Higher congruence companion forms

by

Rajender Adibhatla (Regensburg) and
Jayanta Manoharmayum (Sheffield)

1. Introduction. Let F be a totally real number field and let p be an odd prime. Suppose we are given a Hilbert modular newform f over F of level $\mathfrak{n}_f$, character ψ and (parallel) weight $k \geq 2$. For a prime $\mathfrak{q}$ not dividing $\mathfrak{n}_f$, let $c(\mathfrak{q}, f)$ denote the eigenvalue of the Hecke operator $T(\mathfrak{q})$ acting on f; denote by K_f the number field generated by the $c(\mathfrak{q}, f)$'s and $\psi(\mathrm{Frob}_\mathfrak{q})$'s, and by $\mathcal{O}_f$ the integer ring of K_f. Then for each prime $\wp \,|\, p$ of $\mathcal{O}_f$ one has a continuous, odd, absolutely irreducible representation $\rho_{f,\wp} : G_F \to \mathrm{GL}_2(\mathcal{O}_{f,\wp})$ with the following property: $\rho_{f,\wp}$ is unramified outside primes dividing $p\mathfrak{n}_f$, and at a prime $\mathfrak{q} \nmid p\mathfrak{n}_f$ the characteristic polynomial of $\rho_{f,\wp}(\mathrm{Frob}_\mathfrak{q})$ is $X^2 - c(\mathfrak{q}, f)X + \psi(\mathrm{Frob}_\mathfrak{q})\mathrm{Nm}(\mathfrak{q})^{k-1}$. We denote the p-adic cyclotomic character by χ. Thus the determinant of $\rho_{f,\wp}$ is $\psi\chi^{k-1}$.

From now on we assume that the character ψ is unramified at p. Suppose that f is ordinary at p. Then, by Wiles [17], and Mazur–Wiles [11], for every prime $\mathfrak{p} \,|\, p$ we have

$$\rho_{f,\wp}|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi_{1\mathfrak{p}}\chi^{k-1} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}$$

where $G_\mathfrak{p}$ is a decomposition group at $\mathfrak{p}$ and $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ are unramified characters. In fact, with $a(\mathfrak{p}, f)$ defined to be the unit root of $X^2 - c(\mathfrak{p}, f)X + \psi(\mathrm{Frob}_\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1} = 0$, we have $\psi_{2\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p}) = a(\mathfrak{p}, f)$. A natural question is to ask when the restriction(s) $\rho_{f,\wp}|_{G_\mathfrak{p}}$ actually split(s). If $\rho_{f,\wp}$ mod $\wp$ is absolutely irreducible then the splitting (or not) of $\rho_{f,\wp}|_{G_\mathfrak{p}}$ mod $\wp^n$ is independent of the choice of a lattice used to define $\rho_{f,\wp}$. This is explained in greater detail at the beginning of Section 3.

Now suppose we are given a second newform g which is also ordinary at p. Fix a p-adic integer ring $\mathcal{O}$ in which $\mathcal{O}_f$ and $\mathcal{O}_g$ embed. Given a non-zero, non-unit $\pi \in \mathcal{O}$, we say that g is a *mod π weak companion form* for f if

[159]

$c(\mathfrak{q}, f) \equiv c(\mathfrak{q}, g)\mathrm{Nm}(\mathfrak{q})^{k-1} \bmod \pi$ for all but finitely many primes $\mathfrak{q}$. If the residual representations are absolutely irreducible—which will be the case in this article—then g being a mod π weak companion form to f is equivalent to $\rho_f \sim \rho_g \otimes \chi^{k-1} \bmod \pi$ where $\rho_f, \rho_g : G_F \to \mathrm{GL}_2(\mathcal{O})$ are the p-adic Galois representations associated to f, g. By applying the determinant condition to this companionship criterion, we see that the weight k' of g satisfies the congruence $\chi^{k'-1} \equiv \chi^{1-k} \bmod \pi$ on each decomposition group above p. Note that we do not enforce any optimality requirement on the level of g (and hence the prefix 'weak').

Classically, companion forms mod p played an important part in the weight optimisation part of Serre's Modularity Conjecture. Serre's predicted equivalence between local splitting for the residual modular representation (tame ramification) and the existence of companions was established by Gross in [7]. In much the same spirit, the main result of the present paper, which we now state, proves the equivalence between splitting mod p^n and the existence of mod p^n weak companion forms.

MAIN THEOREM. *Let F be a totally real number field, p be an odd prime unramified in F, and let f be a p-ordinary Hilbert modular newform f of squarefree level $\mathfrak{n}$, character ψ with order coprime to p and unramified at p, and weight $k \geq 2$. Let $n \geq 2$ and set $\boldsymbol{k} := \mathcal{O}_f/\wp$, $\rho_{f,n} := \rho_{f,\wp} \bmod p^n$, $\overline{\rho}_f := \rho_{f,\wp} \bmod \wp$. Assume the following hypotheses:*

- *Global conditions:*

  (GC1) *$\rho_{f,n}$ takes values in $\mathrm{GL}_2(W/p^n)$ where W is the Witt ring of $\boldsymbol{k} := \mathcal{O}_f/\wp$ (under the natural injection $W/p^n \hookrightarrow \mathcal{O}_f/p^n$).*
  (GC2) *The image of $\overline{\rho}_f$ contains $\mathrm{SL}_2(\boldsymbol{k})$. Furthermore, if $p = 3$ then the image $\rho_{f,n}$ contains a transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

- *Local conditions:*

  (LC1) *If $\mathfrak{p}$ is a prime dividing p then $c(\mathfrak{p}, f)^2 \not\equiv \psi(\mathrm{Frob}_{\mathfrak{p}}) \bmod \wp$.*
  (LC2) *Let $\mathfrak{q}$ be a prime dividing the level $\mathfrak{n}$ where $\overline{\rho}_f$ is unramified. If $Nm(\mathfrak{q}) \equiv 1 \bmod p$ then p divides the order of $\overline{\rho}_f(\mathrm{Frob}_{\mathfrak{q}})$.*

Let $k' \geq 2$ be the smallest integer such that $k+k' \equiv 2 \bmod (p-1)p^{n-1}$. Then f splits mod p^n if and only if it has a p-ordinary mod p^n weak companion form g of weight k' and character ψ.

The proof, given in Section 3.2, relies on being able to adapt Taylor's generalisation, [16], of Ramakrishna's methods, [13], to lift $\rho_{f,n} \otimes \chi^{1-k}$ to characteristic 0 with prescribed local properties. Modularity of this lift is established by using results of Skinner and Wiles in [15] along with the existence of companion forms mod p over totally real fields due to Gee ([4, Theorem 2.1]). The construction of characteristic 0 lifts for certain classes

of mod p^n representations is carried out in Section 3.1. (See Theorem 3.2 for the statement.)

The Main Theorem, in practice, is not useful for checking when a given newform fails to split mod p^n because we have very little control over the level of the weak companion form. However, as we show in Section 4, in the situation when the dimension of the tangent space $\mathbf{t}_{\mathcal{D}}$ (associated to a deformation condition $\mathcal{D}$ of $\bar{\rho}$) is 0, we can use higher companion forms to computationally verify a conjecture of Greenberg connecting local splitting with complex multiplication. We conclude by giving examples in support of this conjecture.

2. Toolkit. The method we use for obtaining a fine structure on deformations of a mod p^n representation is an adaptation of the more familiar mod p case and has two key components: the existence of sufficiently well-behaved local deformations, and, for the existence of characteristic 0 liftings, being able to place local constraints so that the dual Selmer group vanishes. Naturally, both of these present difficulties in the general mod p^n case. In this section, we discuss the tools that will enable us to manage the difficulties for certain classes of mod p^n representations.

Throughout this section p is an odd prime, $\mathbf{k}$ is a finite field of characteristic p, and W is the Witt ring of $\mathbf{k}$.

2.1. Deformations and substantial deformation conditions. In the main, we follow Mazur's treatment of deformations and deformation conditions in [10]. Given a residual representation, a *deformation condition* is simply a collection of liftings with some additional properties (closure under projections, a Mayer–Vietoris property etc.). The fundamental consequence then is the existence of a (uni)versal deformation.

We expand on this: Suppose we are given a 'nice' profinite group Γ, and a continuous representation $\bar{\rho} : \Gamma \to \mathrm{GL}_2(\mathbf{k})$. If $\mathcal{D}$ is a deformation condition for $\bar{\rho}$ then there is a complete local Noetherian W-algebra R with residue field $\mathbf{k}$ and a lifting $\rho : \Gamma \to \mathrm{GL}_2(R)$ in $\mathcal{D}$ with the following property: If $\rho' : \Gamma \to \mathrm{GL}_2(A)$ is a lifting of $\bar{\rho}$ in $\mathcal{D}$ then there is a morphism $R \to A$ which gives, on composition with ρ, a representation strictly equivalent to ρ'. In addition, we require that the morphism above is unique when A is the ring of dual numbers $\mathbf{k}[\epsilon]/(\epsilon^2)$. If the projective image of $\bar{\rho}$ has trivial centraliser then R, together with ρ, represents the functor that assigns type $\mathcal{D}$ deformations to a coefficient ring. We shall use the natural identification of the tangent space $\mathbf{t}_{\mathcal{D}}$ with a subspace of $H^1(\Gamma, \mathrm{ad}\,\bar{\rho})$ (and as a subspace of $H^1(\Gamma, \mathrm{ad}^0\,\bar{\rho})$ when considering deformations with a fixed determinant). The (uni)versal deformation ring R then has a presentation $W[[T_1, \ldots, T_n]]/J$

where $n = \dim_{\boldsymbol{k}} \mathbf{t}_{\mathcal{D}}$. We will be particularly interested in smooth deformation conditions (so the ideal of relations J will be (0)).

As hinted at the beginning of this section, the method we use for constructing smooth global deformation conditions depends upon being able to find local (uni)versal deformation rings smooth in a large number of variables. It will be convenient to make the following definition:

DEFINITION 2.1. Let F be a local field and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\boldsymbol{k})$ be a residual representation. We call a deformation condition for $\overline{\rho}$ with fixed determinant a *substantial deformation condition* if it is smooth and its tangent space $\mathbf{t}$ satisfies the inequality

$$\dim_{\boldsymbol{k}} \mathbf{t} \geq \dim_{\boldsymbol{k}} H^0(G_F, \mathrm{ad}^0 \overline{\rho}) + [F : \mathbb{Q}_p]\delta$$

where δ is 1 when F has residue characteristic p and 0 otherwise. Accordingly, any lift $\rho : G_F \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ in this collection of liftings will be called a *substantial deformation.*

We now give examples of substantial deformation conditions. (The reader may compare these with examples E1–E4 in [16].) From now on, for the rest of the section, F is a finite extension of $\mathbb{Q}_l$ for some prime l. As in the definition above, let $\overline{\rho} : G_F \to \mathrm{GL}_2(\boldsymbol{k})$ be a residual representation.

EXAMPLE 2.2. Assume that the residue characteristic of F is different from p. Suppose that the order of $\overline{\rho}(I_F)$ is co-prime to p, and let $d : G_F \to W^\times$ be a character lifting $\det \overline{\rho}$. The collection of liftings of $\overline{\rho}$ which factor through $G_F/(I_F \cap \ker \overline{\rho})$ and have determinant d is a substantial deformation condition. The tangent space has dimension $\dim_{\boldsymbol{k}} H^0(G_F, \mathrm{ad}^0 \overline{\rho})$.

EXAMPLE 2.3. Suppose that

$$\overline{\rho} \sim \begin{pmatrix} \overline{\chi} & * \\ 0 & 1 \end{pmatrix} \overline{\varepsilon}$$

for some character $\overline{\varepsilon} : G_F \to \boldsymbol{k}^\times$. Moreover, assume that if $\overline{\rho}$ is semisimple then $\overline{\chi}$ is non-trivial. Fix a character $\varepsilon : G_F \to W^\times$ lifting $\overline{\varepsilon}$. Then the collection of liftings strictly equivalent to

$$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix} \varepsilon$$

is a substantial deformation condition. Note that $\overline{\rho}$ is equivalent to a representation of the form considered above only if p divides the order of $\overline{\rho}(I_F)$. (See Example 3.3 of [9].)

EXAMPLE 2.4. We now assume that the residue characteristic of F is p. Suppose we are given an integer $k \geq 2$ and a representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\boldsymbol{k})$

such that

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}^{k-1}\overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$$

where $\overline{\psi}_1, \overline{\psi}_2$ are unramified characters. Let ψ be the Teichmüller lift of $\overline{\psi}_1\overline{\psi}_2$. If A is a coefficient ring, we shall call a lifting $\rho_A : G_F \to \mathrm{GL}_2(A)$ of $\overline{\rho}$ a $\overline{\psi}_2$-*good weight* k *lifting with character* ψ if ρ_A is strictly equivalent to a representation of the form

$$\begin{pmatrix} \widetilde{\psi}_1\chi^{k-1} & * \\ 0 & \widetilde{\psi}_2 \end{pmatrix}$$

for some unramified characters $\widetilde{\psi}_1, \widetilde{\psi}_2 : G_F \to A^{\times}$ lifting $\overline{\psi}_1, \overline{\psi}_2$ and $\widetilde{\psi}_1\widetilde{\psi}_2 = \psi$.

We then have the following property of weight k liftings (proof immediate, Example 3.4 in [9]):

PROPOSITION 2.5. *Let* $\overline{\rho} : G_F \to \mathrm{GL}_2(\boldsymbol{k})$ *be as above in Example* 2.4, *and further assume that* $\overline{\chi}^{k-1}\psi_1 \neq \overline{\chi}\psi_2$. *Then the deformation condition consisting of weight* k *liftings of* $\overline{\rho}$ *is a smooth deformation condition. The dimension of its tangent space is equal to* $[F : \mathbb{Q}_p] + \dim_{\boldsymbol{k}} H^0(G_F, \mathrm{ad}^0 \overline{\rho})$.

2.2. Subgroups of $\mathrm{GL}_2(W/p^n)$. We now derive some properties of certain subgroups of $\mathrm{GL}_2(W/p^n)$ which will be of relevance in constructing global deformations. Let us recall that p is an odd prime, and that $\boldsymbol{k}$ is the residue field W/p. We denote by ad^0 the the vector space of trace zero 2×2-matrices over $\boldsymbol{k}$ with $\mathrm{GL}_2(W/p^n)$ acting by conjugation, and by $\mathrm{ad}^0(i)$ its twist by the ith power of the determinant. For convenience, we record the following useful identity:

$$(2.1) \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+cx & b-2ax-cx^2 \\ c & -a-cx \end{pmatrix}.$$

LEMMA 2.6. *For* $p \geq 3$:

(a) $H^1(\mathrm{GL}_2(\boldsymbol{k}), \mathrm{ad}^0(i)) = 0$ *if* $i = 0, 1$ *and* $\boldsymbol{k} = \mathbb{F}_5$.
(b) $H^1(\mathrm{SL}_2(\boldsymbol{k}), \mathrm{ad}^0(i)) = 0$ *if* $i = 0, 1$ *and* $\boldsymbol{k} \neq \mathbb{F}_5$.

Proof. The claim is well known when $\boldsymbol{k} \neq \mathbb{F}_5$ and $i = 0$—see Lemma 2.48 of [1], for instance. The proof, in general, is a simple exercise following the proof of Lemma 1.2 of [3]. ∎

PROPOSITION 2.7. *Let* G *be a subgroup of* $\mathrm{GL}_2(W/p^n)$. *Suppose that the mod* p *reduction of* G *contains* $\mathrm{SL}_2(\boldsymbol{k})$. *Furthermore, assume that if* $p = 3$ *then* G *contains a transvection* $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$. *Then:*

(a) G *contains* $\mathrm{SL}_2(W/p^n)$.

(b) *Suppose that $p \geq 5$. If $\boldsymbol{k} = \mathbb{F}_5$, assume further that G mod $5 = \mathrm{GL}_2(\mathbb{F}_5)$. Then $H^1(G, \mathrm{ad}^0(i)) = 0$ for $i = 0, 1$.*

(c) *The restriction map $H^1(G, \mathrm{ad}^0(i)) \to H^1(\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle, \mathrm{ad}^0(i))$ is an injection (for all $p \geq 3$).*

Proof. (a) We shall only verify that if G is a subgroup of $\mathrm{SL}_2(W/p^n)$ whose mod p^{n-1} reduction is $\mathrm{SL}_2(W/p^{n-1})$ then $G = \mathrm{SL}_2(W/p^n)$. The kernel of the reduction map $G \to \mathrm{SL}_2(W/p^{n-1})$ consists of matrices of the form $I + p^{n-1}A$ with A an element of some additive subgroup of ad^0 stable under the action of G. Consequently, either $G = \mathrm{SL}_2(W/p^n)$ or else the reduction map $G \to \mathrm{SL}_2(W/p^{n-1})$ is an isomorphism. We will now discount the second possibility. So suppose that $G \to \mathrm{SL}_2(W/p^{n-1})$ is an isomorphism. For $p \geq 5$ it follows from [14, IV-23 Lemma 3], that $G \subset \mathrm{GL}_2(W/p^n)$ contains a transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (simply consider $G \cap \mathrm{GL}_2(\mathbb{Z}/p^n)$). For $p = 3$ this is true by hypothesis. However, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has order p^n in G and order p^{n-1} in $\mathrm{SL}_2(W/p^{n-1})$—a contradiction.

(b) The hypothesis and Lemma 2.6 imply that $H^1(G \bmod p, \mathrm{ad}^0(i)) = 0$. So assume that $n \geq 2$ and that $H^1(G \bmod p^{n-1}, \mathrm{ad}^0(i)) = 0$, and suppose that $0 \neq \xi \in H^1(G, \mathrm{ad}^0(i))$. Then the restriction ξ to $H := \ker(G \to G \bmod p^{n-1})$ is a group homomorphism compatible with the action of $\mathrm{SL}_2(W/p^{n-1})$. It follows from part (a) that H is in fact $\ker(\mathrm{SL}_2(W/p^n) \to \mathrm{SL}_2(W/p^{n-1}))$. Since H is naturally identified with ad^0, it follows that $\xi|_H$ is an isomorphism. Denote by W^ν the ring $W/p^{n-1} \oplus \boldsymbol{k}\epsilon$ where $\epsilon^2 = p\epsilon = 0$. (Or equivalently $W^\nu \cong W[\epsilon]/(p^{n-1}, \epsilon^2, p\epsilon)$.) We then see that the homomorphism $\mathrm{SL}_2(W/p^n) \to \mathrm{SL}_2(W^\nu)$ given by

$$g \mapsto (I + \epsilon\xi(g))(g \bmod p^{n-1})$$

is an isomorphism. To finish off, we proceed as in part (a): The transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(W^\nu)$ has order p^{n-1} while its pre-image in $\mathrm{SL}_2(W/p^n)$, a matrix of the form $\left(I + p^{n-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, has order p^n.

(c) Suppose $0 \neq \xi \in H^1(G, \mathrm{ad}^0(i))$ restricts to a trivial cohomology class in $H^1(\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle, \mathrm{ad}^0(i))$. Then the restriction of ξ to $\left(\begin{smallmatrix} 1 & p^{n-1} \\ 0 & 1 \end{smallmatrix}\right)$ is trivial. Set $N := \ker(G \to G \bmod p^{n-1})$. Then $\xi|_N$ has a non-trivial kernel, and hence $\xi|_N$ is trivial. Thus ξ is a non-zero element of $H^1(G \bmod p^{n-1}, \mathrm{ad}^0(i))$. We are thus reduced to the case when $n = 1$. Now $H^1(\mathrm{SL}_2(\boldsymbol{k}), \mathrm{ad}^0) = 0$ except when $\boldsymbol{k} = \mathbb{F}_5$, so we are reduced to the case when G is a subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$ containing $\mathrm{SL}_2(\mathbb{F}_5)$. But in this case $\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ is the Sylow 5-subgroup of G, and hence if $\xi|_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} = 0$ then $\xi = 0$. ∎

3. Constructing characteristic 0 lifts of mod p^n Galois representations. Firstly, we elaborate on the remark made in the introduction that if $\rho_{f,\wp} \bmod \wp$ is absolutely irreducible then the splitting (or not) of

$\rho_{f,\wp}$ mod $\wp^n$ is independent of the choice of a lattice used to define $\rho_{f,\wp}$. Indeed, if for some $M \in \mathrm{GL}_2(K_{f,\wp})$ the conjugate $M\rho_{f,\wp}M^{-1}$ is integral and stabilises the upper triangular decomposition group $G_{\mathfrak{p}}$, then M is a scalar multiple of $\left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ where $u \equiv 1$ mod $\wp$ and $v \equiv 0$ mod $\wp$. If we denote by $c_{\mathfrak{p}} \in H^1(G_{\mathfrak{p}}, \mathcal{O}_{f,\wp}(\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}}^{-1}\chi^{k-1}))$ the cohomology class for $\rho_{f,\wp}|_{G_{\mathfrak{p}}}$, then the cohomology class for the extension at $\mathfrak{p}$ determined by $M\rho_{f,\wp}M^{-1}$ is $uc_{\mathfrak{p}}$. Hence, if $\rho_{f,\wp}$ mod $\wp$ is absolutely irreducible, we can speak of $\rho_{f,\wp}$ mod $\wp^n$ being split without any ambiguity.

We can now formulate precise conditions under which a given mod p^n Galois representation can be lifted to characteristic 0, and use the lifts constructed to prove the existence of weak companion forms.

3.1. Deformations of mod p^n representations to $W(k)$. We now suppose we are given a totally real number field F and continuous odd representations $\overline{\rho} : G_F \to \mathrm{GL}_2(\boldsymbol{k})$, $\rho_n : G_F \to \mathrm{GL}_2(W/p^n)$, $n \geq 2$, with $\overline{\rho} = \rho_n$ mod p. We shall also assume that the $\overline{\rho}, \rho_n$ satisfy the following.

HYPOTHESIS A. *The image of $\overline{\rho}$ contains $\mathrm{SL}_2(\boldsymbol{k})$. Furthermore, if $p = 3$ then the image of ρ_n contains the transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Fix a character $\epsilon : G_F \to W^\times$ lifting the determinant of ρ_n. We wish to consider global deformation conditions $\mathcal{D}$ for $\overline{\rho}$ with determinant ϵ such that ρ_n is a deformation of type $\mathcal{D}$. We shall abbreviate this and call $\mathcal{D}$ a deformation condition for ρ_n. Except for a change in choice of lettering for primes of F we keep the notation of [9]. Thus $\mathcal{D}_{\mathfrak{q}}$ is the local component at a prime q, $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}$ is the tangent space there, and $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}^\perp \subseteq H^1(G_{F_{\mathfrak{q}}}, \mathrm{ad}^0\,\overline{\rho}(1))$ is the orthogonal complement of $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}$ under the pairing induced by

$$\mathrm{ad}^0\,\overline{\rho} \times \mathrm{ad}^0\,\overline{\rho}(1) \xrightarrow{\mathrm{trace}} \boldsymbol{k}(1).$$

The tangent space for $\mathcal{D}$ is the Selmer group $H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}\}}(F, \mathrm{ad}^0\,\overline{\rho})$; the dual Selmer group $H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}^\perp\}}(F, \mathrm{ad}^0\,\overline{\rho}(1))$ is determined by the local conditions $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}^\perp$. (See for instance [12, Definition 8.6.19].) We also set

$$\delta(\mathcal{D}) := \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}\}}(F, \mathrm{ad}^0\,\overline{\rho}) - \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}^\perp\}}(F, \mathrm{ad}^0\,\overline{\rho}(1)).$$

PROPOSITION 3.1. *Suppose we are given a deformation condition $\mathcal{D}$ for ρ_n with determinant ϵ. Let S be a fixed finite set of primes of F including primes where $\mathcal{D}$ is ramified and all the infinite primes. If $\delta(\mathcal{D}) \geq 0$, we can find a deformation condition $\mathcal{E}$ for ρ_n with determinant ϵ such that:*

- *The local conditions $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ are the same at primes $\mathfrak{q} \in S$.*
- *$\mathcal{E}_{\mathfrak{q}}$ is a substantial deformation condition for $\mathfrak{q} \notin S$.*
- *$H^1_{\{\mathbf{t}_{\mathcal{E}_{\mathfrak{q}}}^\perp\}}(F, \mathrm{ad}^0\,\overline{\rho}(1)) = (0)$.*

Proof. Let K be the splitting field of ρ_n adjoined with the p^nth roots of unity. We claim that we can find elements $g, h \in \mathrm{Gal}(K/F)$ such that

(R1) $\rho_n(g) \sim \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\chi(g) = -1 \bmod p^n$;

(R2) $\rho_n(h) \sim a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\chi(h) = 1 \bmod p^n$.

For (R1), we can take g to be complex conjugation. For (R2), by considering $\epsilon = \chi(\epsilon\chi^{-1})$ or otherwise, we can write $\epsilon = \chi\epsilon_0\epsilon_1^2$ where ϵ_0 is a finite order character of order co-prime to p. Our assumptions on the size of $\overline{\rho}$ and ρ_n (when $p = 3$) along with Proposition 2.7 imply that the image of the twist of $\rho_n \otimes \epsilon_1^{-1}$ contains $\mathrm{SL}_2(W/p^n)$. Thus we can find $h_1 \in \mathrm{Gal}(K/F)$ such that $\rho_n(h_1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\epsilon_1(h_1)$ and we get $\epsilon_0(h_1)\chi(h_1) = 1$. We can then take h to be $h_1^{p^k-1}$ where p^k is the cardinality of $\boldsymbol{k}$.

We first adjust $\mathcal{D}$ and define a deformation condition $\mathcal{E}_0$ for ρ_n with determinant ϵ as follows. We make no change if $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains $\mathrm{PSL}_2(\mathbb{F}_5)$; so $\mathcal{E}_0$ is $\mathcal{D}$. Now for the remaining cases: Suppose that either $p = 3$ or the projective image of $\overline{\rho}$ is A_5 (so $\boldsymbol{k}$ is necessarily $\mathbb{F}_5$). Using the Chebotarev Density Theorem and (R2) above, we can find a prime $\mathfrak{q}_0 \notin S$ with $\mathfrak{q}_0 \equiv 1 \bmod p^n$ and $\rho_n(\mathrm{Frob}_{\mathfrak{q}_0}) = a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\mathcal{E}_0$ be the deformation condition for $\overline{\rho}$ with determinant ϵ characterised by the following local conditions:

- at primes $\mathfrak{q} \neq \mathfrak{q}_0$, $\mathcal{E}_{0\mathfrak{q}} = \mathcal{D}_{\mathfrak{q}}$;
- at $\mathfrak{q}_0$, $\mathcal{E}_{0\mathfrak{q}_0}$ consists of deformations of the form

$$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}\epsilon'$$

where $\epsilon' : G_{v_0} \to W^\times$ is unramified and $\epsilon|_{G_{\mathfrak{q}_0}} = \chi\epsilon'^2$.

By our choice of $\mathfrak{q}_0$, $\mathcal{E}_0$ is a deformation condition for ρ_n. Further, $\mathcal{E}_{0\mathfrak{q}_0}$ is a substantial deformation and all non-zero cohomology classes in $\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}_0}}, \mathbf{t}_{\mathcal{E}_{0\mathfrak{q}_0}}^\perp$ are ramified.

We claim that the restriction maps

$$H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\,\overline{\rho}) \to H^1(G_K, \mathrm{ad}^0\,\overline{\rho}),$$

$$H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}^\perp\}}(F, \mathrm{ad}^0\,\overline{\rho}(1)) \to H^1(G_K, \mathrm{ad}^0\,\overline{\rho}(1))$$

are injective. When $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains A_5, an easy calculation using Proposition 2.7 shows $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\,\overline{\rho})$ and $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\,\overline{\rho}(1))$ are trivial, and so the injectivity follows. In the case when $p = 3$ or the projective image of $\overline{\rho}$ is A_5, we argue as follows: If $\xi \in \ker(H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\,\overline{\rho}) \to H^1(G_K, \mathrm{ad}^0\,\overline{\rho}))$, then ξ is naturally an element of $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\,\overline{\rho})$. Thus ξ is unramified at $\mathfrak{q}_0$ and so the restriction of

ξ to the decomposition group at $\mathfrak{q}_0$ must be trivial. From Proposition 2.7 it follows that $\xi \in H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0 \overline{\rho})$ is trivial. A similar argument works for $\mathrm{ad}^0 \overline{\rho}(1)$.

The proof is now standard: If the dual Selmer group for $\mathcal{E}_0$ is non-trivial then we can find

$$0 \neq \xi \in H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0 \overline{\rho}), \qquad 0 \neq \psi \in H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}^{\perp}\}}(F, \mathrm{ad}^0 \overline{\rho}(1)).$$

Take $g \in \mathrm{Gal}(K/L)$ as in (R1), consider pairs $(M_1, N_1), (M_2, N_2)$ where $\left\{ \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix} \right) \right\} = N_1 \subset M_1 = \mathrm{ad}^0 \overline{\rho}$, $\left\{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix} \right) \right\} = N_2 \subset M_2 = \mathrm{ad}^0 \overline{\rho}(1)$ and apply Proposition 2.2 of [9]. One can then find a prime $\mathfrak{r} \notin S \cup \{\mathfrak{q}_0\}$ lifting g such that the restrictions of ξ, ψ to $G_{\mathfrak{r}}$ are not in $H^1(G_{\mathfrak{r}}, N_1), H^1(G_{\mathfrak{r}}, N_2)$.

Now take $\mathcal{E}_1$ to be the deformation condition with determinant ϵ as follows: $\mathcal{E}_1$ and $\mathcal{E}_0$ differ only at $\mathfrak{r}$, and the local component at $\mathfrak{r}$ consists of deformations of the form $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix} \right)(\epsilon/\chi)^{1/2}$ considered in Example 2.3. Here, $(\epsilon/\chi)^{1/2}$ is the unramified character determined by taking the square root of $\epsilon(\mathrm{Frob}_{\mathfrak{r}})\chi^{-1}(\mathrm{Frob}_{\mathfrak{r}})$. Since $\mathrm{Frob}_{\mathfrak{r}}$ lifts g, we have $\chi(\mathrm{Frob}_{\mathfrak{r}}) \equiv -1 \bmod p^n$, and consequently $\mathcal{E}_1$ is a substantial deformation condition for ρ_n. The rest is identical to the proof of Proposition 4.2 of [9]: The dual Selmer group for $\mathcal{E}_1$ has dimension one less than that of the dual Selmer group for $\mathcal{E}_0$. (Of course $\delta(\mathcal{E}_1) = \delta(\mathcal{E}_0) = \delta(\mathcal{D})$.) ∎

We can now prove a general result for lifting a mod p^n representation to characteristic 0.

THEOREM 3.2. *Let $\mathcal{D}$ be a deformation condition for ρ_n with determinant ϵ, and let S be a fixed finite set of primes of F including primes where $\mathcal{D}$ is ramified and all the infinite primes. Suppose that each local component is substantial. We can then find a deformation condition $\mathcal{E}$ for ρ_n with determinant ϵ such that:*

- *The local conditions $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ are the same at primes $\mathfrak{q} \in S$.*
- *Each local component is a substantial deformation condition.*
- *The dual Selmer group $H^1_{\{\mathbf{t}_{\mathcal{E}_{\mathfrak{q}}}^{\perp}\}}(F, \mathrm{ad}^0 \overline{\rho}(1))$ is trivial.*

Furthermore, $\mathcal{E}$ is a smooth deformation condition and the universal deformation ring is a power series ring over W in $\delta(\mathcal{D})$ variables. In particular, there is a representation $\rho : G_F \to \mathrm{GL}_2(W)$ of type $\mathcal{E}$ lifting ρ_n.

Proof. The only verification required is to check that $\delta(\mathcal{D}) \geq 0$ and that $\dim_{\mathbf{k}} H^1_{\{\mathbf{t}_{\mathcal{E}_{\mathfrak{q}}}\}}(F, \mathrm{ad}^0 \overline{\rho}) = \delta(\mathcal{E}) = \delta(\mathcal{D})$. This is done using Wiles' formula (cf. [12, Theorem 8.6.20]). ∎

3.2. Modular characteristic 0 lifts and proof of Main Theorem.
We now look at the question of producing characteristic zero liftings which are modular. Given a mod p^n Galois representation $\rho_n : G_F \to \mathrm{GL}_2(W/p^n)$

with $\rho_n \bmod p$ modular, when can we guarantee the existence of a modular form f with $\rho_{f,p} \bmod p^n \sim \rho_n$? Our answer is a modest attempt using Theorem 3.2 to produce a characteristic 0 lift and then invoking results of Skinner and Wiles [15] to prove that it is modular.

For the rest of this section, F is a totally real field and $\psi : G_F \to W^\times$ is a finite order character of G_F unramified at primes dividing p.

PROPOSITION 3.3. *Let* $\rho_n : G_F \to \mathrm{GL}_2(W/p^n)$ *be a continuous odd representation satisfying Hypothesis A. Suppose* $\epsilon := \psi\chi^a$, $a \geq 1$, *lifts the determinant of* ρ_n. *Assume that:*

(i) *At a prime* $\mathfrak{q} \nmid p$ *where* ρ_n *is ramified, the restriction* $\rho_n|_{G_\mathfrak{q}}$ *is substantial there and a substantial deformation condition* $\mathcal{D}_\mathfrak{q}$ *is specified for* ρ_n.

(ii) *At a prime* $\mathfrak{p}$ *dividing* p,

$$\rho_n|_{G_\mathfrak{p}} \sim \begin{pmatrix} \chi^a\psi_{1\mathfrak{p}} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}$$

where $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ *are unramified,* $\chi^a\psi_{1\mathfrak{p}} \not\equiv \psi_{2\mathfrak{p}} \bmod p$ *and* $\chi^a\psi_{1\mathfrak{p}} \not\equiv \chi\psi_{2\mathfrak{p}} \bmod p$.

(iii) *There is an ordinary, of parallel weight at least 2, modular form which is a* $(\psi_{2\mathfrak{p}} \bmod p)$*-good lift of* $\rho_n \bmod p$.

There is then a modular form f *such that its associated* p*-adic representation* $\rho_{f,p} : G_F \to \mathrm{GL}_2(W)$ *lifts* ρ_n, *has determinant* $\psi\chi^a$, *is of type* $\mathcal{D}_\mathfrak{q}$ *at primes* $\mathfrak{q} \nmid p$ *where* ρ_n *is ramified, and*

$$\rho_{f,p}|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi'_{1\mathfrak{p}}\chi^a & * \\ 0 & \psi'_{2\mathfrak{p}} \end{pmatrix}$$

at primes $\mathfrak{p} \mid p$ *with* $\psi'_{2\mathfrak{p}}$ *an unramified lift of* $\psi_{2\mathfrak{p}} \bmod p$.

Proof. At a prime $\mathfrak{p} \mid p$ take $\mathcal{D}_\mathfrak{p}$ to be the class of deformations of the form

$$\begin{pmatrix} \psi'_{1\mathfrak{p}}\chi^a & * \\ 0 & \psi'_{2\mathfrak{p}} \end{pmatrix}$$

where $\psi'_{1\mathfrak{p}}$ (resp. $\psi'_{2\mathfrak{p}}$) is an unramified lifting of $\psi_{1\mathfrak{p}} \bmod p$ (resp. $\psi_{2\mathfrak{p}} \bmod p$), and $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$. This is a substantial deformation for ρ_n at $\mathfrak{p}$ by Proposition 2.5. By Theorem 3.2, there is a smooth deformation condition $\mathcal{E}$ for ρ_n which agrees with $\mathcal{D}_\mathfrak{p}$ at primes above p and primes where ρ_n is ramified. Thus there is a continuous representation $\rho : G_F \to \mathrm{GL}_2(W)$ with $\rho \bmod p^n = \rho_n$, unramified outside finitely many primes, of determinant $\psi\chi^a$ and with $\rho|_{I_\mathfrak{p}} \sim \begin{pmatrix} \chi^a & * \\ 0 & 1 \end{pmatrix}$ at primes $\mathfrak{p} \mid p$. The proposition now follows from Skinner–Wiles [15]. ∎

Proof of Main Theorem. Let us recall the set-up: We are given a Hilbert modular newform f of weight $k \geq 2$ character ψ which is ordinary at p and whose reduction mod p^n gives $\rho_{f,n} : G_F \to \mathrm{GL}_2(W/p^n)$. For each prime $\mathfrak{p}$ of F over p, let $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ be the unramified characters such that

$$\rho_f|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi_{1\mathfrak{p}}\chi^{k-1} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}.$$

As $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$ and $\psi_{2\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p}) = c(\mathfrak{p}, f)$, hypothesis (LC1) ensures $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ are distinct modulo $\wp$. From this, one deduces easily that if f has a weak companion form mod p^n then $\rho_{f,n}$ splits at p. We now show that 'split at p' implies the existence of a weak companion form.

Let $\rho_n := \rho_{f,n} \otimes \chi^{1-k}$, and set $\overline{\rho} := \rho_n \bmod p$. Recall that $k' \geq 2$ is the smallest integer satisfying the congruence $k + k' \equiv 2 \bmod (p-1)p^{n-1}$. Define a global deformation condition $\mathcal{D}$ for $\overline{\rho} \otimes \chi^{1-k} \bmod p$ by the following requirements:

(a) Deformations are unramified outside primes dividing $p\mathfrak{n}$ and have determinant $\psi\chi^{k'-1}$.

(b) At a prime $\mathfrak{p} \mid p$, the local condition $\mathcal{D}_\mathfrak{p}$ consists of deformations of the form

$$\begin{pmatrix} \psi'_{2\mathfrak{p}}\chi^{k'-1} & * \\ 0 & \psi'_{1\mathfrak{p}} \end{pmatrix}$$

where $\psi'_{1\mathfrak{p}}$ (resp. $\psi'_{2\mathfrak{p}}$) is an unramified lifting of $\psi_{1\mathfrak{p}} \bmod p$ (resp. $\psi_{2\mathfrak{p}} \bmod p$), and $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$.

(c) Let $\mathfrak{q}$ be a prime dividing $\mathfrak{n}$, the level of f. We need to distinguish two cases:

(i) If $\mathfrak{q}$ does not divide the conductor of ψ then $\overline{\rho}|_{G_\mathfrak{q}} \sim \left(\begin{smallmatrix} \overline{\chi} & * \\ 0 & 1 \end{smallmatrix}\right)\overline{\epsilon}$ for some character $\overline{\epsilon}$. Further, hypothesis (LC2) ensures that if $\overline{\rho}|_{G_\mathfrak{q}}$ is semisimple then $\overline{\chi} \neq 1$. We then take $\mathcal{D}_\mathfrak{q}$ to be local liftings with determinant $\psi\chi^{k'-1}$ of the type considered in Example 2.3.

(ii) If $\mathfrak{q}$ divides the conductor of ψ then $\rho_f(I_\mathfrak{q}), \overline{\rho}(I_\mathfrak{q})$ are finite and have the same order. In this case we take $\mathcal{D}_\mathfrak{q}$ as in Example 2.2, i.e. the lifts with determinant $\psi\chi^{k'-1}$ which factor through $G_\mathfrak{q}/(I_\mathfrak{q} \cap \ker \overline{\rho})$.

It then follows that ρ_n is a deformation of type $\mathcal{D}$ and that at each prime $\mathfrak{q} \nmid p$ where ρ_n is ramified the local deformation condition $\mathcal{D}_\mathfrak{q}$ is substantial there. As p is unramified in F, the distinctness of $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ modulo p implies that $\overline{\rho}$ satisfies hypothesis (ii) of Proposition 3.3. From the existence of mod p companion forms ([4, Theorem 2.1]) it follows that $\overline{\rho}$ has an ordinary modular lift which is $(\psi_{1\mathfrak{p}} \bmod p)_{\mathfrak{p}|p}$-good. The existence of a mod p^n weak companion form g for f of weight k' character ψ now follows from Proposition 3.3. ∎

4. Checking local splitting: A computational approach. The lifting result of the previous section is not suitable for computational purposes in general because, except in the case when the dual Selmer group was already trivial, we had no control of the level. There is, however, one case when we do have absolute control. We now describe this situation and go on to verify examples of local splitting.

4.1. A special case. Suppose $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\boldsymbol{k})$ is absolutely irreducible and $\mathcal{D}$ is a deformation condition for $\overline{\rho}$ such that its tangent space is 0-dimensional. Then the universal deformation ring $R_{\mathcal{D}}$ is a quotient of $W(\boldsymbol{k})$. If we also knew that there is a characteristic 0 lift of type $\mathcal{D}$, then we must have $R_{\mathcal{D}} \simeq W$. Consequently, any mod p^n representation of type $\mathcal{D}$ lifts to characteristic 0.

The question now is: How can one check if the tangent space is 0-dimensional? Observe that we must necessarily have exactly one characteristic 0 lift of type $\mathcal{D}$. This alone might not be enough though. For instance, $R_{\mathcal{D}}$ might be $W[X]/(X^2)$.

To proceed further, and with the examples we have in mind, we shall assume that $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\boldsymbol{k})$ is an absolutely irreducible representation with determinant $\overline{\chi}$ such that

- $\overline{\rho}|_{G_p} \sim \left(\begin{smallmatrix} \overline{\chi}\psi^{-1} & * \\ 0 & \psi \end{smallmatrix} \right)$ with ψ unramified and $\psi \neq \psi^{-1}$,
- if $q \nmid p$ then $\#\overline{\rho}(I_q) \,|\, p$.

By Lemma 3.24 of [1], the restriction $\overline{\rho}|_{G_L}$ is absolutely irreducible where $L = \mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$.

Let N be the Artin conductor of $\overline{\rho}$. For an integer $k \geq 2$, let $S(k, N, \overline{\rho})$ be the (possibly empty) set of newforms of level N with $\rho_f \bmod p \simeq \overline{\rho}$.

With notation as in Theorem 3.42 of [1], we then have an isomorphism $R_{\emptyset} \xrightarrow{\sim} \mathbb{T}_{\emptyset}$, where $R_{\emptyset}$ is the universal deformation ring for minimally ramified ordinary lifts and $\mathbb{T}_{\emptyset}$ is the reduced Hecke algebra generated by the Fourier coefficients of the newforms in $S(2, N, \overline{\rho})$. In particular, the dimension of the tangent space in the minimally ramified case is 0 if and only if $\#S(2, N, \overline{\rho}) = 1$.

For $n \geq 1$ set $k_n := (p-1)p^{n-1} - (p-1) + 2$ and define a deformation condition $\mathcal{D}_{k_n}$ for $\overline{\rho}$ as follows: A lift $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ is a deformation of type $\mathcal{D}_{k_n}$ if

- $\det \rho = \chi^{k_n - 1}$ and ρ is unramified outside primes dividing N,
- at primes $q \,|\, N$, $\rho|_{G_q} \sim \left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix} \right)$ up to twist, and
- at p,
$$\rho|_{G_p} \sim \begin{pmatrix} \tilde{\psi}^{-1}\chi^{k_n - 1} & * \\ 0 & \tilde{\psi} \end{pmatrix},$$
where $\tilde{\psi}$ is an unramified lift of ψ.

Note that for $n = 1$ the universal deformation ring $R_{\mathcal{D}_{k_n}}$ is $R_{\emptyset} \simeq \mathbb{T}_{\emptyset}$. Clearly, the type $\mathcal{D}_{k_n}$ deformations to $\boldsymbol{k}[\epsilon]/(\epsilon^2)$ are in bijection with the type $\mathcal{D}_2$ deformations. Hence if the tangent space of $\mathcal{D}_2$ has dimension 0 then so does $\mathcal{D}_{k_n}$. We conclude that $R_{\mathcal{D}_{k_n}} \simeq W$, corresponding to a unique newform in $S(k_n, N, \overline{\rho})$.

PROPOSITION 4.1. *Let f be a newform of weight $k \geq 2$, level N, trivial character and ordinary at p, such that*

- *$\overline{\rho}_f$ is absolutely irreducible,*
- *the conductor of $\overline{\rho}_f$ is N,*
- *$\overline{\rho}|_{G_p} \sim \left(\begin{smallmatrix} * & * \\ 0 & \psi \end{smallmatrix} \right)$ with ψ unramified and $\psi^2 \neq 1$,*
- *if $q \nmid p$ then $\#\overline{\rho}_f(I_q) \,|\, p$.*

Assume that $p - 1$ divides k and that f has exactly one companion form mod p of level N. Then ρ_f splits mod p^n iff f has a companion form mod p^n of level N.

Proof. We need to explain why splitting implies the existence of a companion form. Set $\overline{\rho} := \overline{\rho}_f \otimes \bar{\chi}^{1-k}$. By hypothesis $\overline{\rho}$ is modular and, in fact, $\#S(2, N, \overline{\rho}) = 1$. Therefore, as noted above, the tangent space of $\mathcal{D}_2$ has dimension 0. The preceding discussion thus shows that $R_{\mathcal{D}_{k_n}} \simeq W$, corresponding to a unique newform g_n in $S(k_n, N, \overline{\rho})$. Now, if ρ_f splits mod p^n then $\rho_f \otimes \chi^{1-k}$ is a deformation of type $\mathcal{D}_{k_n}$ and hence $\rho_{g_n} \sim \rho_f \otimes \chi^{1-k}$ mod p^n. ∎

4.2. Greenberg's conjecture. In this section we give some examples of the existence (or non-existence) of higher companion forms. We shall restrict ourselves to the setting of classical elliptic modular forms as we only give examples in this case.

Recall that a newform f is said to have *complex multiplication*, or just CM, by a quadratic character $\phi : G_{\mathbb{Q}} \to \{\pm 1\}$ if $T(q)f = \phi(\mathrm{Frob}_q)c(q, f)f$ for almost all primes q. We will also refer to CM by the corresponding quadratic extension. It is well known that a modular form has CM if and only if its associated p-adic representation is induced from an algebraic Hecke character.

Let $\mathcal{O}$ be a p-adic integer ring with uniformiser π and residue field $\boldsymbol{k}$. Suppose the newform f is p-ordinary and $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$. Then, as indicated in the introduction, $\rho_f|G_p$ can be assumed to be upper triangular with an unramified lower diagonal entry and this leads to the natural question of determining when ρ_f splits at p.

There is a well-known conjectural connection between ρ_f to be split at p, and f to have CM. The antecedents are sketchy, but Hida [8] calls it Greenberg's local non-semisimplicity conjecture; we will simply refer to it as *Greenberg's conjecture* which asserts:

If f is ordinary at p and ρ_f splits at p then f has complex multiplication.

This is satisfactorily known for modular forms over $\mathbb{Q}$ of weight 2. For higher weights, the question remains largely unresolved although some interesting results involving Hida families are shown in Ghate [5], which also has a survey of results for weight 2. The analogous problem for Λ-adic modular forms was resolved in Ghate–Vatsal [6] by using deformation theory but similar methods appear not to bear fruit in the classical case. Emerton [2] shows how this conjecture would follow from a p-adic version of the variational Hodge conjecture. Through the Main Theorem and Proposition 4.1, higher congruence companions offer a slightly different perspective to the question of ρ_f splitting at p.

To describe this further, let N be the level of the p-ordinary newform f. Assume that f has trivial character and has weight $p-1$. For each positive integer n we set
$$k_n := p^{n-1}(p-1) - (p-1) + 2.$$

We then proceed as follows:

(a) Check that f has a companion form mod p. Check congruences to make sure that the residual representation ρ_f mod π is absolutely irreducible and that $c(p, f) \not\equiv \pm 1$ mod π. We can therefore write

$$\rho_f \bmod \pi = \begin{pmatrix} \overline{\chi}^{k-1}\overline{\psi} & 0 \\ 0 & \overline{\psi}^{-1} \end{pmatrix}$$

with $\overline{\psi}^{-1} \neq \overline{\psi}$.

(b) In order to be able to check fewer cases, ensure that ρ_f is minimally ramified, i.e. the Artin conductor of ρ_f mod π is N.

(c) Set $\overline{\rho} := \rho_f \otimes \chi^{1-k}$ mod π. For each $n \geq 1$ let $\mathcal{D}_{k_n}$ be the weight k_n, trivial character deformation condition as described in Section 4.1. Check if the tangent spaces can be taken to be 0-dimensional. Thus we have to check if f has precisely one companion form of type $\mathcal{D}_{k_1}$. We then apply Proposition 4.1 to deduce that ρ_f splits mod p^n if and only if f has a companion form mod p^n, i.e. there is a newform g of level N, trivial character, weight k_n such that $f \equiv g \otimes \chi^{k-1}$ mod p^n.

We check Greenberg's conjecture explicitly for two known non-CM forms of weight 4. The computations were done on MAGMA. In both cases $p = 5$. We note that in these examples, taking N to be the level of f, one may check its companionship with a form g "by hand" by simply verifying the congruences $c(f, m) \equiv m^3 c(g, m)$ mod 5^n for $(m, 5N) = 1$ up to the Sturm bound.

EXAMPLE 4.2. Let f be the newform of weight 4, level 21 and trivial character with the following Fourier expansion:

$$g = q - 3q^2 - 3q^3 + q^4 - 18q^5 + 9q^6 + 7q^7 + 21q^8 + 9q^9 + 54q^{10} - 36q^{11}$$
$$- 3q^{12} - 34q^{13} - 21q^{14} + 54q^{15} - 71q^{16} + 42q^{17} - 27q^{18} - 124q^{19} - 18q^{20}$$
$$- 21q^{21} + 108q^{22} - 63q^{24} + 199q^{25} + 102q^{26} - 27q^{27} + 7q^{28} + 102q^{29}$$
$$- 162q^{30} - 160q^{31} + 45q^{32} + 108q^{33} - 126q^{34} - 126q^{35} + 9q^{36} + 398q^{37}$$
$$+ 372q^{38} + 102q^{39} - 378q^{40} - 318q^{41} + 63q^{42}$$
$$- 268q^{43} - 36q^{44} - 162q^{45} + 240q^{47} + 213q^{48} + 49q^{49} - 597q^{50} + \cdots.$$

MAGMA outputs, modulo 5, a unique companion form g of weight 2, level 21 and trivial character with the following Fourier expansion:

$$f = q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + 3q^8 + q^9 + 2q^{10} + 4q^{11} - q^{12}$$
$$- 2q^{13} + q^{14} - 2q^{15} - q^{16} - 6q^{17} - q^{18} + 4q^{19} + 2q^{20} - q^{21} - 4q^{22}$$
$$+ 3q^{24} - q^{25} + 2q^{26} + q^{27} + q^{28} - 2q^{29} + 2q^{30} - 5q^{32} + 4q^{33}$$
$$+ 6q^{34} + 2q^{35} - q^{36} + 6q^{37} - 4q^{38} - 2q^{39} - 6q^{40} + 2q^{41}$$
$$+ q^{42} - 4q^{43} - 4q^{44} - 2q^{45} - q^{48} + q^{49} + q^{50} + \cdots.$$

Clearly there are no companions of weight 2 and level 3 or 7. Modulo 5^2, f has no companion forms of weight 18, level dividing 21 and trivial character. Thus f does not split mod 5^2.

EXAMPLE 4.3. Let f be the newform of weight 4, level 57 and trivial character with Fourier expansion

$$f = q - q^2 + 3q^3 - 7q^4 - 12q^5 - 3q^6 - 20q^7 + 15q^8 + 9q^9 + 12q^{10} - 4q^{11}$$
$$- 21q^{12} - 76q^{13} + 20q^{14} - 36q^{15} + 41q^{16} + 22q^{17} - 9q^{18} - 19q^{19}$$
$$+ 84q^{20} - 60q^{21} + 4q^{22} + 82q^{23} + 45q^{24} + 19q^{25} + 76q^{26} + 27q^{27}$$
$$+ 140q^{28} + 242q^{29} + 36q^{30} - 126q^{31} - 161q^{32} - 12q^{33} - 22q^{34} + 240q^{35}$$
$$- 63q^{36} - 180q^{37} + 19q^{38} - 228q^{39} - 180q^{40} - 390q^{41} + 60q^{42} + 308q^{43}$$
$$+ 28q^{44} - 108q^{45} - 82q^{46} - 522q^{47} + 123q^{48} + 57q^{49} - 19q^{50} + \cdots.$$

It has a unique mod 5 companion form g of weight 2, level 57 and trivial character with Fourier expansion

$$g = q - 2q^2 - q^3 + 2q^4 - 3q^5 + 2q^6 - 5q^7 + q^9 + 6q^{10} + q^{11} - 2q^{12}$$
$$+ 2q^{13} + 10q^{14} + 3q^{15} - 4q^{16} - q^{17} - 2q^{18} - q^{19} - 6q^{20} + 5q^{21} - 2q^{22}$$
$$- 4q^{23} + 4q^{25} - 4q^{26} - q^{27} - 10q^{28} - 2q^{29} - 6q^{30} - 6q^{31} + 8q^{32}$$
$$- q^{33} + 2q^{34} + 15q^{35} + 2q^{36} + 2q^{38} - 2q^{39} - 10q^{42} - q^{43} + 2q^{44}$$
$$- 3q^{45} + 8q^{46} - 9q^{47} + 4q^{48} + 18q^{49} - 8q^{50} + \cdots.$$

and no other companions of level dividing 57. Modulo 5^2, f has no companion forms of weight 18, level dividing 57 and trivial character.

References

[1] H. Darmon, F. Diamond and R. Taylor, *Fermat's Last Theorem*, in: Current Developments in Mathematics, 1995, R. Bott et al. (eds.), Int. Press, Cambridge, MA, 1995, 1–107.

[2] M. Emerton, *A p-adic variational Hodge conjecture and modular forms with complex multiplication*, preprint.

[3] M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. 109 (1992), 307–327.

[4] T. Gee, *Companion forms over totally real fields, II*, Duke Math. J. 136 (2007), 275–284.

[5] E. Ghate, *On the local behavior of ordinary modular Galois representations*, in: Progr. Math. 224, Birkhäuser, Basel, 2004, 105–124.

[6] E. Ghate and V. Vatsal, *On the local behaviour of ordinary Λ-adic representations*, Ann. Inst. Fourier (Grenoble) 54 (2004), 2143–2162.

[7] B. H. Gross, *A tameness criterion for Galois representations associated to modular forms* (mod p), Duke Math. J. 61 (1990), 445–517.

[8] H. Hida, *CM components of the big Hecke algebra*, a series of lectures at Hokkaido University.

[9] J. Manoharmayum, *Lifting Galois representations of number fields*, J. Number Theory 129 (2009), 1178–1190.

[10] B. Mazur, *An introduction to the deformation theory of Galois representations*, in: Modular Forms and Fermat's Last Theorem, G. Cornell et al. (eds.), Springer, New York, 1997, 243–311.

[11] B. Mazur and A. Wiles, *On p-adic analytic families of Galois representations*, Compos. Math. 59 (1986), 231–264.

[12] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, 2nd ed., Grundlehren Math. Wiss. 323, Springer, Berlin, 2008.

[13] R. Ramakrishna, *Deforming Galois representations and the conjectures of Serre and Fontaine–Mazur*, Ann. of Math. (2) 156 (2002), 115–154.

[14] J.-P. Serre, *Abelian l-Adic Representations and Elliptic Curves*, Res. Notes Math. 7, A K Peters, Wellesley, MA, 1998.

[15] C. M. Skinner and A. J. Wiles, *Nearly ordinary deformations of irreducible residual representations*, Ann. Fac. Sci. Toulouse Math. (6) 10 (2001), 185–215.

[16] R. Taylor, *On icosahedral Artin representations, II*, Amer. J. Math. 125 (2003), 549–566.

[17]   A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. 94 (1988), 529–573.

Rajender Adibhatla
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg, Germany
E-mail: rajender.adibhatla@mathematik.uni-regensburg.de

Jayanta Manoharmayum
School of Mathematics and Statistics
University of Sheffield
Sheffield S3 7RH, United Kingdom
E-mail: j.manoharmayum@sheffield.ac.uk