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# On sums involving products of three binomial coefficients

by

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**1. Introduction.** Let $p$ be an odd prime. It is known that (see, e.g., S. Ahlgren [A], L. van Hamme [vH] and T. Ishikawa [I])

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{-1/2}{k}^3\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x\ \&\ 2\mid y),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Clearly,

$$\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\quad \text{for all } k\in\mathbb{N}=\{0,1,2,\ldots\},$$

and

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod p\quad \text{for any } k=\frac{p+1}{2},\ldots,p-1.$$

After the determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$ (where $m\in\mathbb{Z}$ and $m\not\equiv 0 \pmod p$) in [Su1], the author [Su2, Su3] posed some conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \bmod p^2$ with $m\in\{1,-8,16,-64,256,-512,4096\}$; for example, in [Su2] he conjectured that

$$\text{(1.1)}\quad \sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1, \text{ i.e., } p\equiv 3,5,6 \pmod 7,\end{cases}$$

where $(-)$ denotes the Legendre symbol. (It is known that if $(\frac{p}{7})=1$ then $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$; see, e.g., [C, p. 31].) Quite recently Z.-H. Sun [S2] made a certain progress on those conjectures; in particular,

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he proved (1.1) in the case $\left(\frac{p}{7}\right) = -1$ and confirmed the author's conjecture on $\sum_{k=0}^{p-1} \binom{2k}{k}^3/(-8)^k \bmod p^2$.

Let $p = 2n+1$ be an odd prime. It is easy to see that for any $k = 0, \ldots, n$ we have

$$(1.2) \qquad \binom{n+k}{2k} = \frac{\prod_{j=1}^{k}(-(2j-1)^2)}{4^k(2k)!} \prod_{j=1}^{k} \left(1 - \frac{p^2}{(2j-1)^2}\right) \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.$$

Based on this observation Z.-H. Sun [S2] studied the polynomial

$$f_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 x^k$$

and found the key identity

$$(1.3) \qquad f_n(x(x+1)) = D_n(x)^2$$

in his approach to (1.1), where

$$D_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k.$$

Note that the numbers $D_n = D_n(1)$ ($n \in \mathbb{N}$) are the so-called *central Delannoy numbers* and $P_n(x) := D_n((x-1)/2)$ is the *Legendre polynomial* of degree $n$.

Recall that the *Catalan numbers* are the integers defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \qquad (n \in \mathbb{N}),$$

while the *Schröder numbers* are given by

$$S_n := \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{1}{k+1}.$$

We define the *Schröder polynomial* of degree $n$ by

$$(1.4) \qquad S_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^k.$$

For basic information about $D_n$ and $S_n$, the reader may consult [CHV], [Sl], [St, pp. 178 and 185], and [Su4].

In combinatorics, Zeilberger's algorithm developed in [Z] (see also Chapter 6 of [PWZ, pp. 101–119]) is an algorithm which finds a polynomial recurrence for a terminating hypergeometric sum. For example, if we use `Mathematica 7` and input `Zb[Binomial[n,k]^3,{k,0,n},n,2]`, then we obtain the following second-order recurrence for $S(n)=\sum_{k=0}^n\binom nk^3$:

$$-8(n+1)^2S(n)-(7n^2+21n+16)S(n+1)+(n+2)^2S(n+2)=0.$$

Via the Schröder polynomials and the Zeilberger algorithm, we obtain the following result.

THEOREM 1.1. *Let $p$ be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0 \pmod{p^2} \tag{1.5}$$

*for all $d\in\{0,1,\ldots,p-1\}$ with $d\equiv(p+1)/2 \pmod 2$.*

(ii) *If $p\equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k}\equiv(2p+2-2^{p-1})\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2}. \tag{1.6}$$

Now we state our second theorem the first part of which plays a key role in our proof of the second part.

THEOREM 1.2. *Let $p\equiv 1 \pmod 4$ be a prime and write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$)*

(i) *We can determine $x$ mod $p^2$ in the following way:*

$$\begin{aligned}(-1)^{(p-1)/4}x&\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\\&\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}.\end{aligned} \tag{1.7}$$

*Also,*

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{8^k}&\equiv-2\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{8^k}\\&\equiv(-1)^{(p-1)/4}\left(2x-\frac px\right) \pmod{p^2},\end{aligned} \tag{1.8}$$

$$\begin{aligned}S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}&\equiv-8\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}\\&\equiv(-1)^{(p-1)/4}2\left(2x-\frac px\right) \pmod{p^2},\end{aligned} \tag{1.9}$$

$$\sum_{k=0}^{(p-1)/2}\frac{k^2\binom{2k}{k}^2}{8^k}\equiv(-1)^{(p-1)/4}\left(x-\frac{3p}{4x}\right)\pmod{p^2},\tag{1.10}$$

$$\sum_{k=0}^{(p-1)/2}\frac{k^2\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p+3)/4}\frac{p}{16x}\pmod{p^2}.\tag{1.11}$$

(ii) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv 2p-2x^2\pmod{p^2},\tag{1.12}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -2p\pmod{p^2}.\tag{1.13}$$

Remark 1.1. Let $p$ be an odd prime. We conjecture that

$$\sum_{k=0}^{p-1}\frac{k+1}{8^k}\binom{2k}{k}^2+\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2$$
$$\equiv\begin{cases}2(\frac{2}{p})x\pmod{p^3} & \text{if } p=x^2+y^2\ (4\mid x-1\ \&\ 2\mid y),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Motivated by his study of Gaussian hypergeometric series and Calabi–Yau manifolds, in 2003 F. Rodriguez-Villegas [RV] raised some conjectures on congruences. In particular, he conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)\pmod{p^2},\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2},\tag{1.14}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\left(\frac{p}{3}\right)a(p)\pmod{p^2},\tag{1.15}$$

where

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6=\eta(4z)^6,$$
$$\sum_{n=1}^{\infty}b(n)q^n=q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3=\eta^3(6z)\eta^3(2z),$$
$$\sum_{n=1}^{\infty}c(n)q^n=q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$
$$=\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),$$

and the Dedekind $\eta$-function is given by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad (\mathrm{Im}(z)>0 \text{ and } q=e^{2\pi iz}).$$

In 1892 F. Klein and R. Fricke [KF] proved that (see also [SB])

$$a(p) = \begin{cases} 4x^2-2p & \text{if } p\equiv 1 \pmod 4 \text{ and } p=x^2+y^2\ (2\nmid x),\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

By [SB] we also have

$$b(p) = \begin{cases} 4x^2-2p & \text{if } p\equiv 1 \pmod 3 \text{ and } p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2 \pmod 3,\end{cases}$$

and

$$c(p) = \begin{cases} 4x^2-2p & \text{if } \left(\frac{-2}{p}\right)=1 \text{ and } p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } \left(\frac{-2}{p}\right)=-1, \text{ i.e., } p\equiv 5,7 \pmod 8.\end{cases}$$

Via an advanced approach involving the $p$-adic Gamma function and Gauss and Jacobi sums (see K. Ono [O, Chapter 11] for an introduction to this method), E. Mortenson [M] managed to provide a partial solution of (1.14) and (1.15), with the following congruences still open:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv b(p)=0 \pmod{p^2} \quad \text{if } p\equiv 5 \pmod 6, \tag{1.16}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv c(p) \pmod{p^2} \quad \text{if } p\equiv 3 \pmod 4, \tag{1.17}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv -a(p) \pmod{p^2} \quad \text{if } p\equiv 5 \pmod 6. \tag{1.18}$$

Concerning (1.16)–(1.18), Mortenson's approach [M] only allowed him to show that for each of them the squares of both sides of the congruence are congruent modulo $p^2$.

Our following theorem confirms (1.16)–(1.18) and hence completes the proof of (1.14) and (1.15). So far, all conjectures of Rodriguez-Villegas [RV] involving at most three products of binomial coefficients have been proved!

Theorem 1.3. *Let $p>3$ be a prime.*

(i) *Given $d\in\{0,\ldots,p-1\}$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k}\equiv 0 \pmod{p^2}\quad \text{if } d\equiv \frac{1+(\frac{p}{3})}{2} \pmod 2, \tag{1.19}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2}\quad \text{if } d\equiv \frac{1+(\frac{-2}{p})}{2} \pmod 2, \tag{1.20}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 0 \pmod{p^2}\quad \text{if } d\equiv \frac{1+(\frac{-1}{p})}{2} \pmod 2. \tag{1.21}$$

(ii) *If $p\equiv 3 \pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 4x^2-2p \pmod{p^2}. \tag{1.22}$$

(iii) *If $p\equiv 5 \pmod{12}$ and $p=x^2+y^2$ with $2\nmid x$ and $2\mid y$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 2p-4x^2 \pmod{p^2}. \tag{1.23}$$

In the case $d=1$, Theorem 1.3(i) yields the following new result. (Note that $\binom{2k}{k}\binom{3k}{k+1}=2\binom{2k}{k+1}\binom{3k}{k}$ for any $k\in\mathbb{N}$.)

Corollary 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k+1}}{108^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 3, \tag{1.24}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}\binom{2k}{k+1}}{256^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1,3 \pmod 8, \tag{1.25}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k+1}}{12^{3k}}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 4. \tag{1.26}$$

We will prove Theorems 1.1–1.3 in Sections 2–4 respectively.

## 2. Proof of Theorem 1.1

Lemma 2.1. *For any positive integer $n$ we have*

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}x^{k-1}(x+1)^{k+1}=n(n+1)S_n(x)^2. \tag{2.1}$$

*Proof.* Observe that

$$S_n(x)^2=\sum_{k=0}^n\binom{n+k}{2k}C_kx^k\sum_{l=0}^n\binom{n+l}{2l}C_lx^l=\sum_{m=0}^{2n}a_m(n)x^m,$$

where

$$a_m(n):=\sum_{k=0}^m\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}C_{m-k}.$$

Also, the coefficient of $x^m$ on the left-hand side of (2.1) coincides with

$$\begin{aligned}b_m(n):=&\sum_{k=1}^{m+1}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\binom{k+1}{m+1-k}\\=&\sum_{k=0}^{m}\binom{n+k+1}{2k+2}\binom{2k+2}{k+1}\binom{2k+2}{k}\binom{k+2}{m-k}.\end{aligned}$$

Thus, for the validity of (2.1) it suffices to show that $b_m(n)=n(n+1)a_m(n)$ for all $m=0,1,\ldots$. Obviously, $a_0(n)=1$ and $b_0(n)=n(n+1)$. Also, $a_1(n)=n(n+1)$ and $b_1(n)=n^2(n+1)^2$. By the Zeilberger algorithm via `Mathematica 7` we find that both $u_m=a_m(n)$ and $u_m=b_m(n)$ satisfy the following recursion:

$$\begin{aligned}&(m+2)(m+3)(m+4)u_{m+2}\\&\quad=2(2mn^2+5n^2+2mn+5n-m^3-6m^2-11m-6)u_{m+1}\\&\qquad-(m+1)(m-2n)(m+2n+2)u_m.\end{aligned}$$

So $b_m(n)=n(n+1)a_m(n)$ for all $m\in\mathbb{N}$. This proves (2.1). ∎

*Proof of Theorem 1.1.* We first determine $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{2k}{k+1}/64^k \bmod p^2$ via Lemma 2.1, which actually led the author to the study of (1.5).

Recall the following combinatorial identity (cf. [Su2, (4.3)]):

$$\sum_{k=0}^n\binom{n+k}{2k}\frac{C_k}{(-2)^k}=\begin{cases}(-1)^{(n-1)/2}C_{(n-1)/2}/2^n & \text{if } 2\nmid n,\\ 0 & \text{if } 2\mid n.\end{cases}\tag{2.2}$$

Set $n=(p-1)/2$. Applying (2.1) with $x=-1/2$ we get

$$\sum_{k=1}^n\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-2)^{k-1}2^{k+1}}=n(n+1)S_n\Big(-\frac12\Big)^2.$$

Thus, with the help of (1.2) and (2.2), we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k}\equiv\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-4)^k}=-n(n+1)S_n\left(-\frac{1}{2}\right)^2$$
$$\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ C_{(n-1)/2}^2/2^{2n+2} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Therefore (1.5) with $d=1$ holds if $p\equiv 1 \pmod 4$. In the case $p\equiv 3 \pmod 4$, clearly

$$\frac{C_{(n-1)/2}^2}{2^{2n+2}}=\frac{\left(\binom{(p-1)/2}{(p+1)/4}\frac{2}{p-1}\right)^2}{4\cdot 2^{p-1}}\equiv\frac{1}{(1-2p)(1+p\,q_p(2))}\binom{(p-1)/2}{(p+1)/4}^2$$
$$\equiv(1+2p-p\,q_p(2))\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2},$$

where $q_p(2)=(2^{p-1}-1)/p$, and hence (1.6) holds.

For $d=0,1,2,\ldots$ set

$$u_d=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}=\sum_{d\le k<p}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2u_d-(2d+3)^2u_{d+2}=\frac{(2p-1)^2(d+1)}{64^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1}=pC_{p-1}\equiv 0 \pmod p.$$

If $0\le d<p-2$, then

$$\binom{2p}{p+d+1}=\frac{2p}{p+d+1}\binom{2p-1}{p+d}\equiv 0 \pmod p$$

and hence

$$(2d+1)^2u_d\equiv(2d+3)^2u_{d+2} \pmod{p^2}.$$

For $d\in\{0,\ldots,p-3\}$ with $d\equiv(p+1)/2 \pmod 2$, clearly $p\ne 2d+1<2p$ and hence

$$u_{d+2}\equiv 0 \pmod{p^2}\ \Rightarrow\ u_d\equiv 0 \pmod{p^2}.$$

If $d\in\{p-1,p-2\}$ and $d\equiv(p+1)/2 \pmod 2$, then $d\ge(p+1)/2$ and hence $u_d\equiv 0 \pmod{p^2}$. So (1.5) holds for all $d\in\{0,\ldots,p-1\}$ with $d\equiv(p+1)/2 \pmod 2$.

Thus we have completed the proof of Theorem 1.1. ∎

## 3. Proof of Theorem 1.2

LEMMA 3.1. *For any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k} (-16)^{n-k} = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2. \tag{3.1}$$

*Proof.* For $n = 0, 1$, both sides of (3.1) take the values 1 and 8 respectively. Let $u_n$ denote the left-hand side of (3.1) or the right-hand side of (3.1). Applying the Zeilberger algorithm via `Mathematica 7`, we obtain the recursion

$$(n+2)^3 u_{n+2} = 8(2n+3)(2n^2+6n+5)u_{n+1} - 256(n+1)^3 u_n \quad (n \in \mathbb{N}).$$

So, by induction (3.1) holds for all $n = 0, 1, 2, \ldots$. ∎

LEMMA 3.2. *Let $p$ be an odd prime. Then*

$$\begin{aligned}\sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 &\\ \equiv \sum_{n=0}^{p-1} \frac{2n+1}{(-16)^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 &\\ \equiv p\left(\frac{-1}{p}\right) \pmod{p^3}.&\end{aligned}$$

*Proof.* In view of Lemma 3.1, we have

$$\begin{aligned}&\sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 \\ &= \sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k} (-16)^{n-k} \\ &= \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{8^k} \sum_{j=0}^{p-1-k} (k+j+1) \binom{k}{j} \frac{(-16)^j}{8^j} \\ &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{8^k} \left( (k+1) \sum_{j=0}^{k} \binom{k}{j} (-2)^j - 2k \sum_{j=1}^{k} \binom{k-1}{j-1} (-2)^{j-1} \right) \\ &= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{8^k} ((k+1)(-1)^k - 2k(-1)^{k-1}) \\ &\equiv \sum_{k=0}^{p-1} \frac{3k+1}{(-8)^k} \binom{2k}{k}^3 \pmod{p^3}.\end{aligned}$$

In [Su3] the author conjectured that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\ (\mathrm{mod}\ p^4)$$

provided $p>3$, where $E_0,E_1,E_2,\ldots$ are the Euler numbers given by

$$E_0=1\quad\text{and}\quad\sum_{\substack{k=0\\2|k}}^{n}\binom{n}{k}E_{n-k}=0\quad(n=1,2,\ldots).$$

The last congruence is still open but [GZ] confirmed that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3).$$

So we have

$$\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\equiv p\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3).$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{p-1}&\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\&=\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}\\&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-16)^k}\left((2k+1)\sum_{j=0}^{k}\binom{k}{j}+2k\sum_{j=1}^{k}\binom{k-1}{j-1}\right)\\&\equiv\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3).\ \blacksquare\end{aligned}$$

LEMMA 3.3. *Let $p$ be an odd prime. Then*

$$2\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{8^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{8^k}\equiv 2p^2\left(\frac{2}{p}\right)\ (\mathrm{mod}\ p^3),$$

$$8\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\equiv 2p^2\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3),$$

$$\sum_{k=0}^{(p-1)/2}(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}\equiv p^2\left(\frac{2}{p}\right)\pmod{p^3},$$
$$\sum_{k=0}^{(p-1)/2}(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv p^2\left(\frac{-1}{p}\right)\pmod{p^3}.$$

*Proof.* By induction, for every $n=0,1,2,\ldots$ we have

$$\sum_{k=0}^{n}\left(2k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{8^k}=\frac{(2n+1)^2}{(n+1)8^n}\binom{2n}{n}^2,$$
$$\sum_{k=0}^{n}\left(8k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{(-16)^k}=\frac{(2n+1)^2}{(n+1)(-16)^n}\binom{2n}{n}^2,$$
$$\sum_{k=0}^{n}(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}=\frac{(2n+1)^2}{8^n}\binom{2n}{n}^2,$$
$$\sum_{k=0}^{n}(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}=\frac{(2n+1)^2}{(-16)^n}\binom{2n}{n}^2.$$

Applying these identities with $n=(p-1)/2$ we immediately get the desired congruences. ∎

Let $p\equiv 1 \pmod 4$ be a prime and write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$. In 1828 Gauss showed the congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2x \pmod p$. In 1986, S. Chowla, B. Dwork and R. J. Evans [CDE] used Gauss and Jacobi sums to prove that

$$\binom{(p-1)/2}{(p-1)/4}\equiv\frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.2}$$

which was first conjectured by F. Beukers. (See also [BEW, Chapter 9] and [HW] for further related results.) In 2009, the author (see [Su2]) conjectured that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.3}$$

and this was confirmed by Z.-H. Sun [S1] via (3.2) and the Legendre polynomials.

*Proof of Theorem 1.2(i).* By (1.2),

$$S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\pmod{p^2}.$$

In view of this and Lemma 3.3 and (3.3), it suffices to show (1.7).

As $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$, we have

$$\begin{aligned}\sum_{n=0}^{p-1}&\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\sum_{n=k}^{p-1}\frac{n+1}{8^{n-k}}\binom{2(n-k)}{n-k}^2=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{p-1-k}\frac{k+j+1}{8^j}\binom{2j}{j}^2\\&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{(p-1)/2}\frac{(k+1)+(j+1)-1}{8^j}\binom{2j}{j}^2\\&=2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\sum_{j=0}^{(p-1)/2}\frac{(j+1)\binom{2j}{j}^2}{8^j}-\bigg(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\bigg)^2\pmod{p^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{p-1}&\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\&\equiv2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{(p-1)/2}\frac{(2j+1)\binom{2j}{j}^2}{(-16)^j}-\bigg(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\bigg)^2\pmod{p^2}.\end{aligned}$$

Combining these with Lemma 3.2 and (3.3), we immediately obtain (1.7). ∎

Lemma 3.4. *Let $p \equiv 1 \pmod 4$ be a prime. Write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then*

$$D_{(p-1)/2}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2}.\tag{3.4}$$

*Proof.* By (1.2),

$$D_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\pmod{p^2}.$$

So (3.4) follows from (3.3). ∎

Remark 3.1. If $p$ is a prime with $p \equiv 3 \pmod 4$, then $n = (p-1)/2$ is odd and hence

$$\begin{aligned}D_n&\equiv\sum_{k=0}^{n}(-1)^k\frac{\binom{2k}{k}^2}{16^k}=\sum_{k=0}^{n}(-1)^k\binom{-1/2}{k}^2\\&\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}^2=0\pmod p.\end{aligned}$$

The following result was conjectured by the author [Su2] and confirmed by Z.-H. Sun [S2].

LEMMA 3.5. *Let $p$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 4\mid p-1 \ \&\ p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}\tag{3.5}$$

REMARK 3.2. Fix an odd prime $p=2n+1$. By (1.2) and (1.3) we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 2^k=D_n^2 \pmod{p^2}.$$

Hence (3.5) follows from Lemma 3.4 and Remark 3.1.

LEMMA 3.6. *For any positive integer $n$ we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1} = \frac{S_n(x)}{2}(D_{n-1}(x)+D_{n+1}(x)).\tag{3.6}$$

*Proof.* Note that

$$S_n(x)(D_{n-1}(x)+D_{n+1}(x))=\sum_{m=0}^{2n+1}c_m(n)x^m$$

where

$$\begin{aligned}c_m(n)&=\sum_{k=0}^{m}\Bigg(\binom{n+k}{2k}C_k\binom{2m-2k}{m-k}\\&\qquad\times\bigg(\binom{n-1+m-k}{2m-2k}+\binom{n+1+m-k}{2m-2k}\bigg)\Bigg)\\&=2\sum_{k=0}^{m}\Bigg(\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}\binom{2m-2k}{m-k}\\&\qquad\times\frac{(m+n-k)^2-n(2m-2k-1)}{(m+n-k)(n-m+k+1)}\Bigg).\end{aligned}$$

By the Zeilberger algorithm we find that $u_m=c_m(n)/2$ satisfies the recursion

$$\begin{aligned}&(m+2)(m+3)^2(m^2+5m+6+4n(n+1))u_{m+2}+2P(m,n)u_{m+1}\\&\quad=(m+2)((2n+1)^2-m^2)(m^2+7m+12+4n(n+1))u_m\end{aligned}\tag{3.7}$$

where $P(m,n)$ denotes the polynomial

$$\begin{aligned}&m^5+11m^4+45m^3+83m^2+64m+12+20n^4-40n^3-58n^2-38n\\&\quad-25mn+m^2n+2m^3n-33mn^2+m^2n^2+2m^3n^2-16mn^3-8mn^4.\end{aligned}$$

Clearly the coefficient of $x^m$ on the left-hand side of (3.6) coincides with

$$d_m(n)=\sum_{k=0}^{m}\binom{n+k}{2k}\binom{2k}{k}^2\binom{k+1}{m-k}\frac{2k+1}{(k+1)^2}.$$

By the Zeilberger algorithm $u_m=d_m(n)$ also satisfies the recursion (3.7). Thus we have $d_m(n)=c_m(n)$ by induction on $m$. So (3.6) holds. ■

*Proof of Theorem 1.2(ii).* Write $p=2n+1$. By (2.1),

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k=\frac{n(n+1)}{2}S_n^2.$$

Thus, by (1.2) and (1.9) we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}&\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k\\&\equiv\frac{p^2-1}{8}4(4x^2-4p)\pmod{p^2},\end{aligned}$$

and hence (1.12) holds.

Now we consider (1.13). Observe that

$$\binom{2k}{k+1}^2=\left(1-\frac{2k+1}{(k+1)^2}\right)\binom{2k}{k}^2\quad\text{for } k=0,1,2,\ldots,$$

and

$$\binom{2(p-1)}{p-1}\binom{2(p-1)}{(p-1)+1}^2=\frac{p}{2p-1}\binom{2p-1}{p-1}\binom{2p-2}{p-2}^2\equiv -p \pmod{p^2}.$$

Thus we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -p+\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(-8)^k}-\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\pmod{p^2}.\tag{3.8}$$

By (1.2) and (3.6) with $x=1$,

$$\begin{aligned}\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}&\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{(2k+1)2^k}{(k+1)^2}\\&=\frac{S_n}{4}(D_{n-1}+D_{n+1})\pmod{p^2}.\end{aligned}$$

It is known (cf. [Sl] and [St, p. 191]) that

$$(n+1)D_{n+1}=3(2n+1)D_n-nD_{n-1}\quad\text{and}\quad D_{n+1}-3D_n=2nS_n.$$

Thus

$$\begin{aligned}n(D_{n-1}+D_{n+1})&=3(2n+1)D_n-D_{n+1}\\&=3(2n+1)D_n-(3D_n+2nS_n)=2n(3D_n-S_n)\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv\frac{S_n}{2}(3D_n-S_n)\pmod{p^2}.$$

With the help of (1.9) and (3.4), we have

$$\frac{S_n}{2}(3D_n-S_n)\equiv\left(2x-\frac{p}{x}\right)\left(3\left(2x-\frac{p}{2x}\right)-\left(4x-\frac{2p}{x}\right)\right)\pmod{p^2}$$

and hence

$$\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv 4x^2-p\pmod{p^2}.$$

Combining this with (3.5) and (3.8), we immediately obtain (1.13). ∎

## 4. Proof of Theorem 1.3

Lemma 4.1. *Let $p$ be an odd prime. Then, for any $p$-adic integer $x\not\equiv 0,-1 \pmod p$ we have*

(4.1)
$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\left(\frac{-x}{64}\right)^k \equiv\left(\frac{x+1}{p}\right)\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\left(\frac{x}{64(x+1)^2}\right)^k\pmod p.$$

*Proof.* Taking $n=(p-1)/2$ in the following identity of MacMahon (see, e.g., [G, (6.7)]):

$$\sum_{k=0}^{n}\binom{n}{k}^3x^k=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}x^k(1+x)^{n-2k},$$

and noting (1.2) and the basic facts

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p$$

and

$$\binom{n-k}{k}\equiv\binom{-1/2-k}{k}=\frac{\binom{4k}{2k}}{(-4)^k}\pmod p,$$

we immediately get (4.1). ∎

*Proof of Theorem 1.3.* (i) For $d=0,1,2,\ldots$, we define

$$f(d)=\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k},\qquad g(d)=\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k},$$

and

$$h(d)=\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$

By the Zeilberger algorithm, we find the recursive relations:

$$(4.2)\quad \begin{aligned}&(3d+1)(3d+2)f(d)-(3d+4)(3d+5)f(d+2)\\ &\quad=\frac{(3p-1)(3p-2)(d+1)}{108^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1},\end{aligned}$$

$$(4.3)\quad \begin{aligned}&(4d+1)(4d+3)g(d)-(4d+5)(4d+7)g(d+2)\\ &\quad=\frac{(4p-1)(4p-3)(d+1)}{256^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{4p-4}{2p-2},\end{aligned}$$

$$(4.4)\quad \begin{aligned}&(6d+1)(6d+5)h(d)-(6d+7)(6d+11)h(d+2)\\ &\quad=\frac{(6p-1)(6p-5)(d+1)}{1728^{p-1}p}\binom{2p}{p+d+1}\binom{3p-3}{p-1}\binom{6p-6}{3p-3}.\end{aligned}$$

Recall that $\binom{2p-2}{p-1}=pC_{p-1}\equiv 0 \pmod p$. Also,

$$(3p-2)\binom{3p-3}{p-1}=p\binom{3p-2}{p}\equiv 0 \pmod p,$$

$$(4p-3)\binom{4p-4}{2p-2}=p\binom{4p-2}{2p}\equiv 0 \pmod p,$$

$$(6p-5)\binom{6p-6}{3p-3}=\frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)}\binom{6p-3}{3p}\equiv 0 \pmod p.$$

If $0\le d<p-1$, then

$$\binom{2p}{p+d+1}=\binom{2p}{p-1-d}\equiv 0 \pmod p.$$

So, by (4.2)–(4.4), for any $d\in\{0,\ldots,p-1\}$ we have

$$(4.5)\qquad (3d+1)(3d+2)f(d)\equiv(3d+4)(3d+5)f(d+2) \pmod{p^2},$$

$$(4.6)\qquad (4d+1)(4d+3)g(d)\equiv(4d+5)(4d+7)g(d+2) \pmod{p^2},$$

$$(4.7)\qquad (6d+1)(6d+5)h(d)\equiv(6d+7)(6d+11)h(d+2) \pmod{p^2}.$$

Fix $0\le d\le p-1$. If $d\equiv\big(1+(\frac{p}{3})\big)/2 \pmod 2$, then it is easy to verify that $\{3d+1,3d+2\}\cap\{p,2p\}=\emptyset$, hence $(3d+1)(3d+2)\not\equiv 0 \pmod p$ and thus by (4.5) we have

$$f(d+2)\equiv 0 \pmod{p^2}\ \Rightarrow\ f(d)\equiv 0 \pmod{p^2}.$$

If $d\equiv\big(1+(\frac{-2}{p})\big)/2 \pmod 2$, then $\{4d+1,4d+3\}\cap\{p,3p\}=\emptyset$, hence $(4d+1)(4d+3)\not\equiv 0 \pmod p$ and thus by (4.6) we have

$$g(d+2)\equiv 0 \pmod{p^2}\ \Rightarrow\ g(d)\equiv 0 \pmod{p^2}.$$

If $d \equiv \big(1 + (\frac{-1}{p})\big)/2 \pmod 2$, then $\{6d+1, 6d+3\} \cap \{p, 3p, 5p\} = \emptyset$, hence $(6d+1)(6d+3) \not\equiv 0 \pmod p$ and thus (4.7) yields

$$h(d+2) \equiv 0 \pmod{p^2} \ \Rightarrow\ h(d) \equiv 0 \pmod{p^2}.$$

Since

$$f(p) = f(p+1) = g(p) = g(p+1) = h(p) = h(p+1) = 0,$$

by the last paragraph, for every $d = p+1, p, \ldots, 0$ we have the desired (1.19)–(1.21).

(ii) Assume that $p \equiv 3 \pmod 8$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$. Since $4x^2 \not\equiv 0 \pmod p$ and Mortenson [M] already proved that the squares of both sides of (1.22) are congruent modulo $p^2$, (1.22) is reduced to its mod $p$ form. Applying (4.1) with $x = 1$ we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \pmod p.$$

By [A, Theorem 5(3)], we have

$$\left(\frac{-1}{p}\right) \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} (-1)^k \equiv 4x^2 - 2p \pmod p,$$

where $n = (p-1)/2$. For $k = 0, \ldots, n$ clearly

$$\begin{aligned}\binom{n}{k}^2 \binom{n+k}{k} (-1)^k &= \binom{(p-1)/2}{k}^2 \binom{-(p+1)/2}{k} \\ &\equiv \binom{-1/2}{k}^3 = \frac{\binom{2k}{k}^3}{(-64)^k} \pmod p,\end{aligned}$$

therefore

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod p,$$

and hence (1.22) follows.

(iii) Finally, suppose $p \equiv 5 \pmod{12}$ and write $p = x^2 + y^2$ with $x$ odd and $y$ even. Once again it suffices to show the mod $p$ form of (1.23) in view of Mortenson's work [M]. As Z.-H. Sun observed,

$$\binom{(p-5)/6 + k}{2k} \binom{2k}{k} \equiv \binom{k - 5/6}{2k} \binom{2k}{k} = \frac{\binom{3k}{k}\binom{6k}{3k}}{(-432)^k} \pmod p$$

for all $k = 0, 1, 2, \ldots$. If $p/6 < k < p/3$ then $p \mid \binom{6k}{3k}$; if $p/3 < k < p/2$ then

$p \mid \binom{3k}{k}$; if $p/2 < k < p$ then $p \mid \binom{2k}{k}$. Thus

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv \sum_{k=0}^{(p-5)/6} \binom{(p-5)/6+k}{2k}\binom{2k}{k}^2\left(-\frac{1}{4}\right)^k$$
$$= D_{2n}\left(-\frac{1}{2}\right)^2 \pmod p \quad \text{(by (1.3))},$$

where $n = (p-5)/12$. Note that

$$D_{2n}\left(-\frac{1}{2}\right) = \frac{1}{(-4)^n}\binom{2n}{n}$$

by [G, (3.133) and (3.135)], and

$$\binom{(p-1)/2}{(p-1)/4} \equiv 12(-432)^n\binom{2n}{n} \pmod p$$

by P. Morton [Mo]. Therefore

$$D_{2n}\left(-\frac{1}{2}\right)^2 = \frac{1}{16^n}\binom{2n}{n}^2 \equiv \frac{\binom{(p-1)/2}{(p-1)/4}^2}{12^{6n+2}} \equiv \left(\frac{12}{p}\right)\binom{(p-1)/2}{(p-1)/4}^2 \pmod p.$$

Thus, by applying Gauss' congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$ (cf. [BEW, (9.0.1)] or [HW]) we immediately get the mod $p$ form of (1.23) from the above.

The proof of Theorem 1.3 is now complete. ∎

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