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The 4-rank of the tame kernel versus the 4-rank of the narrow class group in quadratic number fields

by

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1. Introduction. In the paper, we mainly investigate the relation between the 4-rank of the tame kernel of a quadratic number field $F=\mathbb{Q}(\sqrt{d})$ and the 4-rank of the narrow class group of a quadratic number field $E=\mathbb{Q}(\sqrt{-d})$.

Let O_F be the ring of integers of a number field F. For a finite Abelian group A, we shall denote by A_2 its 2-Sylow subgroup, by ${}_2A$ its subgroup consisting of elements of order at most 2, by $r_2(A)$ its 2-rank, and by $r_4(A)$ its 4-rank.

A large number of papers have contributed to determining the structure of the 2-Sylow subgroup of K_2O_F. By [2, 4, 9] we have known 2-ranks and 4-ranks of K_2O_F for general number fields F. Specifically, for quadratic fields F, J. Browkin and A. Schinzel [2] have given 2-rank formulas of K_2O_F, and H. Qin [10, 11] has got a method to calculate 4-ranks of K_2O_F. Recently, J. Hurrelbrink and M. Kolster [8] have generalized and improved the results of [10, 11] and have presented an effective way of computing 4-ranks of K_2O_F for these relative quadratic extensions via the F_2-ranks of certain matrices (the analog of the Rédei matrices) of the local Hilbert symbol.

The aim of this paper is to show two formulas: for a real quadratic field $F=\mathbb{Q}(\sqrt{d})$ and an imaginary quadratic field $E=\mathbb{Q}(\sqrt{-d})$,

$$r_4(K_2O_F)=a(F)+r_4(C(E)),$$

where $C(E)$ is the narrow class group of E and $a(F)=-1,0,$ or 1 is determined by F;

$$r_4(K_2O_E)=a(E)+r_4(C(F)),$$

where $C(F)$ is the narrow class group of F and $a(E)=-1,0,$ or 1 is determined by E.

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We directly use the Rédei matrices to get the values of $a(F)$ and $a(E)$. On the other hand, for some imaginary quadratic fields, we give their Tate kernels.

2. Rédei's criteria. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field and D the discriminant of F. We shall denote the narrow class group of F by $C(F)$ and $N_{F/\mathbb{Q}}(F^*)$ by NF. Then

$$r_4(C(F)) = r_2({}_2C(F) \cap C(F)^2).$$

L. Rédei [12] gave a criterion for $r_4(C(F))$. Let $D(F)$ be the set of all squarefree positive integers $q \mid D$. Then $D(F)$ is an elementary Abelian 2-group with multiplication $q_1 \cdot q_2 = q_1 q_2/(q_1, q_2)^2$. For n $(\neq 0) \in \mathbb{Z}$, we denote by $[n]$ the squarefree rational integer satisfying the relation $n = [n]a^2$ for some $a \in \mathbb{Z}$. Let $q \in D(F)$ and $q' = [qD]$. We call q a *D-norm divisor* if $q \in NF$. Then q is a D-norm divisor if and only if the equation $qx^2 - q'y^2 - z^2 = 0$ has a non-trivial solution $x, y, z \in \mathbb{Z}$ if and only if $\left(\frac{q}{p}\right) = 1$ for every odd prime $p \mid q'$, and $\left(\frac{-q'}{p}\right) = 1$ for every odd prime $p \mid q$.

Let $D(NF)$ be the subgroup of $D(F)$ consisting of all D-norm divisors. For $q \in D(F)$, let Q be the ideal of F such that $(q) = Q^2$ and $\mathrm{cl}(Q) \in {}_2C(F)$ be the narrow ideal class containing Q. Rédei proved that $\mathrm{cl}(Q) \in C(F)^2$ if and only if $q \in D(NF)$ by the Gauss theorem and that the map

$$\alpha : D(NF) \to {}_2C(F) \cap C(F)^2, \qquad q \mapsto \mathrm{cl}(Q),$$

is a surjective homomorphism with $|\ker \alpha| = 2$. Hence

$$r_4(C(F)) = r_2(D(NF)) - 1.$$

In particular, if $D < 0$, then $\ker \alpha = \{1, [-D]\}$, and we have $q \in D(NF)$ if and only if $-q' \in D(NF)$.

Rédei also related a criterion for $r_4(C(F))$ to the rank of a certain matrix with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Suppose that a positive integer n is prime to D; we shall write $a = \left(\frac{D}{n}\right)'$ if the Jacobi symbol $\left(\frac{D}{n}\right) = (-1)^a$ with $a \in \mathbb{Z}/2\mathbb{Z}$. The discriminants $p^* = (-1)^{(p-1)/2}p$ (p odd prime), $-4, 8, -8$ ($p = 2$) are called *prime discriminants*. Let $D = p_1^* \ldots p_t^*$ be the unique decomposition of D into a product of prime discriminants. In the case $2 \mid D$, put $p_t = 2$. We define a $t \times t$ square matrix $A_F = (a_{ij})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ by

$$a_{ij} = \begin{cases} \left(\dfrac{p_i^*}{p_j}\right)' & \text{if } i \neq j, \\ \left(\dfrac{D/p_i^*}{p_i}\right)' & \text{if } i = j. \end{cases} \tag{2.1}$$

Note that the sum of all rows of A_F is 0.

Let A'_F be the $(t-1)\times t$ matrix obtained from A_F by deleting the tth row. Then $\operatorname{rank} A'_F = \operatorname{rank} A_F$. By the reciprocity law, we have

$$A'_F = \begin{pmatrix} \left(\frac{D/p_1^*}{p_1}\right)' & \left(\frac{p_1^*}{p_2}\right)' & \cdots & \left(\frac{p_1^*}{p_{t-1}}\right)' & \left(\frac{p_1^*}{p_t^*}\right)' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{p_{t-1}^*}{p_1}\right)' & \left(\frac{p_{t-1}^*}{p_2}\right)' & \cdots & \left(\frac{D/p_{t-1}^*}{p_{t-1}}\right)' & \left(\frac{p_{t-1}^*}{p_t}\right)' \end{pmatrix} \tag{2.2}$$

$$= \begin{pmatrix} \left(\frac{D/p_1^*}{p_1}\right)' & \left(\frac{p_2}{p_1}\right)' & \cdots & \left(\frac{p_{t-1}}{p_1}\right)' & \left(\frac{p_t}{p_1}\right)' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \left(\frac{p_1}{p_{t-1}}\right)' & \left(\frac{p_2}{p_{t-1}}\right)' & \cdots & \left(\frac{D/p_{t-1}^*}{p_{t-1}}\right)' & \left(\frac{p_t}{p_{t-1}}\right)' \end{pmatrix}.$$

For $q \in D(F)$, we define $X_q = {}^t(x_1,\ldots,x_t) \in (\mathbb{Z}/2\mathbb{Z})^t$ by

$$x_i = \begin{cases} 1 & (p_i \mid q) \\ 0 & (p_i \nmid q) \end{cases} \quad (i=1,\ldots,t).$$

Then we have $A'_F X_q = 0$ if and only if $A_F X_q = 0$ if and only if

$$\begin{cases} \left(\dfrac{q}{p}\right) = 1 & \text{for every odd prime } p \mid q', \\ \left(\dfrac{(q/p)(D/p^*)}{p}\right) = 1 & \text{for every odd prime } p \mid q, \end{cases}$$

if and only if $\left(\frac{q}{p}\right) = 1$ for every odd prime $p \mid q'$, and $\left(\frac{-q'}{p}\right) = 1$ for every odd prime $p \mid q$, if and only if the equation $qx^2 - q'y^2 - z^2 = 0$ has a non-trivial solution $x,y,z \in \mathbb{Z}$. Hence, the map

$$\theta : D(NF) \to \{y \in (\mathbb{Z}/2\mathbb{Z})^t \mid A_F X_q = 0\}, \quad q \mapsto X_q,$$

is an isomorphism, and we have

$$r_4(C(F)) = r_2(D(NF)) - 1 = t - 1 - \operatorname{rank} A_F.$$

3. Real quadratic fields. In the section, let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, and $d > 2$ a squarefree integer. J. Browkin and A. Schinzel [2] have given all elements of order 2 of K_2O_F.

LEMMA 3.1. *Let $F = \mathbb{Q}(\sqrt{d})$, $d > 2$ a squarefree integer, and p a fixed odd prime divisor of d. Then all elements of order at most 2 of K_2O_F are of the form*

$$\{-1, m\gamma_j\},$$

where m is an odd divisor of d positive and negative but $p \nmid m$, $\gamma_1 = 1$, and $\gamma_j = u_j + \sqrt{d}$, $u_j^2 - jw_j^2 = d$, $u_j, w_j \in \mathbb{N}$, $j \in \{-1, \pm 2\} \cap NF$.

In [10], H. Qin has given conditions for K_2O_F to have elements of order 4.

LEMMA 3.2. *Let $F = \mathbb{Q}(\sqrt{d})$, $d > 2$ a squarefree integer, and m an odd positive divisor of d.*

(1) *There is a* $\beta \in K_2O_F$ *such that* $\beta^2 = \{-1, m\}$ *if and only if there is an* $\varepsilon \in \{\pm1, \pm2\}$ *such that*

$$\begin{aligned} \left(\frac{\varepsilon d m^{-1}}{p}\right) &= 1 \quad \text{for every odd prime } p \,|\, m, \\ \left(\frac{\varepsilon m}{p}\right) &= 1 \quad \text{for every odd prime } p \,|\, dm^{-1}. \end{aligned} \tag{3.1}$$

(2) *If* $2 \in NF,\ d = u^2 - 2w^2,\ u, w \in \mathbb{N}$, *then there is a* $\beta \in K_2O_F$ *such that* $\beta^2 = \{-1, m(u+\sqrt{d})\}$ *if and only if there is an* $\varepsilon \in \{\pm1\}$ *such that*

$$\begin{aligned} \left(\frac{\varepsilon d m^{-1}(u+w)}{p}\right) &= 1 \quad \text{for every odd prime } p \,|\, m, \\ \left(\frac{\varepsilon m(u+w)}{p}\right) &= 1 \quad \text{for every odd prime } p \,|\, dm^{-1}. \end{aligned} \tag{3.2}$$

In what follows, we shall investigate the conditions (3.1) and (3.2) to set up the relation between the 4-rank of K_2O_F of the real quadratic field $F = \mathbb{Q}(\sqrt{d})$ and the 4-rank of the narrow class group $C(E)$ of the imaginary field $E = \mathbb{Q}(\sqrt{-d})$.

Definition 3.1. Let $F = \mathbb{Q}(\sqrt{d}),\ d > 2$ a squarefree integer. We define

$$\begin{aligned} S_0 &= \{m \mid m \text{ is an odd positive divisor of } d\}, \\ S_1 &= \{\varepsilon m \mid m \in S_0 \text{ and } \varepsilon \in \{1, 2\} \text{ satisfy (3.1)}\}, \\ S_2 &= \{|\varepsilon| m \mid m \in S_0 \text{ and } \varepsilon \in \{-1, -2\} \text{ satisfy (3.1), but } m, 2m \notin S_1\}. \end{aligned}$$

If $2 \in NF$, $d = u^2 - 2w^2,\ u, w \in \mathbb{N}$, we define

$$\begin{aligned} S_1' &= \{m(u+\sqrt{d}) \mid m \in S_0 \text{ and } \varepsilon = 1 \text{ satisfy (3.2)}\}, \\ S_2' &= \{m(u+\sqrt{d}) \mid m \in S_0 \text{ and } \varepsilon = -1 \text{ satisfy (3.2), but } m \notin S_1'\}. \end{aligned}$$

In fact, if -1 or -2 is in NF, then $S_2 = S_2' = \emptyset$. Similarly to $D(F)$, we define $\overline{S}_1 = S_1$, which is an elementary Abelian 2-group, and $\overline{S}_2 = (S_2 \cup S_1)$ is the group generated by the set $S_2 \cup S_1$ with multiplication $m_1 \cdot m_2 = m_1 m_2/(m_1, m_2)^2$. If $2 \in NF$, $u^2 - 2w^2 = d,\ u, w \in \mathbb{N}$, we define $\overline{S}_1' = (S_1' \cup S_1)$ to be the group generated by the set $S_1' \cup S_1$ and $\overline{S}_2' = (S_2' \cup S_1)$ to be the group generated by the set $S_2' \cup S_1$ with multiplication $(m_1(u+\sqrt{d}))\cdot m_2 = (m_1 \cdot m_2)(u+\sqrt{d})$, $(m_1(u+\sqrt{d}))\cdot(m_2(u+\sqrt{d})) = m_1 \cdot m_2$.

Lemma 3.3. *Notations as above.*

(1) *If* $2 \notin NF$, *then* $r_4(K_2O_F) = r_2(S_1) + s - 1$, *where*

$$s = \begin{cases} 1 & \text{if } S_2 \neq \emptyset, \\ 0 & \text{if } S_2 = \emptyset. \end{cases}$$

(2) *If* $2 \in NF$, *then* $r_4(K_2O_F) = r_2(S_1) + s' - 2$, *where*

$$s' = \begin{cases} 2 & \text{if } S_2, S_1, S_2' \text{ are all non-empty,} \\ 1 & \text{if only one of } S_2, S_1', S_2' \text{ is non-empty,} \\ 0 & \text{if } S_2 = S_1' = S_2' = \emptyset. \end{cases}$$

Proof. (1) Let $2 \notin NF$. Suppose $S_2 \neq \emptyset$, so take $m \in S_2$. Then $mS_1 = \{m \cdot m_1 \mid m_1 \in S_1\} = S_2$ and $mS_2 = \{m \cdot m_2 \mid m_2 \in S_2\} = S_1$. Hence $S = \overline{S}_1\overline{S}_2 = (m) \times S_1$. By Lemmas 3.1 and 3.2, the map $\gamma : S \to {}_2K_2O_F \cap (K_2O_F)^2,\ a \mapsto \{-1, a\}$, is a surjective homomorphism of two groups, and $\ker\gamma = (d) \subset S_1$. Therefore $r_4(K_2O_F) = r_2({}_2K_2O_F \cap (K_2O_F)^2) = r_2(S) - 1 = r_2(S_1) - 1 + s$, where $s = 0$ if $S_2' = \emptyset$ or $s = 1$ if $S_2' \neq \emptyset$.

(2) Let $2 \in NF$. Similarly, if $S_i' \neq \emptyset$, then $\overline{S}_i' = (m(u + \sqrt{d})) \times S_1$, where $m(u + \sqrt{d}) \in S_i',\ i = 1, 2$; if two of S_2, S_1', S_2' are non-empty, then the third is non-empty; if S_2, S_1', S_2' are all non-empty, then $S = \overline{S}_1\overline{S}_2\overline{S}_1'\overline{S}_2' = (m) \times (m_1(u + \sqrt{d})) \times S_1$, where $m \in S_2$ and $m_1(u + \sqrt{d}) \in S_1'$. On the other hand, the map $\gamma' : S \to {}_2K_2O_F \cap (K_2O_F)^2,\ a \mapsto \{-1, a\}$, is a surjective homomorphism and $\ker\gamma' = (2) \times (d) \subset S_1$. Hence $r_4(K_2O_F) = r_2({}_2K_2O_F \cap (K_2O_F)^2) = r_2(S) - 2 = r_2(S_1) + s' - 2$, where $s' = 0$ if $S_2 = S_1' = S_2' = \emptyset$, or $s' = 1$ if only one of S_2, S_1', S_2' is non-empty, or $s' = 2$ if S_2, S_1', S_2' are all non-empty.

LEMMA 3.4. *Notations as above. Suppose* $d \equiv -1 \bmod 8$. *Then* $S_2 = \emptyset$ *and* $S_2' = \emptyset$ *if* $2 \in NF$.

Proof. Suppose odd $m \in S_2$. Then $\left(\frac{dm^{-1}}{p}\right) = \left(\frac{-1}{p}\right)$ for every odd prime $p \mid m$, and $\left(\frac{m}{p}\right) = \left(\frac{-1}{p}\right)$ for every odd prime $p \mid dm^{-1}$. By $d \equiv -1 \bmod 8$ and the quadratic reciprocity law, $\left(\frac{dm^{-1}}{m}\right) = \left(\frac{m}{dm^{-1}}\right)$, so $\left(\frac{-1}{m}\right) = \left(\frac{-1}{dm^{-1}}\right)$, which is contradictory. Similarly, we can prove that there is no even $2m \in S_2$.

Let $2 \in NF,\ u^2 - 2w^2 = d,\ u, w \in \mathbb{N}$. Suppose $m(u + \sqrt{d}) \in S_2'$. Then $\left(\frac{m(u+w)}{p}\right) = \left(\frac{-1}{p}\right)$ for every odd prime $p \mid dm^{-1}$ and $\left(\frac{dm^{-1}(u+w)}{p}\right) = \left(\frac{-1}{p}\right)$ for every odd prime $p \mid m$. By $d \equiv -1 \bmod 8$ and the quadratic reciprocity law, $\left(\frac{dm^{-1}}{m}\right) = \left(\frac{m}{dm^{-1}}\right)$. Also $2(u+w)^2 = d + (u+2w)^2$ and let $u + w = 2^i(\overline{u + w})$, where $\overline{u + w}$ is odd. Then $1 = \left(\frac{-d}{\overline{u+w}}\right) = \left(\frac{-mdm^{-1}}{\overline{u+w}}\right)$. Hence $\left(\frac{\overline{u+w}}{dm^{-1}}\right) = \left(\frac{\overline{u+w}}{m}\right)$ by $d \equiv -1 \bmod 8$ and the quadratic reciprocity. Therefore $\left(\frac{-1}{m}\right) = \left(\frac{-1}{dm^{-1}}\right)$, contrary to $d \equiv -1 \bmod 8$.

It is clear that S_1 is related to the group $D(NE)$ of the quadratic field $E = \mathbb{Q}(\sqrt{-d})$, which is defined as in the second section, so we can get the following formula.

THEOREM 3.1. *Let* $F = \mathbb{Q}(\sqrt{d}),\ E = \mathbb{Q}(\sqrt{-d}),\ d > 2$ *a squarefree integer*, *and* $C(E)$ *the* (*narrow*) *class group of* E.

(1) *If* $2 \notin NF$, *then* $r_4(K_2O_F) = r_4(C(E)) + s$, *where*

$$s = \begin{cases} 1 & \text{if } S_2 \neq \emptyset, \text{ or } d \equiv -1 \bmod 8 \text{ and even } 2m \in S_1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If* $2 \in NF$, *then*

$$r_4(K_2O_F) = \begin{cases} r_4(C(E)) + s' - 1 & \text{if } d \not\equiv -1 \bmod 8, \\ r_4(C(E)) + s' & \text{if } d \equiv -1 \bmod 8, \end{cases}$$

where

$$s' = \begin{cases} 2 & \text{if } S_2, S_1, S_2' \text{ are all empty}, \\ 1 & \text{if only one of } S_2, S_1', S_2' \text{ is non-empty}, \\ 0 & \text{if } S_2 = S_1' = S_2' = \emptyset. \end{cases}$$

Moreover, $r_4(K_2O_F) = r_4(C(E)) + a(F)$, *where* $a(F) = -1,\ 0,$ *or* 1 *is determined by* F.

Proof. By Lemmas 3.3 and 3.4, it is sufficient to find the relation between $r_2(S_1)$ and $r_4(C(E))$.

(1) Let $2 \notin NF$. Suppose $d \not\equiv -1 \bmod 4$. Then $2 \mid D$, where D is the discriminant of $E = \mathbb{Q}(\sqrt{-d})$, so $D(NE) = S_1$. Hence $r_4(C(E)) = r_2(D(NF)) - 1 = r_2(S_1) - 1$.

Suppose $d \equiv -5 \bmod 8$. Then $2 \nmid D$. Also there is no even $2m \in S_1$ by the quadratic reciprocity law (or $\left(\frac{2dm^{-1}}{m}\right) = \left(\frac{2m}{dm^{-1}}\right)$, which is contradictory). Hence $D(NE) = S_1$, so $r_4(C(E)) = r_2(S_1) - 1$.

Suppose that $d \equiv -1 \bmod 8$ and there is an even $2m \in S_1$. Then $S_1 = (2m) \times D(NE)$, so $r_4(C(E)) = r_2(S_1) - 2$.

(2) If $2 \in NF$, then $2 \in S_1$. Suppose $d \not\equiv -1 \bmod 4$. Then $2 \mid D$, where D is the discriminant of E, and $D(NE) = S_1$, so $r_4(C(E)) = r_2(S_1) - 1$. Suppose $d \equiv -1 \bmod 8$. Then $2 \nmid D$ $(= -d)$ and $S_1 = (2) \times D(NE)$, so $r_4(C(E)) = r_2(S_1) - 2$.

In Theorem 3.1, in order to get the value of $r_4(K_2O_F)$ clearly, we use the Rédei matrix to determine if S_2, S_1', S_2' are empty.

THEOREM 3.2. *Let* $F = \mathbb{Q}(\sqrt{d})$, $E = \mathbb{Q}(\sqrt{-d})$, *and* $d > 2$ *a squarefree integer.*

(1) *If* $2 \notin NF$ *and* $d \equiv -1 \bmod 8$, *then there is an even* $2m \in S_1$ *if and only if the system of equations*

$$A_E' X = B' \tag{3.3}$$

is solvable, where $B' = {}^t\left(\left(\frac{2}{p_1}\right)', \ldots, \left(\frac{2}{p_{t-1}}\right)'\right)$ *and* A_E' *is defined as* (2.2).

(2) *If* $-1, -2 \notin NF$, *then* $S_2 = \emptyset$ *if and only if the system* (3.3) *has no solution, where* $B' = {}^t\left(\left(\frac{-1}{p_1}\right)', \ldots, \left(\frac{-1}{p_{t-1}}\right)'\right)$ *if* $d \not\equiv -1 \bmod 4$ *and* $B' = {}^t\left(\left(\frac{-2}{p_1}\right)', \ldots, \left(\frac{-2}{p_{t-1}}\right)'\right)$ *if* $d \equiv 3 \bmod 8$.

(3) *If* $2 \in NF$, *then* $S_1' = \emptyset$ *if and only if the system* (3.3) *has no solution*, *where* $B' = {}^t\big(\big(\frac{u+w}{p_1}\big)', \ldots, \big(\frac{u+w}{p_{t-1}}\big)'\big)$.

(4) *If* $2 \in NF$, $-1 \notin NF$, *and* $d \not\equiv -1 \bmod 8$, *then* $S_2' = \emptyset$ *if and only if the system* (3.3) *has no solution*, *where* $B' = {}^t\big(\big(\frac{-u-w}{p_1}\big)', \ldots, \big(\frac{-u-w}{p_{t-1}}\big)'\big)$.

Proof. (1) If $d \equiv -1 \bmod 8$ and $2 \notin NF$, then $D = -d$ is the discriminant of E and $1 = \big(\frac{2}{d}\big) = \big(\frac{2}{p_1}\big) \ldots \big(\frac{2}{p_t}\big)$. For $2m \in S_1$, we define $X_m = {}^t(x_1, \ldots, x_t) \in (\mathbb{Z}/2\mathbb{Z})^t$ by

$$x_i = \begin{cases} 1 & \text{if } p_i \mid m, \\ 0 & \text{if } p_i \nmid m, \end{cases}$$

where $i = 1, \ldots, t$. So we have $A_E' X_m = B'$, where $B' = {}^t\big(\big(\frac{2}{p_1}\big)', \ldots, \big(\frac{2}{p_{t-1}}\big)'\big)$, if and only if $A_E X_m = B$, where $B = {}^t\big(\big(\frac{2}{p_1}\big)', \ldots, \big(\frac{2}{p_t}\big)'\big)$, if and only if

$$\begin{cases} \left(\dfrac{m}{p}\right) = \left(\dfrac{2}{p}\right) & \text{for every prime } p \mid dm^{-1}, \\ \left(\dfrac{dm^{-1}}{p}\right) = \left(\dfrac{2}{p}\right) & \text{for every prime } p \mid m, \end{cases}$$

if and only if $2m \in S_1$.

(2) Suppose $d \not\equiv -1 \bmod 4$ and $-1, -2 \notin NF$. Then $D = -4d$ is the discriminant of E and $p_t = 2$. For $m \in S_0$ and $\varepsilon \in \{1, 2\}$, we have $A_E' X_{\varepsilon m} = B'$, where $X_{\varepsilon m}$ is defined as above and $B' = {}^t\big(\big(\frac{-1}{p_1}\big)', \ldots, \big(\frac{-1}{p_{t-1}}\big)'\big)$, if and only if

$$\begin{cases} \left(\dfrac{\varepsilon m}{p}\right) = \left(\dfrac{-1}{p}\right) & \text{for every prime } p\ (\neq p_t) \mid dm^{-1}, \\ \left(\dfrac{4d(\varepsilon m)^{-1}}{p}\right) = \left(\dfrac{-1}{p}\right) & \text{for every prime } p\ (\neq p_t) \mid m, \end{cases}$$

if and only if $\varepsilon m \in S_2$.

Suppose $d \equiv 3 \bmod 8$ and $-1, -2 \notin NF$. Then $D = -d$ is the discriminant of E, odd $m \notin S_2$ by the quadratic reciprocity law, and $1 = \big(\frac{-2}{d}\big) = \big(\frac{-2}{p_1}\big) \ldots \big(\frac{-2}{p_t}\big)$. Similarly to (1), we can get the second part of (2).

(3) If $2 \in NF$, $d = u^2 - 2w^2$, $u, w \in \mathbb{N}$, and $2(u+w)^2 = d + (u+2w)^2$, we need only consider the case of $d \equiv -1 \bmod 8$. Let $u + w = 2^i \overline{u+w}$, where $\overline{u+w}$ is odd. Then

$$1 = \left(\frac{-d}{\overline{u+w}}\right) = \left(\frac{\overline{u+w}}{d}\right) = \left(\frac{u+w}{p_1}\right) \cdots \left(\frac{u+w}{p_t}\right)$$

by $2 \in NF$, $d \equiv -1 \bmod 8$, and the quadratic reciprocity law. Similarly to (1), we can get (3).

(4) It is clear.

Corollary 3.1. *Let* $F = \mathbb{Q}(\sqrt{d})$, $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ *a squarefree integer*, -1 *or* $-2 \in NF$, *and* $C(E)$ *the* (*narrow*) *class group of* E.

(1) *If* $2 \notin NF$, *then the* 2-*Sylow subgroup of* K_2O_F *is elementary Abelian if and only if* $r_4(C(E)) = 0$.

(2) *If* $2 \in NF$, *then the* 2-*Sylow subgroup of* K_2O_F *is elementary Abelian if and only if* $r_4(C(E)) = 1$ *and the system* (3.3) *is not solvable*, *where* $B' = {}^t\big(\big(\frac{u+w}{p_1}\big)', \ldots, \big(\frac{u+w}{p_{t-1}}\big)'\big)$.

Proof. Since -1 or -2 is in NF, $d \not\equiv -1 \bmod 8$ by the quadratic reciprocity law. If $2 \notin NF$, by Theorem 3.1, we can get (1). If $2 \in NF$, then $d \equiv 1$ or $2 \bmod 8$ and $r_4(C(E)) \geq 1$, so we can get (2) by Theorem 3.1.

4. Imaginary quadratic field. For an imaginary quadratic field $E = \mathbb{Q}(\sqrt{-d})$, by [14], we have $[\Delta_E : E^{*2}] = 4$, where $\Delta_E = \{a \in E^* \mid \{-1, a\} = 1\}$ is called the *Tate kernel* of E.

J. Browkin and A. Schinzel [2] have given all elements of order 2 of K_2O_E.

Lemma 4.1. *Let* $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ *a squarefree integer. Then all elements of order at most* 2 *of* K_2O_F *are of the form*

$$\{-1, m\gamma_j\}, \qquad j = 1, 2,$$

where m *is an odd positive divisor of* D, $\gamma_1 = 1$, *and* $\gamma_2 = u + \sqrt{-d}$, $-d = u^2 - 2w^2$, $u, w \in \mathbb{N}$. *Moreover there is a unique* $m\gamma_j$ $(\neq 1) \in \Delta_E$.

In [11], H. Qin has given conditions for K_2O_E to have elements of order 4.

Lemma 4.2. *Let* $E = \mathbb{Q}(\sqrt{-d})$, $F = \mathbb{Q}(\sqrt{d})$, $d > 2$ *a squarefree integer*, *and* m *an odd positive divisor of* d.

(1) *There is a* $\beta \in K_2O_E$ *such that* $\beta^2 = \{-1, m\}$ *if and only if there is* $\varepsilon \in \{1, 2\}$ *such that* $\varepsilon m \in NF$.

(2) *If* $2 \in NE$, $-d = u^2 - 2w^2$, $u, w \in \mathbb{N}$, *then there is a* $\beta \in K_2O_E$ *such that* $\beta^2 = \{-1, m(u + \sqrt{-d})\}$ *if and only if* $m(u + w) \in NF$.

Definition 4.1. Let $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ a squarefree integer. We define

$$S_0 = \{m \mid m \text{ is an odd positive divisor of } d\},$$
$$T = \{\varepsilon m \in NF \mid m \in S_0 \text{ and } \varepsilon \in \{1, 2\}\}.$$

If $2 \in NE$, $-d = u^2 - 2w^2$, $u, w \in \mathbb{N}$, we define

$$T' = \{m(u + \sqrt{-d}) \mid m \in S_0 \text{ and } m(u + w) \in NF\}.$$

Similarly, T is the group with multiplication $m_1 \cdot m_2 = m_1 m_2/(m_1, m_2)^2$, and $\overline{T}' = (T' \cup T)$ is the group generated by the set $T' \cup T$ with multiplication $m_1(u + \sqrt{-d}) \cdot m_2(u + \sqrt{-d}) = m_1 \cdot m_2$, $m_1 \cdot m_2(u + \sqrt{-d}) = (m_1 \cdot m_2) \cdot (u + \sqrt{-d})$. In fact, if $T' \neq \emptyset$, then $\overline{T}' = (m(u + \sqrt{-d})) \times T$, where

$m(u+\sqrt{-d}) \in T'$. Note that, by [11], there is a δ $(\neq 1,2) \in T \cup T'$ such that $\delta \in \Delta_E$.

LEMMA 4.3. *Notations as above.*

(1) *If* $2 \notin NE$, *then* $r_4(K_2O_E) = r_2(T) - 1$.
(2) *If* $2 \in NE$, *then* $r_4(K_2O_E) = r_2(T) + s - 2$, *where*

$$s = \begin{cases} 1 & \text{if } T' \neq \emptyset, \\ 0 & \text{if } T' = \emptyset. \end{cases}$$

Proof. (1) If $2 \notin NE$, then $\alpha : T \to {}_2K_2O_E \cap (K_2O_E)^2,\ a \mapsto \{-1, a\}$, is a surjective homomorphism and $\ker \alpha = \{1, \varepsilon m\}$, where $\{-1, \varepsilon m\} = 1$ and $\varepsilon m \neq 1, 2$. Hence $r_4(K_2O_F) = r_2(T) - 1$.

(2) If $2 \in NF$, then $\alpha : \overline{T}' \to {}_2(K_2O_E) \cup (K_2O_E)^2$, $\varepsilon m\gamma_j \mapsto \{-1, \varepsilon m\gamma_j\}$, $j = 1, 2$, is surjective homomorphism and $\ker \alpha = \{1, 2, 2m\gamma_j, m\gamma_j\}$, where $\{-1, m\gamma_j\} = 1$ and $m\gamma_j \neq 1, 2$. Hence $r_4(K_2O_E) = r_2(\overline{T}') - 2 = r_2(T) + s - 2$, where $s = 1$ if $T' \neq \emptyset$ or $s = 0$ if $T' = \emptyset$.

THEOREM 4.1. *Let* $F = \mathbb{Q}(\sqrt{d})$, $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ *a squarefree integer*, *and* $C(F)$ *the narrow class group of* F.

(1) *If* $2 \notin NE$, *then* $r_4(K_2O_E) = r_4(C(F)) + s$, *where*

$$s = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 8 \text{ and } 2m \in T, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If* $2 \in NE$, *then*

$$r_4(K_2O_E) = \begin{cases} r_4(C(F)) + s' & \text{if } d \equiv 1 \bmod 8, \\ r_4(C(F)) + s' - 1 & \text{if } d \not\equiv 1 \bmod 8, \end{cases}$$

where

$$s' = \begin{cases} 1 & \text{if } T' \neq \emptyset, \\ 0 & \text{if } T' = \emptyset. \end{cases}$$

Moreover, $r_4(K_2O_E) = r_4(C(F)) + a(E)$, *where* $a(E) = -1, 0$, *or* 1 *is determined by* E.

Proof. By Lemma 4.3, the relation between T and $D(NF)$, and by $r_4(C(F)) = r_2(D(NF)) - 1$, we get the result.

COROLLARY 4.1. *Notations as above.*

(1) *If* $2 \notin NE$, *then* $r_4(K_2O_E) = 0$ *if and only if* $r_4(C(F)) = 0$, *and* $2m \notin T$ *if* $d \equiv 1 \bmod 8$.

(2) *If* $2 \in NE$ *and* $d \equiv 1 \bmod 8$, *then* $r_4(K_2O_E) = 0$ *if and only if* $r_4(C(F)) = 0$ *and* $T' = \emptyset$.

(3) *If* $2 \in NE$ *and* $d \not\equiv 1 \bmod 8$, *then* $r_4(K_2O_F) = 0$ *if and only if* $r_4(C(F)) = 1$ *and* $T' = \emptyset$, *or* $r_4(C(F)) = 0$ *and* $T' \neq \emptyset$.

THEOREM 4.2. *Let* $F = \mathbb{Q}(\sqrt{d})$, $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ *a squarefree integer*, $C(F)$ *the narrow class group of* F, *and* A'_F *defined as in* (2.2).

(1) *If* $2 \notin NE$ *and* $d \equiv 1 \bmod 8$, *then there is even* $2m \in T$ *if and only if the system of equations*

$$A'_F X = B' \tag{4.1}$$

is solvable, *where* A'_F *is defined as in* (2.2) *and* $B' = {}^t\big(\big(\frac{2}{p_1}\big)', \ldots, \big(\frac{2}{p_{t-1}}\big)'\big)$.

(2) *If* $2 \in NF$, $-d = u^2 - 2w^2$, $u, w \in \mathbb{N}$, *then* $T' \neq \emptyset$ *if and only if the system* (4.1) *is solvable*, *where* $B' = {}^t\big(\big(\frac{u+w}{p_1}\big)', \ldots, \big(\frac{u+w}{p_{t-1}}\big)'\big)$.

Proof. Proceed as in the proof of Theorem 3.2.

Let $F = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. By genus theory, there is a unique q $(\neq 1) \in D(NF)$ such that $Q^2 = (q)$ and $\mathrm{cl}(Q) = 1$ in the narrow class group $C(F)$. We call the q the *dependent divisor* of ambiguous ideals of F. Suppose $r_4(K_2O_E) = 0$. We set up a relation between the Tate kernel of K_2O_E and the dependent divisor of ambiguous ideals of F.

THEOREM 4.3. *Let* $F = \mathbb{Q}(\sqrt{d})$, $E = \mathbb{Q}(\sqrt{-d})$, $d > 2$ *a squarefree integer. Suppose* $r_4(K_2O_E) = 0$. *Then*, *if* q $(\neq 2)$ *is the dependent divisor of ambiguous ideals of* F, $\Delta_E = (\{2, q\})E^{*2}$; *if* 2 *is the dependent divisor of ambiguous ideals of* F, $\Delta_E = (\{2, m(u + \sqrt{-d})\})E^{*2}$, *where* $m(u + \sqrt{-d}) \in T'$.

Proof. If $2 \notin NE$ and $r_4(K_2O_E) = 0$, then, by Corollary 4.1, $r_4(C(F)) = 0$, and $2m \notin T$ if $d \equiv 1 \bmod 8$. Hence $\operatorname{rank} A_F = t - 1$ and there is a unique q $(\neq 1, 2) \in D(NF) = T$ such that $A_F X_q = 0$. Therefore q is the dependent divisor of ambiguous ideals of F and $q \in \Delta_E$.

If $2 \in NE$, $d \equiv 1 \bmod 8$ and $r_4(K_2O_E) = 0$, then by Corollary 4.1, we have the same result as above.

If $2 \in NE$, $d \not\equiv 1 \bmod 8$ and $r_4(K_2O_E) = 0$, then by Corollary 4.1, we need to consider two cases.

The first case: $r_4(C(F)) = 1$ and $T' = \emptyset$. Then $\operatorname{rank} A_F = t - 2$ and $\overline{T}' = T = D(NF)$. Hence $D(NF) = \{1, 2, q, 2q\}$. Suppose that 2 is the dependent divisor of ambiguous ideals of F. Since $2(u+w)^2 = (u+2w)^2 - d$, we have $((u+2w) + \sqrt{d}) = Q_2 Q_{u+w}^2$, where Q_2 and Q_{u+w} are ideals of F with $Q_2^2 = (2)$ and $Q_{u+w}Q'_{u+w} = (u+w)$. Then $\mathrm{cl}(Q_{u+w}2)^2 = \mathrm{cl}(Q_2) = 1$. Hence, by genus theory, $\mathrm{cl}(Q_{u+w}) = \mathrm{cl}(Q_m)$, where Q_m is an ideal of F with $Q_m^2 = (m)$ and $m \in D(F)$. So $\mathrm{cl}(Q_{u+w}Q_m) = 1 \in C(F)^2$ and $m(u+w) = N_{F/\mathbb{Q}}(Q_{u+w}Q_m) \in NF$, contrary to $T' = \emptyset$. Therefore, q or $2q$ is the dependent divisor of ambiguous ideals of F and q, $2q \in \Delta_E$.

The second case: $r_4(C(F)) = 0$ and $T' \neq \emptyset$. Then $\operatorname{rank} A_F = t - 1$, $D(NF) = T = \{1, 2\}$, and $\overline{T}' = (m(u+\sqrt{-d})) \times T$. Hence 2 is the dependent divisor of ambiguous ideals of F and $m(u + \sqrt{-d}) \in \Delta_E$.

Question. Suppose $r_4(K_2O_E) \geq 1$. Do we have results similar to Theorem 4.3?

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