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Some problems about the ternary quadratic form $m_1^2+m_2^2+m_3^2$

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1. Introduction. The ternary quadratic form

$$\mathrm{g}(m_1,m_2,m_3):=m_1^2+m_2^2+m_3^2,$$

important in number theory, attracts the interest of many authors and has been extensively studied. For example, the well-known sphere problem is to evaluate the number of lattice points in the 3-dimensional ball $u_1^2+u_2^2+u_3^2 \le x$. The asymptotic formula

$$\sum_{\substack{m_1^2+m_2^2+m_3^2\le x\\ m_j\in\mathbb{Z}}} 1=\frac{4}{3}\pi x^{3/2}+O(x^{2/3})$$

was proved by Vinogradov [11] and by Chen [3] independently. The exponent 2/3 was improved to 29/44 in Chamizo and Iwaniec [2], and to 21/32 in Heath-Brown [5].

Recently several authors studied some problems connected with this ternary quadratic form by different methods.

C. Calderón and M. J. de Velasco [1] studied the divisors of the quadratic form $m_1^2+m_2^2+m_3^2$ and proved the asymptotic formula

$$S(x):=\sum_{1\le m_1,m_2,m_3\le x} d(m_1^2+m_2^2+m_3^2)=\frac{8\zeta(3)}{5\zeta(4)}x^3\log x+O(x^3). \tag{1.1}$$

To prove (1.1), they write

$$S(x)=2S_1-S_2, \tag{1.2}$$

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where

$$S_1 = \sum_{d\le\sqrt{3}x} \sum_{\substack{1\le m_1,m_2,m_3\le x\\ m_1^2+m_2^2+m_3^2\equiv 0\ (\mathrm{mod}\ d)}} 1, \qquad S_2 = \sum_{d\le\sqrt{3}x} \sum_{\substack{1\le m_1,m_2,m_3\le x\\ m_1^2+m_2^2+m_3^2\equiv 0\ (\mathrm{mod}\ d)\\ m_1^2+m_2^2+m_3^2\le dx\sqrt{3}}} 1,$$

and use a mean value result for $\rho(n)$, the number of solutions of the congruence

$$m_1^2+m_2^2+m_3^2 \equiv 0 \ (\mathrm{mod}\ n), \qquad 1\le m_1,m_2,m_3\le n.$$

Friedlander and Iwaniec [4] studied the number of prime vectors among integer lattice points in the 3-dimensional ball. They proved that the number $\pi_3(x)$ of integer points $(m_1,m_2,m_3)\in\mathbb{Z}^3$ with

$$m_1^2+m_2^2+m_3^2 = p \le x \tag{1.3}$$

satisfies

$$\pi_3(x) \sim \frac{4\pi}{3}\,\frac{x^{3/2}}{\log x}, \tag{1.4}$$

which can be viewed as a generalization of the prime number theorem.

The asymptotic formula (1.4) is proved by using Gauss's formula for the function $r_3(p)$ and the properties of $L(1,\chi_p)$, where $r_3(p)$ denotes the number of ways p can be written as a sum of three squares and $L(s,\chi_p)$ is the Dirichlet L-function with the Kronecker symbol $\chi_p(n) = \left(\frac{-4p}{n}\right)$.

In this paper we shall use the classical circle method to study the above problems. Our main results are as follows.

Theorem 1. *We have the asymptotic formula*

$$S(x) = 2C_1I_1x^3\log x + (C_1I_2+C_2I_1)x^3 + O(x^{8/3+\varepsilon}), \tag{1.5}$$

where

$$C_1 := \sum_{q=1}^{\infty} q^{-4} \sum_{\substack{0\le a<q\\ (a,q)=1}} G^3(a,0,q), \qquad G(a,0,q) = \sum_{r=1}^{q} e(ar^2/q),$$

$$C_2 := \sum_{q=1}^{\infty} \frac{-2\log q + 2\gamma}{q^4} \sum_{\substack{0\le a<q\\ (a,q)=1}} G^3(a,0,q),$$

$$I_1 := \int_{-\infty}^{\infty} \mathcal{H}_1(z)\,dz, \qquad I_2 := \int_{-\infty}^{\infty} \mathcal{H}_2(z)\,dz,$$

$$\mathcal{H}_1(z) := \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^3 e(-uz)\,du,$$

and

$$\mathcal{H}_2(z) := \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^3 e(-uz)\log u\,du.$$

THEOREM 2. *Define*

$$\pi_\Lambda(x) := \sum_{m_1^2+m_2^2+m_3^2\le x} \Lambda(m_1^2+m_2^2+m_3^2).$$

Then for any fixed constant $A>0$, we have

$$\pi_\Lambda(x) = 8C_3I_3x^{3/2} + O(x^{3/2}\log^{-A}x), \tag{1.6}$$

where

$$C_3 := \sum_{q=1}^{\infty} \frac{1}{q^3\varphi(q)} \sum_{\substack{0\le a<q\\(a,q)=1}} G^3(a,0,q)C_q(-a),$$

$$I_3 := \int_{-\infty}^{\infty} \mathcal{H}_3(z)\,dz, \qquad \mathcal{H}_3(z) := \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^1 e(-uz)\,du.$$

COROLLARY. *For any positive constant $A>0$, we have*

$$\pi_3(x) = 12C_3I_3\int_2^x \frac{t^{1/2}}{\log t}\,dt + O(x^{3/2}\log^{-A}x). \tag{1.7}$$

NOTATION. Throughout this paper ε denotes a small positive constant. $\mathbb{R},\mathbb{Z},\mathbb{N},\mathbb{P}$ denote the sets of all real numbers, integers, natural numbers, and primes, respectively; $d(n)$ denotes the Dirichlet divisor function, $\mu(n)$ the Möbius function, and $\Lambda(n)$ the von Mangoldt function. $C_q(r)$ denotes the Ramanujan sum. For any real number t, $[t]$ denotes its integer part, $\{t\}$ its fractional part, $\psi_1(t)=\{t\}-1/2$, $\|t\|=\min(\{t\},1-\{t\})$. The functions $\psi_j(u)$ $(j\ge 2)$ are defined by

$$\begin{cases} \psi_{j+1}(u)-\psi_{j+1}(0) = \int_0^u \psi_j(t)\,dt, & j\ge 1,\\ \int_0^1 \psi_{j+1}(u)\,du = 0, & j\ge 1.\end{cases} \tag{1.8}$$

Finally, $G(a,b,q)$ denotes the quadratic Gauss sum

$$G(a,b,q) = \sum_{r=1}^{q} e\Big(\frac{ar^2+br}{q}\Big).$$

2. The circle method.

2. The circle method. Throughout this paper, x is a large positive integer. For any $\alpha\in\mathbb{R}$ and $y>1$, define

$$S_1(\alpha;y) := \sum_{1\le m\le y} e(m^2\alpha), \qquad S_2(\alpha;y) := \sum_{1\le n\le y} d(n)e(n\alpha). \tag{2.1}$$

By the definition of $S(x)$ and the well-known identity

$$\int_0^1 e(u\alpha)\,d\alpha = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \in \mathbb{Z}, u \neq 0, \end{cases} \tag{2.2}$$

we have

$$S(x) = \int_0^1 S_1^3(\alpha; x) S_2(-\alpha; 3x^2)\,d\alpha. \tag{2.3}$$

Suppose Q and τ are two large real numbers to be determined later, which satisfy

$$\log x < Q < x, \qquad 2Q^2 < \tau, \qquad \tau > x^{1+\varepsilon}, \qquad Q\tau \le x^2. \tag{2.4}$$

For any $0 \le a < q \le Q$ with $(q, a) = 1$, define

$$I(a,q) := [a/q - 1/q\tau, a/q + 1/q\tau],$$
$$E_1 := \bigcup_{1\le q\le Q} \bigcup_{\substack{0\le a<q \\ (a,q)=1}} I(a,q), \qquad E_2 := [-1/\tau, 1 - 1/\tau] \setminus E_1.$$

We call E_1 the *major arc* and E_2 the *minor arc*.

We then have

$$S(x) = S_1(x) + S_2(x), \tag{2.5}$$

where

$$S_1(x) = \int_{E_1} S_1^3(\alpha; x) S_2(-\alpha; 3x^2)\,d\alpha, \qquad S_2(x) = \int_{E_2} S_1^3(\alpha; x) S_2(-\alpha; 3x^2)\,d\alpha.$$

The problem is now reduced to evaluating $S_1(x)$ and giving an upper bound of $S_2(x)$.

3. Some classical lemmas. In this subsection we quote some classical results which are needed for our proof. Lemmas 3.1 and 3.7 are well-known. Lemmas 3.2, 3.3, 3.5, 3.6, 3.9 can be found in [9], Lemma 3.4 in [8], and Lemma 3.8 in [7].

LEMMA 3.1.

$$\sum_{h=1}^{q} e\left(\frac{hr}{q}\right) = \begin{cases} q & \textit{if } q \mid r, \\ 0 & \textit{otherwise.} \end{cases}$$

LEMMA 3.2.

$$\sum_{1\le n\le y} e(n\alpha) \ll \min(y, 1/\|\alpha\|).$$

LEMMA 3.3. *Suppose $\alpha = a/q + z$ with $(a,q)=1$, $q \ge 3$ and $|z| \le 1/q^2$. Then*

$$\sum_{n\le N} \min\left(y, \frac{1}{\|\alpha n\|}\right) \ll (y + q\log q)(1+N/q).$$

LEMMA 3.4. *Suppose $q \in \mathbb{N}$, $a,b \in \mathbb{Z}$, $q \ge 3$, $(a,q)=1$. Then the Gauss sum satisfies*

$$G(a,b,q) := \sum_{h=1}^{q} e\left(\frac{ah^2+bh}{q}\right) \ll \sqrt{q}.$$

LEMMA 3.5. *Suppose $f(t)$ is a real-valued continuously differentiable function defined on the interval $[t_1,t_2]$ such that $|f'(t)| \gg \Delta > 0$. Then*

$$\int_{t_1}^{t_2} e(f(t))\,dt \ll 1/\Delta.$$

LEMMA 3.6. *Suppose $f(t)$ is a real-valued twice continuously differentiable function defined on $[t_1,t_2]$ such that $|f''(t)| \gg \Delta > 0$. Then*

$$\int_{t_1}^{t_2} e(f(t))\,dt \ll 1/\sqrt{\Delta}.$$

LEMMA 3.7. *Suppose $b_n \in \mathbb{C}$ $(n \ge 1)$ are such that their summation function $B(u)$ can be written as*

$$B(u) := \sum_{n\le u} b_n = M(u) + E(u),$$

where $M(u)$ is continuously differentiable on $(0,\infty)$. Suppose $f(u)$ is continuously differentiable on $[u_1,u_2]$, where $u_1 \ge 0$. Then

$$\sum_{u_1<n\le u_2} b_n f(n) = \int_{u_1}^{u_2} f(u)M'(u)\,du + \int_{u_1}^{u_2} f(u)\,dE(u).$$

LEMMA 3.8. *For any $H \ge 2$, we have*

$$\psi_1(u) = \sum_{1\le|h|\le H} \frac{e(hu)}{2\pi h} + O\left(\min\left(1, \frac{1}{H\|u\|}\right)\right),$$

$$\min\left(1, \frac{1}{H\|u\|}\right) = \sum_{h=-\infty}^{\infty} a(h)e(hu),$$

$$a(0) \ll \frac{\log H}{H}, \qquad a(h) \ll \min(1/|h|, H/h^2).$$

LEMMA 3.9. *For fixed $l \ge 1$, we have*

$$\psi_{2l}(u) = (-1)^{l-1} \sum_{h=1}^{\infty} \frac{2}{(2h\pi)^{2l}} \cos(2\pi u),$$

$$\psi_{2l+1}(u) = (-1)^{l-1} \sum_{h=1}^{\infty} \frac{2}{(2h\pi)^{2l+1}} \sin(2\pi u).$$

4. The estimate of $S_1(\alpha; x)$ on the major arc. The exponential sum $S_1(\alpha; x)$ is of importance in the quadratic Waring problem. For completeness, we give a detailed argument for $S_1(\alpha; x)$ both on the major arc and the minor arc. Our approach on the major arc is slightly different from previous ones.

Suppose $\alpha = a/q + z \in E_1$ with $0 \le a < q \le Q$, $(a, q) = 1$, $|z| \le 1/q\tau$. We have

$$S_1(\alpha; x) = \sum_{1 \le n \le x} e(n^2(a/q + z)) = \sum_{1 \le n \le x} e(n^2 a/q) e(n^2 z) \tag{4.1}$$
$$= \sum_{r=1}^{q} e(r^2 a/q) \sum_{\substack{1 \le n \le x \\ n \equiv r (mod\ q)}} e(n^2 z).$$

The counting function of the integers $n \equiv r \pmod q$ is

$$1 + \left[\frac{u - r}{q}\right] = \frac{u - r}{q} + 1/2 - \psi_1\left(\frac{u - r}{q}\right).$$

Taking

$$M(u) = \frac{u - r}{q} + 1/2, \qquad E(u) = -\psi_1\left(\frac{u - r}{q}\right), \qquad u_1 = 0,\ u_2 = x$$

in Lemma 3.7, we get

$$\sum_{\substack{1 \le n \le x \\ n \equiv r \pmod q}} e(n^2 z) = \int_1 - \int_2, \tag{4.2}$$

where

$$\int_1 = \frac{1}{q} \int_0^x e(u^2 z)\, du, \qquad \int_2 = \int_0^x e(u^2 z)\, d\psi_1\left(\frac{u - r}{q}\right).$$

By the change of variable $u = xv$ we get

$$\sum_{r=1}^{q} e(r^2 a/q) \int_1 = \frac{G(a, 0, q)}{q} x \int_0^1 e(v^2 x^2 z)\, dv. \tag{4.3}$$

Let $g(u) = e(u^2 z)$ and $l \ge 1$ be a fixed positive integer to be determined later. Repeated partial integration gives

$$\int_2 = g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^x - \int_0^x \psi_1\Big(\frac{u-r}{q}\Big) g'(u)\,du \tag{4.4}$$
$$= g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^x - q\int_0^x g'(u)\,d\psi_2\Big(\frac{u-r}{q}\Big)$$
$$= g(u)\psi_1\Big(\frac{u-r}{q}\Big)\Big|_0^x - qg'(u)\psi_2\Big(\frac{u-r}{q}\Big)\Big|_0^x + q\int_0^x g''(u)\psi_2\Big(\frac{u-r}{q}\Big)\,du$$
$$= \cdots$$
$$= \sum_{j=0}^{l}(-1)^j q^j g^{(j)}(u)\psi_{j+1}\Big(\frac{u-r}{q}\Big)\Big|_0^x + (-1)^{l+1}q^l\int_0^x g^{(l+1)}(u)\psi_{l+1}\Big(\frac{u-r}{q}\Big)\,du.$$

Thus we have

$$\sum_{r=1}^{q} e(r^2a/q)\int_2 = \sum_{j=0}^{l}(-1)^j q^j g^{(j)}(u)\sum_{r=1}^{q} e(r^2a/q)\psi_{j+1}\Big(\frac{u-r}{q}\Big)\Big|_0^x + (-1)^{l+1}q^l\int_0^x g^{(l+1)}(u)\sum_{r=1}^{q} e(r^2a/q)\psi_{l+1}\Big(\frac{u-r}{q}\Big)\,du. \tag{4.5}$$

We now show for any fixed $j \ge 0$ and uniformly for $0 \le u \le x$ that

$$\sum_{r=1}^{q} e(r^2a/q)\psi_{j+1}\Big(\frac{u-r}{q}\Big) \ll \begin{cases} \sqrt{q}\log(q+1) & \text{if } j=0,\\ \sqrt{q} & \text{if } j\ge 1.\end{cases} \tag{4.6}$$

The estimate (4.6) is trivial for $q = 1, 2$. Suppose now $q \ge 3$ and $j = 0$. Taking $H = q^2$ in Lemma 3.8 we can write

$$\sum_{r=1}^{q} e(r^2a/q)\psi_1\Big(\frac{u-r}{q}\Big) = \varSigma_1 + \varSigma_2, \tag{4.7}$$

where

$$\varSigma_1 = \sum_{r=1}^{q} e(r^2a/q)\sum_{1\le|h|\le q^2}\frac{1}{2\pi ih}e(h(u-r)/q),$$
$$\varSigma_2 = \sum_{r=1}^{q} O\Big(\min\Big(1, \frac{1}{q^2\|\frac{u-r}{q}\|}\Big)\Big).$$

By Lemma 3.4 we get

$$\Sigma_1 = \sum_{1\le |h|\le q^2} \frac{1}{2\pi i h} G(a,-h,q) \ll \sqrt{q}\log q. \tag{4.8}$$

For Σ_2 by the second expression in Lemma 3.8, and Lemma 3.1, we get

$$\Sigma_2 \ll \sum_{h\in\mathbb{Z}} a(h)e(hu/q)\sum_{r=1}^{q} e(-hr/q) \ll q\sum_{\substack{h\in\mathbb{Z}\\ q|h}} |a(h)| \ll \log q. \tag{4.9}$$

From (4.7)–(4.9) we see that (4.6) holds for $j=0$. Similarly, the case $j\ge 1$ follows from Lemmas 3.4 and 3.9.

Now we bound $g(u)$ for $0\le u\le x$. We shall show that for any fixed $s\ge 0$, the estimate

$$g^{(s)}(u) \ll_s \left(\frac{x}{q\tau}\right)^s \tag{4.10}$$

holds uniformly for $0\le u\le x$. Let $f(u)=4\pi i u z$. Obviously (4.10) holds for $s=0$. Since $g'(u)=g(u)f(u)$, we see that (4.10) holds for $s=1$. For any fixed $k\ge 1$, by Leibniz's formula we get

$$g^{(k+1)}(u) = f(u)g^{(k)}(u) + kf'(u)g^{(k-1)}(u),$$

hence

$$\begin{aligned} |g^{(k+1)}(u)| &\le |f(u)g^{(k)}(u)| + k|f'(u)g^{(k-1)}(u)| \\ &\le 4\pi x|z|\,|g^{(k)}(u)| + 4k\pi|z|\,|g^{(k-1)}(u)|. \end{aligned} \tag{4.11}$$

The condition $Q\tau\le x^2$ implies that $1/q\tau \le (x/q\tau)^2$. Now (4.10) follows from (4.11) by induction.

Inserting (4.6) and (4.10) into (4.5) we get

$$\begin{aligned} \sum_{r=1}^{q} e(r^2a/q)\int_2 &\ll \sqrt{q}\log(q+1) + \sqrt{q}\sum_{j=1}^{l} q^j(x/q\tau)^j + q^{l+1/2}x(x/q\tau)^{l+1} \\ &\ll \sqrt{q}\log(q+1) + \sqrt{q}\sum_{j=1}^{l}(x/\tau)^j + q^{-1/2}x(x/\tau)^{l+1}. \end{aligned}$$

Taking $l=[2/\varepsilon]$ and recalling the condition $\tau > x^{1+\varepsilon}$ we get

$$\sqrt{q}\sum_{j=1}^{l}(x/\tau)^j \ll \sqrt{q}, \qquad q^{-1/2}x(x/\tau)^{l+1} \ll 1.$$

From the above two estimates we have

$$\sum_{r=1}^{q} e(r^2a/q)\int_2 \ll \sqrt{q}\log(q+1). \tag{4.12}$$

From (4.1)–(4.3) and (4.12) we get the following

LEMMA 4.1. *Suppose $\alpha\in E_1$ with $Q\tau\le x^2$ and $\tau>x^{1+\varepsilon}$. Then*

$$S_1(\alpha;x)=\frac{G(a,0,q)}{q}x\int_0^1 e(v^2x^2z)\,dv+O(\sqrt{q}\log(q+1)). \tag{4.13}$$

5. The estimate of $S_1(\alpha;x)$ on the minor arc. By Dirichlet's theorem, for any $\alpha\in E_2$, there exist integers a and q such that

$$\alpha=a/q+z,\qquad |z|\le 1/q\tau,\ 1\le a\le q,\ (a,q)=1,\ Q<q\le\tau.$$

Opening the square we get

$$|S_1(\alpha;x)|^2=\sum_{1\le m,n\le x}e((m^2-n^2)\alpha)=[x]+T(x)+\overline{T(x)}, \tag{5.1}$$

where

$$T(x)=\sum_{1\le n<m\le x}e((m^2-n^2)\alpha).$$

By Lemmas 3.2 and 3.3 we have

$$\begin{aligned}T(x)&=\sum_{1\le v\le x-1}e(v^2\alpha)\sum_{1\le n\le x-v}e(2nv\alpha)\\&\ll\sum_{1\le v\le x}\min(x,1/\|2v\alpha\|)\ll\sum_{1\le v\le 2x}\min(x,1/\|v\alpha\|)\\&\ll(x+q\log q)(1+x/q)\ll x^2q^{-1}+q\log q+x\log q.\end{aligned} \tag{5.2}$$

From (5.1) and (5.2) we get

LEMMA 5.1. *Suppose $\alpha\in E_2$. Then*

$$S_1(\alpha;x)\ll xQ^{-1/2}+\tau^{1/2}\log^{1/2}x.$$

6. Heath-Brown's results on $D(y;q,r)$. Let $1\le r\le q$ be integers. For any $y>0$ define

$$D(y;q,r):=\sum_{\substack{n\le y\\ n\equiv r\ (\mathrm{mod}\ q)}}d(n).$$

In this subsection we quote some results about $D(y;q,r)$ from Heath-Brown [6]. First,

$$D(y;q,r)=R(y;q,r)+\Delta(y;q,r), \tag{6.1}$$

where

$$R(y;q,r)=\frac{yA(q,r)}{q^2}(\log y-2\log q+2\gamma-1)+\frac{2yB(q,r)}{q^2},$$

$$A(q,r)=\sum_{d|(q,r)}\sum_{\delta|q/d}d\delta\mu\Big(\frac{q}{d\delta}\Big),$$

$$B(q,r)=\sum_{d|(q,r)}\sum_{\delta|q/d}d\delta\mu\Big(\frac{q}{d\delta}\Big)\log\delta,$$

where γ is the Euler constant. For $A(q,r)$ and $B(q,r)$ we have

$$A(q,r)=\sum_{h=1}^{q}e\Big(-\frac{hr}{q}\Big)(h,q), \tag{6.2}$$

$$B(q,r)=\sum_{h=1}^{q}e\Big(-\frac{hr}{q}\Big)(h,q)\log(h,q). \tag{6.3}$$

Define

$$t(n;q,r)=\sum_{n=n_1n_2}\sum_{\substack{1\le m_1,m_2\le q\\ m_1m_2\equiv r\ (\mathrm{mod}\ q)}}e\Big(\frac{n_1m_1+n_2m_2}{q}\Big).$$

Then for $\varepsilon>0$, we have (Theorem 3 of [6])

$$\begin{aligned}\Delta(y;q,r)=\frac{y^{1/4}q^{-1/2}}{\pi\sqrt{2}}&\sum_{1\le n\le N}t(n;q,r)n^{-3/4}\cos(4\pi\sqrt{ny}/q-\pi/4)\\&+O((Ny)^{\varepsilon}q^{1/2}+(Ny)^{\varepsilon}y^{1/2}N^{-1/2})\end{aligned} \tag{6.4}$$

uniformly for $q\ge 1, y>1, N\ge q^2y^{-1}$.

For $t(n;q,r)$ the formula (51) of Heath-Brown [6] reads

$$t(n;q,r)=\sum_{n=n_1n_2}\sum_{\delta|(q,r,n_2)}\delta S(n_1,n_2\delta^{-1}r^*,q^*), \tag{6.5}$$

where $r^*=r/\delta, q^*=q/\delta$ and $S(u,v,q)$ is the Kloosterman sum

$$S(u,v,q):=\sum_{\substack{1\le n\le q\\ (n,q)=1}}e\Big(\frac{un+vn'}{q}\Big),$$

in which $nn'\equiv 1\ (\mathrm{mod}\ q)$.

7. The estimate of $S_2(-\alpha;3x^2)$ on the major arc

7.1. Some estimates involving $D(y;q,r)$. Suppose a,r,q are integers such that $1\le r\le q$, $0\le a<q$, $(a,q)=1$. The function $R(y;q,r)$ is the main term of the summation function $D(y;q,r)$. We may write it in the form

$$R(y;q,r)=c_1(q,r)y\log y+c_2(q,r)y, \tag{7.1}$$

where

$$c_1(q,r)=\frac{A(q,r)}{q^2}, \tag{7.2}$$

$$c_2(q,r)=\frac{A(q,r)(-2\log q+2\gamma-1)}{q^2}+\frac{2B(q,r)}{q^2}. \tag{7.3}$$

From (6.2) and Lemma 3.1 we have

$$\begin{aligned}\sum_{r=1}^{q}A(q,r)e(-ar/q)&=\sum_{h=1}^{q}(h,q)\sum_{r=1}^{q}e(-(h+a)r/q)\\&=(q-a,q)q=q.\end{aligned} \tag{7.4}$$

Similarly from (6.3) and Lemma 3.1 we have

$$\begin{aligned}\sum_{r=1}^{q}B(q,r)e(-ar/q)&=\sum_{h=1}^{q}(h,q)\log(h,q)\sum_{r=1}^{q}e(-(h+a)r/q)\\&=q(q-a,q)\log(q-a,q)=0.\end{aligned} \tag{7.5}$$

From (7.2)–(7.5) we get

$$\sum_{r=1}^{q}c_1(q,r)e(-ar/q)=q^{-1}=:B_1(q), \tag{7.6}$$

$$\sum_{r=1}^{q}c_2(q,r)e(-ar/q)=q^{-1}(-2\log q+2\gamma-1)=:B_2(q). \tag{7.7}$$

By (6.5) and the definition of Kloosterman sum we get

$$\begin{aligned}&\sum_{r=1}^{q}t(n;q,r)e(-ar/q)\\&=\sum_{r=1}^{q}\sum_{n=n_1n_2}\sum_{\delta\mid(q,r,n_2)}\delta S(n_1,n_2\delta^{-1}r^*,q^*)e(-ar^*/q^*)\\&=\sum_{n=n_1n_2}\sum_{\delta\mid(q,n_2)}\delta\sum_{r^*=1}^{q^*}S(n_1,n_2\delta^{-1}r^*,q^*)e(-ar^*/q^*)\\&=\sum_{n=n_1n_2}\sum_{\delta\mid(q,n_2)}\delta\sum_{r^*=1}^{q^*}\sum_{\substack{1\le h\le q^*\\(h,q^*)=1}}e\left(\frac{n_1h+n_2\delta^{-1}r^*h'-ar^*}{q^*}\right)\\&=\sum_{n=n_1n_2}\sum_{\delta\mid(q,n_2)}\delta\sum_{\substack{1\le h\le q^*\\(h,q^*)=1}}e(n_1h/q^*)\sum_{r^*=1}^{q^*}e(r^*(n_2\delta^{-1}h'-a)/q^*).\end{aligned} \tag{7.8}$$

The innermost sum is q^* if $n_2\delta^{-1}h' \equiv a \pmod{q^*}$ and 0 otherwise. Inserting this into (7.8) we get

$$\Big|\sum_{r=1}^{q} t(n;q,r)e(-ar/q)\Big| \le qd(q)d(n). \tag{7.9}$$

We shall prove that

$$F(y;q,a) := \sum_{r=1}^{q} \Delta(y;q,r)e(-ar/q) \ll (q^{3/2}+q^{2/3}y^{1/3})(qy)^{\varepsilon} \tag{7.10}$$

uniformly for $y>0$. For $0<y<q$, (7.10) follows from the trivial estimate $\Delta(y;q,r) \ll q^{\varepsilon}$. Now suppose $y \ge q$. From (6.4) and (7.9) we get

$$\begin{aligned} &\sum_{r=1}^{q} \Delta(y;q,r)e(-ar/q) \\ &= \frac{y^{1/4}q^{-1/2}}{\pi\sqrt{2}} \sum_{1\le n\le N} n^{-3/4}\cos(4\pi\sqrt{ny}/q-\pi/4)\sum_{r=1}^{q} t(n;q,r)e(-ar/q) \\ &\quad + O((Ny)^{\varepsilon}q^{3/2}+(Ny)^{\varepsilon}qy^{1/2}N^{-1/2}) \\ &\ll y^{1/4}q^{-1/2}\sum_{1\le n\le N} n^{-3/4}\Big|\sum_{r=1}^{q} t(n;q,r)e(-ar/q)\Big| \\ &\quad + O((Ny)^{\varepsilon}q^{3/2}+(Ny)^{\varepsilon}qy^{1/2}N^{-1/2}) \\ &\ll (Nyq)^{\varepsilon}(y^{1/4}q^{1/2}N^{1/4}+q^{3/2}+qy^{1/2}N^{-1/2}) \ll (q^{3/2}+q^{2/3}y^{1/3})(qy)^{\varepsilon} \end{aligned} \tag{7.11}$$

for $q \le y$, where we take $N=q^{2/3}y^{1/3}$.

Now we shall prove that

$$H(T;q,a) := \int_0^T F(y;q,a)\,dy \ll q^{3/2+\varepsilon}T \tag{7.12}$$

uniformly for $T>0$. When $T \ll q$, (7.12) follows from $\Delta(y;q,r) \ll q^{\varepsilon}$. Now suppose $T \gg q$. We shall bound the integral $\int_U^{2U} F(y;q,a)\,dy$. Taking $N=U$ in (6.4), we get

$$\begin{aligned} &\int_U^{2U} F(y;q,a)\,dy \\ &= \frac{q^{-1/2}}{\sqrt{2}\pi}\sum_{n\le U} n^{-3/4}\Big(\sum_{r=1}^{q} t(n;q,r)e(-ar/q)\Big)\int_U^{2U} y^{1/4}\cos(4\pi\sqrt{ny}\,q^{-1}-\pi/4)\,dy \\ &\quad + O(q^{3/2+\varepsilon}U) \\ &\ll q^{-1/2}\sum_{n\le U} n^{-3/4}qd(n)d(q)U^{3/4}qn^{-1/2} + O(q^{3/2+\varepsilon}U) \ll q^{3/2+\varepsilon}U \end{aligned}$$

with the help of (7.9) and Lemma 3.5, which implies (7.12) via a splitting argument.

7.2. Estimate of $S_2(-\alpha;3x^2)$ on the major arc. Suppose $\alpha = a/q + z \in E_1$ with $0 \le a < q \le Q$, $(a,q)=1, |z| \le 1/q\tau$. We have

$$\sum_{1\le n\le 3x^2} d(n)e(-n\alpha) = \sum_{r=1}^{q} e(-ar/q) \sum_{\substack{1\le n\le 3x^2\\ n\equiv r \,(\mathrm{mod}\, q)}} d(n)e(-nz).$$

So taking $M(u)=R(u;q,r)$ and $E(u)=\Delta(u;q,r)$ in Lemma 3.7 we get

$$\sum_{1\le n\le 3x^2} d(n)e(-n\alpha) = J_1 + J_2, \tag{7.13}$$

where

$$J_1 = \sum_{r=1}^{q} e(-ar/q) \int_0^{3x^2} e(-uz)\, dR(u;q,r),$$

$$J_2 = \sum_{r=1}^{q} e(-ar/q) \int_0^{3x^2} e(-uz)\, d\Delta(u;q,r).$$

We first treat J_1. By (7.1), (7.6), (7.7) and the change of variable $u = x^2 v$ we have

$$\begin{aligned}
J_1 &= \sum_{r=1}^{q} e(-ar/q) \int_0^{3x^2} e(-uz)(c_1(q,r)\log u + c_1(q,r) + c_2(q,r))\, du \qquad (7.14)\\
&= \sum_{r=1}^{q} e(-ar/q)c_1(q,r) \int_0^{3x^2} e(-uz)\log u\, du\\
&\quad + \sum_{r=1}^{q} e(-ar/q)(c_1(q,r)+c_2(q,r)) \int_0^{3x^2} e(-uz)\, du\\
&= B_1(q) \int_0^{3x^2} e(-uz)\log u\, du + (B_1(q)+B_2(q)) \int_0^{3x^2} e(-uz)\, du\\
&= 2B_1(q)x^2 \log x \int_0^{3} e(-ux^2 z)\, du + B_1(q)x^2 \int_0^{3} e(-ux^2 z)\log u\, du\\
&\quad + (B_1(q)+B_2(q))x^2 \int_0^{3} e(-ux^2 z)\, du.
\end{aligned}$$

For J_2, by partial integration twice and then using (7.11) and (7.12) we get

$$J_2 = \sum_{r=1}^{q} e(-ar/q)\Delta(3x^2;q,r)e(-3x^2z) + 2\pi iz\sum_{r=1}^{q} e(-ar/q)\int_0^{3x^2} \Delta(u;q,r)e(-uz)\,du$$
$$= F(3x^2;q,a)e(-3x^2z) + 2\pi iz\int_0^{3x^2} e(-uz)F(u;q,a)\,du$$
$$= F(3x^2;q,a)e(-3x^2z) + 2\pi iz\int_0^{3x^2} e(-uz)dH(u;q,a)$$
$$= F(3x^2;q,a)e(-3x^2z) + 2\pi izH(3x^2;q,a)e(-3x^2z) + (2\pi iz)^2\int_0^{3x^2} e(-uz)H(u;q,a)\,du$$
$$\ll x^{\varepsilon}(q^{3/2}x^2|z| + q^{3/2} + q^{2/3}x^{2/3})$$
$$\ll x^{\varepsilon}(q^{1/2}x^2\tau^{-1} + q^{3/2} + q^{2/3}x^{2/3})$$
$$\ll x^{\varepsilon}(q^{1/2}x^2\tau^{-1} + q^{2/3}x^{2/3}), \tag{7.15}$$

where the term $q^{3/2}$ was absorbed into $q^{1/2}x^2\tau^{-1}$ since $Q\tau \le x^2$.

Thus we get the following lemma.

LEMMA 7.1. *Suppose $\alpha \in E_1$ is such that $Q\tau \le x^2, \tau > x^{1+\varepsilon}$. Then*

$$S_2(-\alpha;3x^2) = \frac{2x^2\log x}{q}\int_0^3 e(-ux^2z)\,du + \frac{x^2}{q}\int_0^3 e(-ux^2z)\log u\,du + \frac{-2\log q + 2\gamma}{q}x^2\int_0^3 e(-ux^2z)\,du + O(x^{\varepsilon}(q^{1/2}x^2\tau^{-1} + q^{2/3}x^{2/3})). \tag{7.16}$$

8. Proof of Theorem 1. In this section we take $Q = x^{2/3}/8$, $\tau = 2x^{4/3}$.

We first treat the integral on the major arc. We have

$$\int_{E_1} S_1^3(\alpha;x)S_2(-\alpha;3x^2)\,d\alpha = \sum_{1\le q\le Q}\ \sum_{\substack{0\le a<q\\(a,q)=1}}\ \int_{a/q-1/q\tau}^{a/q+1/q\tau} S_1^3(\alpha;x)S_2(-\alpha;3x^2)\,d\alpha. \tag{8.1}$$

Suppose $\alpha = a/q + z \in E_1$. From Lemmas 4.1 and 7.1 we get

$$\begin{aligned} &S_1^3(\alpha; x) S_2(-\alpha; 3x^2) \\ &\quad = 2x^5 \log x \frac{G^3(a,0,q)}{q^4} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\,du \\ &\qquad + x^5 \frac{G^3(a,0,q)}{q^4} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\log u\,du \\ &\qquad + x^5 \frac{G^3(a,0,q)(-2\log q + 2\gamma)}{q^4} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\,du \\ &\qquad + O(x^{5+\varepsilon} q^{-1}\tau^{-1} + x^{11/3+\varepsilon} q^{-5/6} + x^{4+\varepsilon} q^{-3/2}). \end{aligned}$$

Thus

$$(8.2)\qquad \int_{a/q-1/q\tau}^{a/q+1/q\tau} S_1^3(\alpha; x) S_2(-\alpha; 3x^2)\,d\alpha$$

$$\begin{aligned} &= 2x^5 \log x \frac{G^3(a,0,q)}{q^4} \int_{-1/q\tau}^{1/q\tau} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\,du\,dz \\ &\quad + x^5 \frac{G^3(a,0,q)}{q^4} \int_{-1/q\tau}^{1/q\tau} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\log u\,du\,dz \\ &\quad + x^5 \frac{G^3(a,0,q)(-2\log q + 2\gamma)}{q^4} \int_{-1/q\tau}^{1/q\tau} \Big(\int_0^1 e(u^2x^2z)\,du\Big)^3 \int_0^3 e(-ux^2z)\,du\,dz \\ &\quad + O(x^{5+\varepsilon} q^{-2}\tau^{-2} + x^{11/3+\varepsilon} q^{-11/6}\tau^{-1} + x^{4+\varepsilon} q^{-5/2}\tau^{-1}) \\ &= 2x^3 \log x \frac{G^3(a,0,q)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^3 e(-uz)\,du\,dz \\ &\quad + x^3 \frac{G^3(a,0,q)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^3 e(-uz)\log u\,du\,dz \\ &\quad + x^3 \frac{G^3(a,0,q)(-2\log q + 2\gamma)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \Big(\int_0^1 e(u^2z)\,du\Big)^3 \int_0^3 e(-uz)\,du\,dz \\ &\quad + O(x^{5+\varepsilon} q^{-2}\tau^{-2} + x^{11/3+\varepsilon} q^{-11/6}\tau^{-1} + x^{4+\varepsilon} q^{-5/2}\tau^{-1}) \end{aligned}$$

$$= 2x^3 \log x \frac{G^3(a,0,q)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \mathcal{H}_1(z)\,dz + x^3 \frac{G^3(a,0,q)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \mathcal{H}_2(z)\,dz$$
$$+ x^3 \frac{G^3(a,0,q)(-2\log q + 2\gamma)}{q^4} \int_{-x^2/q\tau}^{x^2/q\tau} \mathcal{H}_1(z)\,dz$$
$$+ O(x^{5+\varepsilon}q^{-2}\tau^{-2} + x^{11/3+\varepsilon}q^{-11/6}\tau^{-1} + x^{4+\varepsilon}q^{-5/2}\tau^{-1}),$$

where $\mathcal{H}_1(z)$ and $\mathcal{H}_2(z)$ were defined in Theorem 1.

By our choice of Q and τ, $x^2/q\tau > 2$. So we first give upper bounds of $\mathcal{H}_1(z)$ and $\mathcal{H}_2(z)$ for $|z| \geq 2$. By Lemmas 3.5 and 3.6 we directly have

$$G_z(y) := \int_0^y e(-uz)\,du \ll 1/|z| \quad (y > 0), \qquad \int_0^1 e(u^2 z)\,du \ll 1/\sqrt{|z|}. \tag{8.3}$$

By partial summation and the first estimate in (8.3) we have

$$\begin{aligned}
&\int_0^1 e(-uz)\log u\,du \\
&\quad = \int_0^{1/|z|} e(-uz)\log u\,du + \int_{1/|z|}^1 e(-uz)\log u\,du \\
&\quad = \int_0^{1/|z|} e(-uz)\log u\,du + \int_{1/|z|}^1 \log u\,dG_z(u) \\
&\quad = \int_0^{1/|z|} e(-uz)\log u\,du + G_z(u)\log u\big|_{1/|z|}^1 - \int_{1/|z|}^1 G_z(u)u^{-1}\,du \\
&\quad \ll |z|^{-1}\log |z|.
\end{aligned} \tag{8.4}$$

Hence we get

$$\mathcal{H}_1(z) \ll |z|^{-5/2}, \qquad \mathcal{H}_2(z) \ll |z|^{-5/2}\log |z| \qquad (|z| \geq 2),$$

and correspondingly for $U \geq 2$ we have

$$\begin{aligned}
&\int_{|z|>U} \mathcal{H}_1(z)\,dz \ll \int_{|z|>U} z^{-5/2}\,dz \ll U^{-3/2}, \\
&\int_{|z|>U} \mathcal{H}_2(z)\,dz \ll \int_{|z|>U} z^{-5/2}\log z\,dz \ll U^{-3/2}\log U,
\end{aligned} \tag{8.5}$$

which means that the infinite integrals $\int_{-\infty}^{\infty} \mathcal{H}_j(z)\,dz$ $(j = 1,2)$ converge.

Taking $U=x^2/q\tau$ in (8.5), we get

$$\int\limits_{|z|>x^2/q\tau}\mathcal{H}_1(z)\,dz\ll\int\limits_{|z|>x^2/q\tau}z^{-5/2}\,dz\ll q^{3/2}\tau^{3/2}x^{-3},$$
$$\int\limits_{|z|>x^2/q\tau}\mathcal{H}_2(z)\,dz\ll\int\limits_{|z|>x^2/q\tau}z^{-5/2}\log z\,dz\ll q^{3/2}\tau^{3/2}x^{-3}\log x.$$

Inserting the above two estimates into (8.2) we have

$$(8.6)\qquad \int\limits_{a/q-1/q\tau}^{a/q+1/q\tau}S_1^3(\alpha;x)S_2(-\alpha;3x^2)\,d\alpha$$
$$=2x^3\log x\frac{G^3(a,0,q)}{q^4}\int\limits_{-\infty}^{\infty}\mathcal{H}_1(z)\,dz+x^3\frac{G^3(a,0,q)}{q^4}\int\limits_{-\infty}^{\infty}\mathcal{H}_2(z)\,dz$$
$$+x^3\frac{G^3(a,0,q)(-2\log q+2\gamma)}{q^4}\int\limits_{-\infty}^{\infty}\mathcal{H}_1(z)\,dz$$
$$+O(\tau^{3/2}q^{-1}\log x+x^{5+\varepsilon}q^{-2}\tau^{-2}+x^{11/3+\varepsilon}q^{-11/6}\tau^{-1}+x^{4+\varepsilon}q^{-5/2}\tau^{-1}).$$

Combining (8.1) and (8.6) we get

$$(8.7)\qquad \int\limits_{E_1}S_1^3(\alpha;x)S_2(-\alpha;3x^2)\,d\alpha$$
$$=2x^3\log x\sum_{1\le q\le Q}q^{-4}\sum_{\substack{0\le a<q\\(a,q)=1}}G^3(a,0,q)\int\limits_{-\infty}^{\infty}\mathcal{H}_1(z)\,dz$$
$$+x^3\sum_{1\le q\le Q}q^{-4}\sum_{\substack{0\le a<q\\(a,q)=1}}G^3(a,0,q)\int\limits_{-\infty}^{\infty}\mathcal{H}_2(z)\,dz$$
$$+x^3\sum_{1\le q\le Q}\frac{-2\log q+2\gamma}{q^4}\sum_{\substack{0\le a<q\\(a,q)=1}}G^3(a,0,q)\int\limits_{-\infty}^{\infty}\mathcal{H}_1(z)\,dz$$
$$+O(\tau^{3/2}Q\log x+x^{5+\varepsilon}\tau^{-2}+x^{11/3+\varepsilon}Q^{1/6}\tau^{-1}+x^{4+\varepsilon}\tau^{-1})$$
$$=2C_1I_1x^3\log x+(C_1I_2+C_2I_1)x^3$$
$$+O(x^3Q^{-1/2}\log Q+\tau^{3/2}Q\log x+x^{5+\varepsilon}\tau^{-2}+x^{11/3+\varepsilon}Q^{1/6}\tau^{-1}+x^{4+\varepsilon}\tau^{-1})$$
$$=2C_1I_1x^3\log x+(C_1I_2+C_2I_1)x^3+O(x^{8/3+\varepsilon}),$$

where C_j $(j=1,2)$ and I_j $(j=1,2)$ were defined in Theorem 1.

Now we study the integral on the minor arc. We have

$$\begin{aligned}(8.8)\qquad &\int_{E_2} S_1^3(\alpha;x)S_2(-\alpha;3x^2)\,d\alpha\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|\int_0^1 |S_1(\alpha;x)|^2|S_2(-\alpha;3x^2)|\,d\alpha\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|\Big(\int_0^1 |S_1(\alpha;x)|^4\,d\alpha\Big)^{1/2}\Big(\int_0^1 |S_2(-\alpha;3x^2)|^2\,d\alpha\Big)^{1/2}\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|\times\Big(\sum_{\substack{m_1^2+m_2^2=m_3^2+m_4^2\\ 1\le m_1,m_2,m_3,m_4\le x}} 1\Big)^{1/2}\times\Big(\sum_{1\le n\le 3x^2} d(n)\Big)^{1/2}\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|\times\Big(\sum_{\substack{m_1^2-m_3^2=m_4^2-m_2^2\\ 1\le m_1,m_2,m_3,m_4\le x}} 1\Big)^{1/2}\times\Big(\sum_{1\le n\le 3x^2} d(n)\Big)^{1/2}\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|\times\Big(\sum_{n\le 2x^2} d^2(n)\Big)^{1/2}\times\Big(\sum_{1\le n\le 3x^2} d(n)\Big)^{1/2}\\ &\ll \max_{\alpha\in E_2}|S_1(\alpha;x)|x^2\log^2 x \ll x^{8/3}\log^{2.5} x,\end{aligned}$$

where we used Lemma 5.1 and the well-known estimates

$$\sum_{n\le t} d^2(n)\ll t\log^3 t,\qquad \sum_{n\le t} d(n)\ll t\log t.$$

From (2.5), (8.7) and (8.8) the proof of Theorem 1 is complete.

9. Outline of the proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1, so we only give an outline.

For any $\alpha\in\mathbb{R}$ and $y>1$, define

$$(9.1)\qquad S_3(\alpha;y):=\sum_{|m|\le y} e(m^2\alpha),\qquad S_4(\alpha;y):=\sum_{1\le n\le y} \Lambda(n)e(n\alpha).$$

It is easily seen that

$$(9.2)\qquad S_3(\alpha;y)=2S_1(\alpha;y)+1.$$

By (2.2) and (9.2) we have

$$\begin{aligned}(9.3)\qquad \pi_\Lambda(x)&=\int_0^1 S_3^3(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha\\ &=8\int_0^1 S_1^3(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha+12\int_0^1 S_1^2(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha\end{aligned}$$

$$+6\int_0^1 S_1(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha+\int_0^1 S_4(-\alpha;x)\,d\alpha$$
$$=8\int_0^1 S_1^3(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha+O(x\log x).$$

Suppose $A>0$ is a fixed positive constant. Let $Q_1=\log^{A+2}x$ and $\tau_1=x\log^{-10A-10}x$ and define

$$E_3:=\bigcup_{1\le q\le Q_1}\ \bigcup_{\substack{0\le a<q\\(a,q)=1}} I(a,q),\qquad E_4:=[-1/\tau_1,1-1/\tau_1]\setminus E_3.$$

Suppose $\alpha=a/q+z\in E_3$, $(a,q)=1$. Similar to Lemma 4.1 and more easily we have

$$S_1(\alpha;\sqrt{x})=\sqrt{x}\frac{G(a,0,q)}{q}\int_0^1 e(u^2xz)\,du+O((1+x|z|)\sqrt{q}\log(q+1)).\tag{9.4}$$

For $S_4(-\alpha;x)$, similar to the formula (6.21) of [10] we have

$$S_4(-\alpha;x)=x\frac{C_q(-a)}{\varphi(q)}\int_0^1 e(-uxz)\,du+O(xe^{-c\sqrt{\log x}}),\tag{9.5}$$

where $c>0$ is an absolute positive constant and $C_q(r)$ is the Ramanujan sum. From (9.4) and (9.5) we get, similar to the proof of Theorem 1,

$$\begin{aligned}\int_{E_1} & S_1^3(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha \\ &= x^{3/2}\sum_{q=1}^{\infty}\frac{1}{q^3\varphi(q)} \\ &\quad\times\sum_{\substack{a=0\\(a,q)=1}}^{q-1} G^3(a,0,q)C_q(-a)\int_{-\infty}^{\infty}\Big(\int_0^1 e(u^2z)\,du\Big)^3\int_0^1 e(-uz)\,du\,dz \\ &\quad+O(x^{3/2}Q_1^{-1/2}+\tau_1^{3/2}Q_1+x^{5/2}\tau_1^{-1}e^{-c\sqrt{\log x}}+x^3\tau_1^{-2}) \\ &= C_3I_3x^{3/2}+O(x^{3/2}Q_1^{-1/2}+\tau_1^{3/2}Q_1+x^{5/2}\tau_1^{-1}e^{-c\sqrt{\log x}}+x^3\tau_1^{-2}),\end{aligned}\tag{9.6}$$

where C_3 and I_3 were defined in Theorem 2.

Now consider the integral on E_4. According to Dirichlet's lemma, each $\alpha\in E_4$ can be written as $\alpha=a/q+z$ with $(a,q)=1$, $Q_1\le q\le\tau_1$, $|z|\le 1/q\tau_1$. Lemma 5.1 still holds. So we have (x should be replaced by $\sqrt{x}$)

$$S_1(\alpha;\sqrt{x})\ll\sqrt{x}\,Q_1^{-1/2}+\tau_1^{1/2}\log^{1/2}x\ll\sqrt{x}\,Q_1^{-1/2}.$$

Hence similar to (8.8) we have

$$(9.7)\quad \int_{E_4} S_1^3(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha$$
$$\ll \max_{\alpha\in E_4}|S_1(\alpha;\sqrt{x})|\int_0^1 |S_1(\alpha;\sqrt{x})|^2|S_4(-\alpha;x)|\,d\alpha$$
$$\ll \max_{\alpha\in E_4}|S_1(\alpha;\sqrt{x})|\Big(\int_0^1 |S_1(\alpha;\sqrt{x})|^4\,d\alpha\Big)^{1/2}\Big(\int_0^1 |S_4(-\alpha;x)|^2\,d\alpha\Big)^{1/2}$$
$$\ll \max_{\alpha\in E_4}|S_1(\alpha;\sqrt{x})|\times\Big(\sum_{n\le x} d^2(n)\Big)^{1/2}\times\Big(\sum_{1\le n\le x}\Lambda^2(n)\Big)^{1/2}$$
$$\ll \max_{\alpha\in E_4}|S_1(\alpha;\sqrt{x})|x\log^2 x$$
$$\ll x^{3/2}Q_1^{-1/2}\log^2 x \ll x^{3/2}\log^{2-\lambda} x \ll x^{3/2}\log^{-A} x.$$

Theorem 2 follows from (9.3), (9.6) and (9.7).

10. Concluding remark. The circle method is very powerful for dealing with many sums involving the quadratic form g. It not only can be used for evaluating sums of the form

$$\sum_{m_1,m_2,m_3} f(m_1^2+m_2^2+m_3^2),$$

but also of the form

$$\sum_{m_1,m_2,m_3} f(m_1^2+m_2^2+m_3^2)g_1(m_1)g_2(m_2)g_3(m_3),$$

where f, g_1, g_2, g_3 are arithmetic functions which have good value distribution in residue classes to large moduli.

Now we list some results without proofs. All these results can be proved by the classical circle method. For some of them we can get even better asymptotic formulas.

$$(10.1)\quad S_{\mathbb{N}}(x;\mu) := \sum_{1\le m_1,m_2,m_3\le x}\mu(m_1^2+m_2^2+m_3^2) \ll x^3\log^{-A} x,$$

$$(10.2)\quad S_{\mathbb{Z}}(x;\mu) := \sum_{m_1^2+m_2^2+m_3^2\le x}\mu(m_1^2+m_2^2+m_3^2) \ll x^{3/2}\log^{-A} x,$$

$$(10.3)\quad S_{\mathbb{P}}(x;\mu) := \sum_{1\le p_1,p_2,p_3\le x}\mu(p_1^2+p_2^2+p_3^2) \ll x^3\log^{-A} x,$$

$$(10.4)\quad \pi_{3,\mathbb{P}}(x) := \sum_{p=p_1^2+p_2^2+p_3^2\le x} 1 \sim c_0x^{3/2}\log^{-4} x,$$

$$(10.5)\qquad S_{\mathbb{N}}(x;d) := \sum_{1\le p_1,p_2,p_3\le x} d(p_1^2+p_2^2+p_3^2) \sim c_1 x^3 \log^{-2} x,$$

$$(10.6)\qquad S_{\mathbb{N}}(x;d) := \sum_{1\le m_1,m_2,m_3\le x} d(m_1^2+m_2^2+m_3^2)d(m_1)d(m_2)d(m_3) \sim c_2 x^3 \log^4 x.$$

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References

[1] C. Calderón and M. J. de Velasco, *On divisors of a quadratic form*, Bol. Soc. Brasil. Mat. 31 (2000), 81–91.
[2] F. Chamizo and H. Iwaniec, *On the sphere problem*, Rev. Mat. Iberoamer. 11 (1995), 417–429.
[3] J. Chen, *Improvement of asymptotic formulas for the number of lattice points in a region of three dimensions (II)*, Sci. Sinica 12 (1963), 751–764.
[4] J. B. Friedlander and H. Iwaniec, *Hyperbolic prime number theorem*, Acta Math. 202 (2009), 1–19.
[5] D. R. Heath-Brown, *Lattice points in the sphere*, in: Number Theory in Progress, Vol. 2, K. Győry et al. (eds.), de Gruyter, 1999, 883–892.
[6] D. R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. (3) 38 (1979), 385–422.
[7] D. R. Heath-Brown, *The Pjateckii–Shapiro prime number theorem*, J. Number Theory 16 (1983), 242–266.
[8] L. K. Hua, *Introduction to Number Theory*, Science Press, Beijing, 1957 (in Chinese).
[9] Chengdong Pan and Chengbiao Pan, *Foundations of the Analytic Number Theory*, Science Press, Beijing, 1991 (in Chinese).
[10] Chengdong Pan and Chengbiao Pan, *Goldbach Conjecture*, Science Press, Beijing, 1981 (in Chinese).
[11] I. M. Vinogradov, *On the number of integer points in a sphere*, Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963), 957–968 (in Russian).

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