sum_{n\le x,\,q\mid n}\Lambda(n)$$
$$=\psi(x)-\sum_{q^m\le x}\log q=\psi(x)+O(\log x).$$

Combining this with Lemma 5, we have

$$\psi(x;q,l)=\frac{\psi(x)}{q-1}+\frac{1}{q-1}\sum_{\chi\ne\chi^0}\overline{\chi}(l)\psi(x,\chi)+O\Big(\frac{1}{q}\log x\Big)$$
$$=\frac{\psi(x)}{q-1}-\frac{\widetilde{E}(q)\widetilde{\chi}(l)}{q-1}\cdot\frac{x^{\widetilde{\beta}}}{\widetilde{\beta}}-\frac{1}{q-1}\sum_{\chi\ne\chi^0}\overline{\chi}(l)\sideset{}{'}\sum_{|\gamma|\le T}\frac{x^{\varrho}}{\varrho}$$
$$+O\Big(\frac{x\log^2(xqT)}{T}+\log^2(xq)+\widetilde{E}(q)\,\frac{x^{1/4}}{q}\Big),$$

which proves the lemma by using $\theta(x;q,l)=\psi(x;q,l)+O(x^{1/2})$.

Lemma 7. *Let $A=\min(c_2,c_3,c_4)$, where c_2,c_3,c_4 are defined in Lemmas 2 and 3. Then for every sufficiently large integer q, we have*

$$\sum_{\chi\bmod q}\sideset{}{'}\sum_{|\gamma|\le T}u^{\beta-1}\ll\Big(\frac{u}{(qT)^3}\Big)^{-A/\log(qT)}+u^{-1/2}qT\log(qT)\tag{12}$$

for $u\ge x_1=q^{\log\log q}$ and $T=q^4$; $\sum'$ and β are defined in Lemma 5.

Proof. From Lemmas 2 and 3, we have:

(i) If the exceptional zero $\widetilde{\beta}$ exists and satisfies $\widetilde{\delta}\log(qT)\le A$, then $\prod_{\chi\bmod q}L(s,\chi)$ does not vanish in the region

$$\sigma\ge 1-\frac{A}{\log(qT)}\log\frac{Ae}{\widetilde{\delta}\log(qT)},\qquad |t|\le T$$

except at $s=\widetilde{\beta}$.

(ii) If $\widetilde{\delta}\log(qT)>A$ or the exceptional zero does not exist, then $\prod_{\chi\bmod q}L(s,\chi)$ does not vanish in the region $\sigma\ge 1-A/\log(qT)$, $|t|\le T$.

Hence if we choose

$$\eta_0=A\log\frac{Ae}{\delta_0\log(qT)},$$

where

$$\delta_0=\begin{cases}\widetilde{\delta}, & \widetilde{\delta}\log(qT)\le A,\\ A/\log(qT), & \widetilde{\delta}\log(qT)>A\text{ or the exceptional zero does not exist,}\end{cases}$$

then $\eta_0 \geq A$, and $\prod_{\chi \bmod q} L(s,\chi) \neq 0$ in the region $\sigma \geq 1 - \eta_0/\log(qT)$, $|t| \leq T$ except at one point $s = \widetilde{\beta}$. Hence

$$\sum_{\chi \bmod q} \sideset{}{'}\sum_{|\gamma|\leq T} u^{\beta-1} \ll \sum_{\chi \bmod q} \sideset{}{'}\sum_{\substack{|\gamma|\leq T \\ \beta \geq 1/2}} u^{\beta-1} = -\int_{1/2}^{1-\eta_0/\log(qT)} u^{\alpha-1}\, d_\alpha N(\alpha,T,q).$$

Making use of $N(1/2,T,q) \ll qT\log(qT)$ and Lemma 4, we obtain

$$\begin{aligned}
\sum_{\chi \bmod q} \sideset{}{'}\sum_{|\gamma|\leq T} u^{\beta-1} &\ll \int_{1/2}^{1-\eta_0/\log(qT)} N(\alpha,T,q) u^{\alpha-1} \log u\, d\alpha + u^{-1/2} qT\log(qT) \\
&\ll (\log u) \int_{1/2}^{1-\eta_0/\log(qT)} \left(\frac{u}{(qT)^3}\right)^{\alpha-1} d\alpha + u^{-1/2} qT\log(qT) \\
&\ll \frac{\log u}{\log u - 3\log(qT)} \left(\frac{u}{(qT)^3}\right)^{-\eta_0/\log(qT)} + u^{-1/2} qT\log(qT),
\end{aligned}$$

and (12) follows at once from the choice of u, T and $\eta_0 \geq A$.

Lemma 8. *For every sufficiently large prime $q \equiv 1 \pmod 4$, we have*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} \ll (\log q)^{-A/5} (\log\log q)^{-1}, \tag{13}$$

where A is defined in Lemma 7.

Proof. Making use of (7) and Lemma 6, we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \sum_{x_1 < p \leq x_2} \frac{\chi(p)}{p} = \sum_{x_1 < p \leq x_2} \frac{1}{p} \sum_{\chi(-1)=-1} \chi(p) \\
&= \frac{q-1}{2} \Bigg(\sum_{\substack{x_1 < p \leq x_2 \\ p \equiv 1 \pmod q}} \frac{1}{p} - \sum_{\substack{x_1 < p \leq x_2 \\ p \equiv -1 \pmod q}} \frac{1}{p} \Bigg) \\
&= \frac{q-1}{2} \int_{x_1}^{x_2} \frac{1}{u\log u}\, d\{\theta(u;q,1) - \theta(u;q,-1)\} \\
&= \int_{x_1}^{x_2} \frac{1}{u\log u}\, d\bigg\{ \frac{\widetilde{E}(q)}{2} (\widetilde{\chi}(-1)-1) \frac{u^{\widetilde{\beta}}}{\widetilde{\beta}} - \frac{1}{2} \sum_{\chi \neq \chi^0} (1-\overline{\chi}(-1)) \sideset{}{'}\sum_{|\gamma|\leq T} \frac{u^\varrho}{\varrho} \bigg\} \\
&\quad + O\bigg(\frac{q\log^2(x_2 qT)}{T\log x_2} + \frac{q}{\sqrt{x_1}\log x_1} + \int_{x_1}^{x_2} \bigg(\frac{uq\log^2(uqT)}{T} + u^{1/2} q \bigg) \frac{\log u + 1}{u^2\log^2 u}\, du \bigg).
\end{aligned}$$

Notice that q is a prime and $\widetilde{\chi}$ is nonprincipal real character, so that

$$\widetilde{\chi}(-1) = \left(\frac{-1}{q}\right) = (-1)^{(q-1)/2} = 1,$$

where $\left(\frac{n}{q}\right)$ is the Legendre symbol modulo q. Hence

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \sum_{x_1<p\le x_2} \frac{\chi(p)}{p} = -\int_{x_1}^{x_2} \Big(\sum_{\chi(-1)=-1} \sideset{}{'}\sum_{|\gamma|\le T} u^{\varrho-1}\Big) \frac{du}{u\log u}$$
$$+ O\bigg(\frac{q\log^2(x_2qT)}{T\log x_2} + \frac{q}{\sqrt{x_1}\log x_1} + \int_{x_1}^{x_2} \bigg(\frac{uq\log^2(uqT)}{T} + u^{1/2}q\bigg)\frac{\log u+1}{u^2\log^2 u}\,du\bigg)$$
$$= -\int_{x_1}^{x_2} \Big(\sum_{\chi(-1)=-1} \sideset{}{'}\sum_{|\gamma|\le T} u^{\varrho-1}\Big) \frac{du}{u\log u} + O\bigg(\frac{q\log^2(qT)\log^2 x_2}{T} + \frac{q}{\sqrt{x_1}\log x_1}\bigg).$$

If we choose $T = q^4$ and make use of Lemma 7 as well as the definition of x_1, x_2, we obtain

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \sum_{x_1<p\le x_2} \frac{\chi(p)}{p} \ll \int_{x_1}^{x_2} \Big(\sum_{\chi \bmod q} \sideset{}{'}\sum_{|\gamma|\le T} u^{\varrho-1}\Big) \frac{du}{u\log u} + \frac{\log^2 q}{q}$$
$$\ll \int_{x_1}^{x_2} \bigg\{\bigg(\frac{u}{q^{15}}\bigg)^{-A/5\log q} + u^{-1/2}q^5\log q\bigg\}\frac{du}{u\log u} + \frac{\log^2 q}{q}$$
$$\ll \int_{x_1}^{x_2} u^{-1-A/5\log q}\frac{du}{\log u} + \frac{\log^2 q}{q}$$
$$\ll \frac{1}{\log x_1}\cdot\frac{5\log q}{A}\,x_1^{-A/5\log q} + \frac{\log^2 q}{q}$$
$$\ll (\log q)^{-A/5}(\log\log q)^{-1}.$$

This completes the proof of the lemma.

Lemma 9 (see [5, Theorem 28.6.1]). *Assume* $(q,l)=1$ *and* $1\le l<q<y\le x$. *Then*

$$\pi(x;q,l) - \pi(x-y;q,l) < \frac{2}{\phi(q)}\cdot\frac{y}{\log(y/q)}. \tag{14}$$

Lemma 10. *Let* $q\ge 3$ *be a prime number*, $1\le l<q$, l *is not a prime*, *and* $x\ge q^2$. *Then*

$$\sum_{\substack{p\le x \\ p\equiv l \,(\mathrm{mod}\, q)}} \frac{1}{p} \le \frac{2}{q-1}\bigg(\log\log\frac{x}{q} + \frac{1}{2} + \frac{1}{\log q}\bigg). \tag{15}$$

Proof. Using Abel's identity, we obtain

$$\sum_{\substack{2q<p\le x\\ p\equiv l\,(\mathrm{mod}\,q)}} \frac{1}{p} = \frac{1}{x}\,\pi(x;q,l) - \frac{1}{2q}\,\pi(2q;q,l) + \int_{2q}^{x} \frac{1}{u^2}\,\pi(u;q,l)\,du. \tag{16}$$

If we choose $x = y > q$ in Lemma 9, we have

$$\pi(x;q,l) < \frac{2}{\phi(q)}\cdot\frac{x}{\log(x/q)} \quad (x>q).$$

Combining this with (16), we find that

$$\sum_{\substack{2q<p\le x\\ p\equiv l\,(\mathrm{mod}\,q)}} \frac{1}{p} \le \frac{1}{x}\cdot\frac{2}{q-1}\cdot\frac{x}{\log(x/q)} + \int_{2q}^{x} \frac{1}{u^2}\cdot\frac{2}{q-1}\cdot\frac{u}{\log(u/q)}\,du$$
$$\le \frac{2}{q-1}\log\log\frac{x}{q} + \frac{2}{q-1}\cdot\frac{1}{\log q};$$

we have used $x \ge q^2$ in the last inequality. Since l is not a prime, we have

$$\sum_{\substack{p\le 2q\\ p\equiv l\,(\mathrm{mod}\,q)}} \frac{1}{p} \le \frac{1}{q},$$

and we easily deduce (15) from the discussion above.

LEMMA 11. *Assume q is a sufficiently large prime with $q \equiv 1 \pmod 4$, and define*

$$\sum\nolimits_1 = \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \sum_{p} \frac{\chi(p)}{p}.$$

Then

$$\Big|\sum\nolimits_1\Big| \le \log\log q + \log\log\log q + \frac{1}{2} + O\bigg(\frac{1}{\log\log q}\bigg). \tag{17}$$

Proof. From (7) we get

$$\sum_{\chi(-1)=-1}\sum_{p\le x_1}\frac{\chi(p)}{p} = \sum_{p\le x_1}\frac{1}{p}\sum_{\chi(-1)=-1}\chi(p)$$
$$= \frac{q-1}{2}\bigg(\sum_{\substack{p\le x_1\\ p\equiv 1\,(\mathrm{mod}\,q)}}\frac{1}{p} - \sum_{\substack{p\le x_1\\ p\equiv -1\,(\mathrm{mod}\,q)}}\frac{1}{p}\bigg).$$

Thus

$$-\frac{q-1}{2}\sum_{\substack{p\le x_1\\ p\equiv -1\,(\mathrm{mod}\,q)}}\frac{1}{p} \le \sum_{\chi(-1)=-1}\sum_{p\le x_1}\frac{\chi(p)}{p} \le \frac{q-1}{2}\sum_{\substack{p\le x_1\\ p\equiv 1\,(\mathrm{mod}\,q)}}\frac{1}{p}. \tag{18}$$

If we choose $x = x_1$, $l = 1$ and $x = x_1$, $l = q-1$ respectively in Lemma 10, and put the results into (18), we obtain

$$\Big|\sum_{\chi(-1)=-1}\sum_{p\le x_1}\frac{\chi(p)}{p}\Big| \le \log\log\frac{x_1}{q}+\frac{1}{2}+\frac{1}{\log q}$$
$$\le \log\log q+\log\log\log q+\frac{1}{2}+\frac{1}{\log q}.$$

Applying Lemmas 1 and 8 yields (17).

LEMMA 12. *Let q be a sufficiently large prime, and define*

$$\sum\nolimits_2 = \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\sum_{p}\sum_{j=2}^{\infty}\frac{\chi(p^j)}{jp^j}.$$

Then

$$\sum\nolimits_2 \ge -\frac{1}{3}\log\log q-\frac{1}{6}\Big(2+\log\frac{8}{\log^2 2}+\gamma\Big)+O\Big(\frac{1}{\log q}\Big), \tag{19}$$

$$\sum\nolimits_2 \le \frac{1}{6}\log\log q+\frac{1}{12}\Big(2+\log\frac{2}{\log^2 2}+\gamma\Big)+O\Big(\frac{1}{\log q}\Big), \tag{20}$$

where γ is the Euler constant.

Proof. We have

$$\sum\nolimits_2 = \sum_{p}\sum_{j=2}^{\infty}\frac{1}{jp^j}\sum_{\chi(-1)=-1}\chi(p^j) \tag{21}$$
$$=\frac{q-1}{2}\Big\{\sum_{p}\sum_{\substack{j=2\\ p^j\equiv 1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}-\sum_{p}\sum_{\substack{j=2\\ p^j\equiv -1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}\Big\}.$$

Since

$$\sum_{p}\sum_{\substack{j=2\\ p^j\equiv 1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}=\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv 1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}+O\Big(\sum_{p>q}\sum_{j=2}^{\infty}\frac{1}{jp^j}\Big)$$
$$=\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv 1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}+O\Big(\sum_{p>q}\frac{1}{p^2}\Big)$$
$$=\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv 1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}+O\Big(\frac{1}{q\log q}\Big),$$

and in the same way

$$\sum_{p}\sum_{\substack{j=2\\ p^j\equiv -1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}=\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv -1\ (\mathrm{mod}\ q)}}^{\infty}\frac{1}{jp^j}+O\Big(\frac{1}{q\log q}\Big),$$

we deduce from (21) that

$$\sum\nolimits_2 = \frac{q-1}{2}\Bigg(\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv 1 \,(\mathrm{mod}\, q)}}^{\infty}\frac{1}{jp^j}-\sum_{p<q}\sum_{\substack{j=2\\ p^j\equiv -1 \,(\mathrm{mod}\, q)}}^{\infty}\frac{1}{jp^j}\Bigg)+O\Big(\frac{1}{\log q}\Big).$$

Let g be a primitive root modulo q, $\delta(p)$ be the exponent order of p modulo q for $p<q$ (that is to say, $\delta(p)=\min\{d>0 : p^d\equiv 1 \pmod q\}$), and let $k(p)$ $(1\le k(p)\le q-1)$ denote the integer satisfying $p\equiv g^{k(p)} \pmod q$ for $p<q$. Then

$$\delta(p)=\frac{q-1}{(k(p),q-1)},$$

and thus

$$\begin{aligned}\sum\nolimits_2 &= \frac{q-1}{2}\Bigg(\sum_{p<q}\sum_{\substack{j=2\\ k(p)j\equiv 0 \,(\mathrm{mod}\, q-1)}}^{\infty}\frac{1}{jp^j}-\sum_{p<q}\sum_{\substack{j=2\\ k(p)j\equiv (q-1)/2 \,(\mathrm{mod}\, q-1)}}^{\infty}\frac{1}{jp^j}\Bigg)\\ &\quad+O\Big(\frac{1}{\log q}\Big)\\ &=\frac{q-1}{2}\Bigg(\sum_{p<q}\sum_{\substack{j=2\\ j\equiv 0 \,(\mathrm{mod}\, \delta(p))}}^{\infty}\frac{1}{jp^j}-\sum_{\substack{p<q\\ 2\mid\delta(p)}}\sum_{\substack{j=2\\ j\equiv \delta(p)/2 \,(\mathrm{mod}\, \delta(p))}}^{\infty}\frac{1}{jp^j}\Bigg)+O\Big(\frac{1}{\log q}\Big).\end{aligned}$$

Since $\delta(p)\ge 3$ when $p<q$ because q is a sufficiently large prime, we have

$$\begin{aligned}(22)\quad &\sum\nolimits_2\\ &=\frac{q-1}{2}\Bigg(\sum_{p<q}\sum_{j=1}^{\infty}\frac{1}{j\delta(p)p^{j\delta(p)}}-\sum_{\substack{p<q\\ 2\mid\delta(p)}}\sum_{j=1}^{\infty}\frac{1}{(j-1/2)\delta(p)p^{(j-1/2)\delta(p)}}\Bigg)\\ &\quad+O\Big(\frac{1}{\log q}\Big)\\ &=\frac{q-1}{2}\Bigg(-\sum_{p<q}\frac{1}{\delta(p)}\log\Big(1-\frac{1}{p^{\delta(p)}}\Big)-\sum_{\substack{p<q\\ 2\mid\delta(p)}}\frac{1}{\delta(p)}\log\frac{1+p^{-\delta(p)/2}}{1-p^{-\delta(p)/2}}\Bigg)\\ &\quad+O\Big(\frac{1}{\log q}\Big)\\ &=\frac{q-1}{2}\Bigg\{\sum_{\substack{p<q\\ 2\nmid\delta(p)}}\frac{1}{\delta(p)}\log\Big(1+\frac{1}{p^{\delta(p)}-1}\Big)-2\sum_{\substack{p<q\\ 2\mid\delta(p)}}\frac{1}{\delta(p)}\log\Big(1+\frac{1}{p^{\delta(p)/2}}\Big)\Bigg\}\\ &\quad+O\Big(\frac{1}{\log q}\Big)\end{aligned}$$

and this implies that

$$\sum\nolimits_2 \geq -(q-1) \sum_{\substack{p<q \\ 2|\delta(p)}} \frac{1}{\delta(p)} \log\left(1+\frac{1}{p^{\delta(p)/2}}\right) + O\left(\frac{1}{\log q}\right)$$
$$\geq -(q-1) \sum_{\substack{p<q \\ 2|\delta(p)}} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)/2}} + O\left(\frac{1}{\log q}\right).$$

Notice that $2 \mid \delta(p)$ implies $p^{\delta(p)/2} \equiv -1 \pmod q$, thus $p^{\delta(p)/2} \geq q-1$ and

$$\frac{1}{p^{\delta(p)/2}} \leq \frac{1}{p^{\delta(p)/2}+1} \cdot \frac{q}{q-1},$$

so that

$$\sum\nolimits_2 \geq -q \sum_{\substack{p<q \\ 2|\delta(p)}} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)/2}+1} + O\left(\frac{1}{\log q}\right).$$

On the other hand, we can also deduce from (22) that

$$\sum\nolimits_2 \leq \frac{q-1}{2} \sum_{\substack{p<q \\ 2\nmid\delta(p)}} \frac{1}{\delta(p)} \cdot \frac{1}{p^{\delta(p)}-1} + O\left(\frac{1}{\log q}\right).$$

Since $\delta(p) \geq 3$ for $p<q$, we have

$$\sum\nolimits_2 \geq -\frac{q}{3} \sum_{\substack{p<q \\ 2|\delta(p)}} \frac{1}{p^{\delta(p)/2}+1} + O\left(\frac{1}{\log q}\right), \tag{23}$$

$$\sum\nolimits_2 \leq \frac{q}{6} \sum_{\substack{p<q \\ 2\nmid\delta(p)}} \frac{1}{p^{\delta(p)}-1} + O\left(\frac{1}{\log q}\right). \tag{24}$$

We write $p^{\delta(p)} = l(p)q+1$. Because $2 \mid \delta(p)$ implies $p^{\delta(p)/2} \equiv -1 \pmod q$, we can write $p^{\delta(p)/2} = h(p)q-1$ if $2 \mid \delta(p)$. Then

$$\sum_{\substack{p<q \\ 2\nmid\delta(p)}} \frac{1}{p^{\delta(p)}-1} = \sum_{\substack{p<q \\ 2\nmid\delta(p)}} \frac{1}{l(p)q} \tag{25}$$
$$= \frac{1}{q}\left(\sum_{\substack{p<q \\ 2\nmid\delta(p),\, l(p)<q}} \frac{1}{l(p)} + \sum_{\substack{p<q \\ 2\nmid\delta(p),\, l(p)\geq q}} \frac{1}{l(p)}\right)$$
$$= \frac{1}{q} \sum_{\substack{p<q \\ 2\nmid\delta(p),\, l(p)<q}} \frac{1}{l(p)} + O\left(\frac{1}{q\log q}\right).$$

Using the same method, we have

$$\sum_{\substack{p<q\\ 2|\delta(p)}} \frac{1}{p^{\delta(p)/2}+1} = \frac{1}{q} \sum_{\substack{p<q\\ 2|\delta(p),\, h(p)<q}} \frac{1}{h(p)} + O\left(\frac{1}{q\log q}\right). \tag{26}$$

Since $l(p) < q \Rightarrow \delta(p) \le n_1 = [(2\log q)/\log 2]$ and $h(p) < q \Rightarrow \delta(p) \le n_2 = [(4\log q)/\log 2]$, we deduce from (25), (26) that

$$\sum_{\substack{p<q\\ 2\nmid\delta(p)}} \frac{1}{p^{\delta(p)}-1} \le \frac{1}{q} \sum_{\substack{p<q\\ 2\nmid\delta(p),\, \delta(p)\le n_1}} \frac{1}{l(p)} + O\left(\frac{1}{q\log q}\right), \tag{27}$$

$$\sum_{\substack{p<q\\ 2|\delta(p)}} \frac{1}{p^{\delta(p)/2}+1} = \frac{1}{q} \sum_{\substack{p<q\\ 2|\delta(p),\, \delta(p)\le n_2}} \frac{1}{h(p)} + O\left(\frac{1}{q\log q}\right). \tag{28}$$

It is well known that for every $d \mid q-1$, there are exactly $\phi(d)$ integers in the reduced residue class modulo q with d as their exponent order modulo q, so the number of terms of the sum on the right hand side of (27) is less than

$$\sum_{\substack{n\le n_1\\ n|q-1,\, 2\nmid n}} \phi(n) \le n_1 \sum_{\substack{n\le n_1\\ n|q-1,\, 2\nmid n}} 1 \le \frac{1}{2}\, n_1^2 \le \frac{2}{\log^2 2}\log^2 q.$$

It is obvious that $l(p)$ is different for each p, and

$$l(p) \begin{cases} \text{is even} & \text{if } p>2,\\ \ge 1 & \text{if } p=2, \end{cases}$$

so we infer from (27) that

$$\begin{aligned} \sum_{\substack{p<q\\ 2\nmid\delta(p)}} \frac{1}{p^{\delta(p)}-1} &\le \frac{1}{q}\Bigg(1 + \sum_{n\le \frac{2}{\log^2 2}\log^2 q} \frac{1}{2n}\Bigg) + O\left(\frac{1}{q\log q}\right)\\ &= \frac{1}{q}\left\{1 + \frac{1}{2}\left(\log\left(\frac{2}{\log^2 2}\log^2 q\right) + \gamma + O\left(\frac{1}{\log^2 q}\right)\right)\right\}\\ &\quad + O\left(\frac{1}{q\log q}\right)\\ &= \frac{1}{q}\left(\log\log q + 1 + \frac{1}{2}\log\frac{2}{\log^2 2} + \frac{\gamma}{2}\right) + O\left(\frac{1}{q\log q}\right). \end{aligned} \tag{29}$$

Also, the number of terms of the sum on the right hand side of (28) is less than

$$\sum_{\substack{n\le n_2\\ n|q-1,\, 2|n}} \phi(n) \le \frac{1}{2}\, n_2^2 \le \frac{8}{\log^2 2}\log^2 q.$$

Because $h(p)$ is different for each p, and

$$h(p) \begin{cases} \text{is even} & \text{if } p > 2, \\ \geq 1 & \text{if } p = 2, \end{cases}$$

we deduce from (28) that

$$\sum_{\substack{p<q \\ 2|\delta(p)}} \frac{1}{p^{\delta(p)/2}+1} \leq \frac{1}{q}\Big(1 + \sum_{n \leq \frac{8}{\log^2 2}\log^2 q} \frac{1}{2n}\Big) + O\Big(\frac{1}{q\log q}\Big) \tag{30}$$

$$= \frac{1}{q}\Big(\log\log q + 1 + \frac{1}{2}\log\frac{8}{\log^2 2} + \frac{\gamma}{2}\Big) + O\Big(\frac{1}{q\log q}\Big).$$

The lemma now follows at once from (23), (24), (29), (30).

3. Proof of the theorems

Proof of Theorem 1. It is easy to verify that $\prod_{\chi(-1)=-1} L(1,\chi)$ is a positive real number. Making use of the identity

$$L(1,\chi) = \prod_p \Big(1 - \frac{\chi(p)}{p}\Big)^{-1} \qquad (\chi \neq \chi^0),$$

we have

$$\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) = \log\Big(\prod_{\chi(-1)=-1}\prod_p \Big(1 - \frac{\chi(p)}{p}\Big)^{-1}\Big) \tag{31}$$

$$= -\sum_{\chi(-1)=-1}\sum_p \log\Big(1 - \frac{\chi(p)}{p}\Big)$$

$$= \sum_{\chi(-1)=-1}\sum_p\sum_{j=1}^{\infty} \frac{\chi(p^j)}{jp^j} = \sum\nolimits_1 + \sum\nolimits_2,$$

so that Lemmas 11 and 12 imply

$$\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \geq -\frac{4}{3}\log\log q - \log\log\log q + \log A_1 + O\Big(\frac{1}{\log\log q}\Big),$$

$$\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \leq \frac{7}{6}\log\log q + \log\log\log q + \log A_2 + O\Big(\frac{1}{\log\log q}\Big),$$

with

$$A_1 = \exp\Big(-\frac{1}{6}\Big(5 + \log\frac{8}{\log^2 2} + \gamma\Big)\Big),$$

$$A_2 = \exp\Big(\frac{1}{12}\Big(8 + \log\frac{2}{\log^2 2} + \gamma\Big)\Big).$$

A simple calculation shows that

$$A_1 > e^{-1.4}, \quad A_2 < e^{0.84}, \tag{32}$$

and Theorem 1 follows at once.

Proof of Theorem 2. Inspecting the proof of Lemma 8 shows that if the exceptional zero does not exist, then the assertion of Lemma 8 holds for every sufficiently large prime q, and therefore so does the assertion of Lemma 11. If we now invoke Lemma 12, the result follows at once.

Proof of Theorem 3. We separate $\log(\prod_{\chi(-1)=-1} L(1,\chi))$ into two parts as in (31). Because we are assuming GRH, we have

$$\theta(x;q,l) = \frac{\psi(x)}{q-1} + O(x^{1/2}\log^2 x) \tag{33}$$

for $x \geq 2$, $q \geq 3$, $(q,l)=1$. This implies

$$\sum_{\chi(-1)=-1} \sum_{p>q^3} \frac{\chi(p)}{p} \ll \frac{\log q}{\sqrt{q}}. \tag{34}$$

Choosing $x = q^3$ in Lemma 10 yields

$$\sum_{\substack{p \leq q^3 \\ p \equiv l \,(\mathrm{mod}\, q)}} \frac{1}{p} \leq \frac{2}{q-1}\left(\log\log q + \frac{1}{2} + \log 2 + \frac{1}{\log q}\right) \tag{35}$$

for $l = \pm 1$. Similar to Lemma 11, from (34), (35) we deduce that assuming GRH, for every sufficiently large prime q,

$$\left|\sum\nolimits_1\right| \leq \log\log q + \frac{1}{2} + \log 2 + O\left(\frac{1}{\log q}\right). \tag{36}$$

By using (36) and Lemma 12, we have

$$\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \geq -\frac{4}{3}\log\log q + \log A_3 + O\left(\frac{1}{\log q}\right),$$

$$\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \leq \frac{7}{6}\log\log q + \log A_4 + O\left(\frac{1}{\log q}\right),$$

with

$$A_3 = \exp\left(-\frac{1}{6}\,(5 + 9\log 2 - 2\log\log 2 + \gamma)\right),$$

$$A_4 = \exp\left(\frac{1}{12}\,(8 + 13\log 2 - 2\log\log 2 + \gamma)\right).$$

A simple calculation shows that

$$A_3 > e^{-2.1}, \quad A_4 < e^{1.53},$$

which completes the proof of Theorem 3.

Proof of Theorem 4. The procedure is similar to the proof of Theorem 3, but now we have to use the Siegel–Walfisz theorem instead of (33). Hence, for every $\varepsilon > 0$, write $x_3 = \exp(q^{\varepsilon})$. We have the following formula corresponding to (34):

$$\sum_{\chi(-1)=-1} \sum_{p>x_3} \frac{\chi(p)}{p} \ll q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}), \tag{37}$$

where c_5 is a constant dependent on ε. Choosing $x = x_3$ in Lemma 10 yields

$$\sum_{\substack{p\le x_3 \\ p\equiv l \,(\mathrm{mod}\, q)}} \frac{1}{p} \le \frac{2}{q-1}\left(\varepsilon \log q + \frac{1}{2} + \frac{1}{q}\right) \tag{38}$$

for $l = \pm 1$. Similar to Lemma 11, from (37), (38) we infer that for every sufficiently large prime q and all $\varepsilon > 0$,

$$\Big|\sum\nolimits_1\Big| \le \varepsilon \log q + \frac{1}{2} + O\left(q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q}\right). \tag{39}$$

Combining this with Lemma 12 and (31), we conclude that for all $\varepsilon > 0$ and every sufficiently large prime q,

$$\begin{aligned}\log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \ge & -\varepsilon \log q - \frac{1}{3}\log\log q + \log A_1 \\ & + O\left(q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q}\right),\\ \log\Big(\prod_{\chi(-1)=-1} L(1,\chi)\Big) \le & \ \varepsilon \log q + \frac{1}{6}\log\log q + \log A_2 \\ & + O\left(q^{1+\varepsilon} \exp(-c_5 q^{\varepsilon/2}) + \frac{1}{\log q}\right).\end{aligned}$$

Together with (32) this completes the proof of Theorem 4.

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