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On the Iwasawa λ-invariants of real quadratic fields

by

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1. Introduction. Let k be a number field and $k_\infty = \bigcup_{n\ge 0} k_n$ be a $\mathbb{Z}_p$-extension of k for a prime p. Let A_n be the Sylow p-subgroup of the ideal class group of k_n and e_n be the exact power of p of ${}^{\#}A_n$, i.e., ${}^{\#}A_n = p^{e_n}$. It is well known that there are integers $\mu, \lambda \ge 0$ and ν, called Iwasawa invariants of k_∞/k, such that $e_n = \mu p^n + \lambda n + \nu$ for $n \gg 0$ ([4]).

In [3], Greenberg conjectured that $\mu = \lambda = 0$ if k is a totally real field and gave several examples supporting the conjecture. Then in 1979, Ferrero and Washington ([1]) proved that $\mu = 0$ if k is an abelian field and k_∞ is the basic $\mathbb{Z}_p$-extension of k. Since then, a lot of results have been published on the Iwasawa invariants including some recent work on the λ-invariant ([8], [12]). Greenberg's conjecture on the λ-invariant, however, still remains open even when k is a real quadratic field.

In this paper we will study the λ-invariant when k is a real quadratic field. One of the advantages of studying ideal class groups of abelian fields as compared with other fields is that the former have circular units which carry information about the class number. More precisely, let E_n be the group of units and C_n the group of circular units of k_n defined by Sinnott ([10]). Then the index theorem of Sinnott says that ${}^{\#}A_n = {}^{\#}B_n$ if p is an odd prime, where B_n is the Sylow p-subgroup of E_n/C_n.

From now on, we let k be a real quadratic field and k_∞ be its $\mathbb{Z}_p$-extension. Let S be the set of primes consisting of 2 and the prime factors of the conductor and the class number of k. The aim of this paper is to find a criterion for the vanishing of the λ-invariant λ_p of the $\mathbb{Z}_p$-extension k_∞ over k for $p \notin S$.

If p remains inert in k, then $A_n = 0$ for all $n \ge 0$ since $p \nmid h_0$ ([11]), and so $\lambda_p = 0$. Thus, throughout this paper, we will always assume that p splits in k. When p splits in k, indivisibility of h_0 by p does not imply $A_n = 0$

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for all $n \geq 0$. For instance, if $k = \mathbb{Q}(\sqrt{85})$ and $p = 3$, then $A_0 = \{0\}$ but $A_1 \neq \{0\}$ ([6]). Despite this example, we can still hope that $\lambda_3 = 0$.

We briefly explain the main theorems of this paper. Let $G_n = \mathrm{Gal}(k_n/k)$ and $\Gamma = \varprojlim G_n = \mathrm{Gal}(k_\infty/k)$. It is known that the Tate cohomology group $\widehat{H}^0(G_n, C_n)$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ ([6]). Let δ be a circular unit of k which generates $\widehat{H}^0(G_n, C_n)$. We may assume that $\delta \equiv 1 \bmod p$, for otherwise we can replace δ by δ^{p-1}. To be more precise, we may let $\delta = \prod_{\tau\in\Delta}(1-\zeta_d^\tau)^{\chi(\tau)(p-1)}$, where $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ and ζ_d is a primitive dth root of 1 ([6]). Let v_p be the p-adic valuation normalized by $v_p(p) = 1$. Then we have

THEOREM 1. *Let $p \notin S$. If $v_p(\delta - 1) = \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} + 1$, then $\lambda_p = 0$.*

As an application of Theorem 1, we examine the following special case. As was mentioned, $^{\#}A_n = {}^{\#}B_n$. The groups A_n and B_n are known to be more deeply related. Indeed, as $\mathrm{Gal}(k_\infty/k)$-modules, $\varprojlim A_n$ and $\varprojlim B_n$ have the same characteristic ideals by the main conjecture which was proved by Mazur and Wiles ([7]). And it is an open question if $A_n \simeq B_n$ as abelian groups or as G_n-modules. We prove

THEOREM 2. *Let $p \notin S$. Let M be the integer such that $v_p(\delta-1) = M+1$. If $^{\#}A_M^{G_M} = {}^{\#}B_M^{G_M}$, then $\lambda_p = 0$.*

Finally, in Theorem 3, we consider the p-adic L-function $L(s,\chi)$ attached to the nontrivial character χ of k. It is known that if $v_p(L_p(1,\chi)) = 0$ (i.e., $p \nmid L_p(1,\chi)$), then $A_n = \{0\}$ for all $n \geq 0$ and thus $\lambda_p = 0$ ([6]). In Theorem 3, we generalize this.

THEOREM 3. *Let $p \notin S$. If $v_p(L_p(1,\chi)) \leq 1$, then $\lambda_p = 0$.*

2. Lemmas. In this section, we prove several lemmas on cohomology groups of units and circular units in the $\mathbb{Z}_p$-extension of a real quadratic field k. We keep assuming that p splits in k and $p \notin S$. In particular, p does not divide the class number h_0 of k.

LEMMA 1. *Let $G_n = \mathrm{Gal}(k_n/k)$ and $\Gamma = \varprojlim G_n = \mathrm{Gal}(k_\infty/k)$. Let $C_\infty = \bigcup_{n\geq 0} C_n$ and $E_\infty = \bigcup_{n\geq 0} E_n$. Then*

(1) $C_n^{G_n} = C_0$,
$\widehat{H}^0(G_n, C_n) \simeq \mathbb{Z}/p^n\mathbb{Z}$,
$H^1(G_n, C_n) \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$,
$H^1(\Gamma, C_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2$,
$H^2(\Gamma, C_\infty) \simeq \mathbb{Q}_p/\mathbb{Z}_p$.

(2) $H^1(\Gamma, E_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2$,
$H^2(\Gamma, E_\infty) \simeq \mathbb{Q}_p/\mathbb{Z}_p$.

Proof. For (1), we refer to [6]. In [5], Iwasawa proved that $H^2(\Gamma, E_\infty) \simeq \mathbb{Q}_p/\mathbb{Z}_p$ and that $H^1(\Gamma, E_\infty) = (\mathbb{Q}_p/\mathbb{Z}_p)^2 \oplus H$ for some finite group H.

Let I_n be the ideal group of k_n and P_n the subgroup of I_n generated by the principal ideals.

From $0 \to E_n \to k_n^\times \to P_n \to 0$, we have

$$0 \to E_n^{G_n} \to {k_n^\times}^{G_n} \to P_n^{G_n} \to H^1(G_n, E_n) \to H^1(G_n, k_n^\times) \to \\ H^1(G_n, P_n) \to \widehat{H}^0(G_n, E_n) \to \widehat{H}^0(G_n, k_n^\times) \to \ldots$$

Note that $E_n^{G_n} \simeq E_0$, ${k_n^\times}^{G_n} \simeq k_0^\times$ and $H^1(G_n, k_n^\times) \simeq 0$. Hence

$$0 \to k_0^\times/E_0 \to P_n^{G_n} \to H^1(G_n, E_n) \to 0$$

and

$$0 \to H^1(G_n, P_n) \to \widehat{H}^0(G_n, E_n) \to \widehat{H}^0(G_n, k_n^\times) \to \ldots$$

Thus we have

$$H^1(G_n, E_n) \simeq P_n^{G_n}/P_0$$

and

$$H^1(G_n, P_n) \simeq \ker(\widehat{H}^0(G_n, E_n) \to \widehat{H}^0(G_n, k_n^\times)).$$

From $0 \to P_n \to I_n \to I_n/P_n \to 0$, we have

$$0 \to P_n^{G_n} \to I_n^{G_n} \to (I_n/P_n)^{G_n} \to H^1(G_n, P_n) \to H^1(G_n, I_n) \simeq 0 \to \ldots$$

Hence

$$0 \to P_n^{G_n}/P_0 \to I_n^{G_n}/P_0 \to (I_n/P_n)^{G_n} \to H^1(G_n, P_n) \to 0.$$

Therefore

$$0 \to H^1(G_n, E_n) \to I_n^{G_n}/P_0 \to (I_n/P_n)^{G_n} \to \\ \ker(\widehat{H}^0(G_n, E_n) \to \widehat{H}^0(G_n, k_n^\times)) \to 0.$$

We also have an exact sequence

$$0 \to I_0/P_0 \to I_n^{G_n}/P_0 \to I_n^{G_n}/I_0 \to 0.$$

By tensoring the above sequence with $\mathbb{Z}_p$, we get

$$I_n^{G_n}/P_0 \otimes \mathbb{Z}_p \simeq I_n^{G_n}/I_0 \otimes \mathbb{Z}_p \simeq I_n^{G_n}/I_0 \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$$

since $p \nmid h_0$. Therefore, we obtain

$$(*) \quad 0 \to H^1(G_n, E_n) \to I_n^{G_n}/I_0 \to (I_n/P_n)^{G_n} \to \\ \ker(\widehat{H}^0(G_n, E_n) \to \widehat{H}^0(G_n, k_n^\times)) \to 0.$$

For $m > n$, we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(G_n, E_n) & \longrightarrow & I_n^{G_n}/I_0 \simeq (\mathbb{Z}/p^n\mathbb{Z})^2 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(G_m, E_m) & \longrightarrow & I_m^{G_m}/I_0 \simeq (\mathbb{Z}/p^m\mathbb{Z})^2 \end{array}$$

By taking direct limits, we have

$$0 \to \varinjlim H^1(G_n, E_n) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2 \oplus H \to \varinjlim (\mathbb{Z}/p^n\mathbb{Z})^2 \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2.$$

Hence H must be trivial, and so

$$H^1(\Gamma, E_\infty) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2.$$

LEMMA 2. *The induced homomorphism $H^1(\Gamma, C_\infty) \to H^1(\Gamma, E_\infty)$ is surjective.*

Proof. Let $B_n = E_n/C_n \otimes \mathbb{Z}_p$ be the Sylow p-subgroup of E_n/C_n. It is known ([2]) that the natural map $E_n/C_n \to E_m/C_m$ for $m > n$ is injective. Let $B_\infty = \varinjlim B_n$ be the direct limit under the natural injection. So $B_\infty = \varinjlim (E_n/C_n \otimes \mathbb{Z}_p) = E_\infty/C_\infty \otimes \mathbb{Z}_p$. From $0 \to C_\infty \to E_\infty \to E_\infty/C_\infty \to 0$, we have $0 \to C_\infty \otimes \mathbb{Z}_p \to E_\infty \otimes \mathbb{Z}_p \to B_\infty \to 0$. Then we obtain a long exact sequence

$$0 \to B_\infty^\Gamma/B_0 \to H^1(\Gamma, C_\infty) \to H^1(\Gamma, E_\infty) \to H^1(\Gamma, B_\infty) \to H^2(\Gamma, C_\infty)$$
$$\to H^2(\Gamma, E_\infty) \to H^2(\Gamma, B_\infty) \to \ldots$$

Note that $B_\infty^{\Gamma_n}$ is finite ([11], Lemma 15.39, Proposition 15.44), where $\Gamma_n = \mathrm{Gal}(k_\infty/k_n)$. Thus B_∞^Γ is also finite. Also note that $H^2(\Gamma, B_\infty) = \{0\}$, since B_∞ is a torsion group ([9], Proposition 3.25, Example 17). Hence the above long exact sequence reads

$$0 \to \text{finite} \to (\mathbb{Q}_p/\mathbb{Z}_p)^2 \to (\mathbb{Q}_p/\mathbb{Z}_p)^2 \to H^1(\Gamma, B_\infty) \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Therefore $H^1(\Gamma, B_\infty)$ is a finite group. Since $(\mathbb{Q}_p/\mathbb{Z}_p)^2$ has no nontrivial cokernel, $H^1(\Gamma, C_\infty) \to H^1(\Gamma, E_\infty)$ is surjective.

LEMMA 3. *The induced maps $H^1(G_n, C_n) \to H^1(G_n, E_n)$ and $\widehat{H}^0(G_n, C_n) \to \widehat{H}^0(G_n, E_n)$ are surjective for all $n \geq 1$.*

Proof. From $0 \to C_n \to E_n \to E_n/C_n \to 0$, we get a long exact sequence

$$0 \to B_0 \to B_n^{G_n} \to H^1(G_n, C_n) \to H^1(G_n, E_n) \xrightarrow{f} H^1(G_n, B_n) \to .$$

Since $p \nmid h_0$, we have $B_0 = \{0\}$. Then consider the following commutative diagram:

$$\begin{array}{ccccccc}
\longrightarrow H^1(G_n, C_n) & \longrightarrow & H^1(G_n, E_n) & \xrightarrow{f} & H^1(G_n, B_n) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\longrightarrow H^1(\mathit{\Gamma}, C_\infty) & \longrightarrow & H^1(\mathit{\Gamma}, E_\infty) & \xrightarrow{\text{zero map}} & H^1(\mathit{\Gamma}, B_\infty) & \longrightarrow
\end{array}$$

where vertical maps are inflations. From the injectivity of the inflation map $H^1(G_n, B_n) \to H^1(\mathit{\Gamma}, B_\infty)$, one can easily see that f is a zero map. Hence $H^1(G_n, C_n) \to H^1(G_n, E_n)$ is surjective. The surjectivity of $\widehat{H}^0(G_n, C_n) \simeq C_0/N_nC_n \to \widehat{H}^0(G_n, E_n) \simeq E_0/N_nE_n$ follows immediately from the assumption $p \nmid h_0$, where N_n is the norm map from k_n to k_0.

3. λ-invariant. In this section, we will prove the main theorems stated in the introduction. Since p splits in k, the completion of k at a prime $\wp$ above p is $\mathbb{Q}_p$. We denote the completion of k_n at the prime of k_n above $\wp$ by $\mathbb{Q}_{p,n}$. Let $\delta = \prod_{\tau \in \mathit{\Delta}} (1 - \zeta_d^\tau)^{\chi(\tau)(p-1)}$ be a circular unit of k which generates $\widehat{H}^0(G_n, C_n)$ as in the introduction.

Theorem 1. *Let* $p \notin S$. *If* $v_p(\delta - 1) = \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} + 1$, *then* $\lambda_p = 0$.

Remark. Since $N_mE_m \subset N_nE_n$ for $m > n$, $\widehat{H}^0(G_n, E_n)$ is a quotient group of $\widehat{H}^0(G_m, E_m)$. Thus if $\widehat{H}^0(G_m, E_m) = 0$, then $\widehat{H}^0(G_n, E_n) = 0$. And since $E_0/\bigcap_{n \ge 0} N_nE_n \simeq \mathbb{Q}_p/\mathbb{Z}_p$ (we are assuming that p splits in k), $\widehat{H}^0(G_m, E_m) \ne 0$ for sufficiently large m. Thus $\max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\}$ is well defined. Also note that $\widehat{H}^0(G_t, E_t)$ is generated by δ since $\widehat{H}^0(G_t, C_t) \to \widehat{H}^0(G_t, E_t)$ is surjective. Hence, if $\widehat{H}^0(G_t, E_t) = \{0\}$, then $\delta = N_t(\eta_t)$ for some $\eta_t \in k_t$. By reading this equation in the completion, i.e., $\eta_t \in \mathbb{Q}_{p,t}$ and $N_t = N_{\mathbb{Q}_{p,t}/\mathbb{Q}_p}$, we get $\delta \equiv 1 \bmod p^{t+1}$. Therefore $v_p(\delta - 1) \ge \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} + 1$.

Proof of Theorem 1. We will prove the vanishing of λ_p by showing that every ideal in k_n capitulates in k_∞. Let D_n be the kernel of the natural map $j_n : A_n \to \varinjlim A_m$. Since G_n acts on D_n, A_n/D_n is a G_n-module. To prove $A_n/D_n = \{0\}$, it is enough to show that $(A_n/D_n)^{G_n} = \{0\}$. Let C be a class in A_n such that $C^{\sigma - 1} \in D_n$, where σ is a topological generator of $\mathit{\Gamma} = \mathrm{Gal}(k_\infty/k)$. Thus $C^{\sigma-1} = 0$ in A_m for $m \gg n$. Let $j_{n,m} : A_n \to A_m$ be the natural map and write $j_{n,m}(C) = C'$. Let $\mathfrak{a}$ be an ideal of k_m which represents C'. Thus $\mathfrak{a}^{\sigma-1} = (\alpha)$ for some $\alpha \in k_m$.

Since $(N_m(\alpha)) = N_m\mathfrak{a}^{\sigma-1} = (1)$, $N_m(\alpha) = \varepsilon$ is a unit in k. Let a, $0 \le a < p^m$, be such that $\varepsilon \bmod N_mE_m = \delta^a \bmod N_mE_m$. That is, $\varepsilon = \delta^a N_m(\eta_m)$

for some unit $\eta_m \in E_m$. This is possible by Lemma 3. Then $\delta^a = N_m(\alpha\eta_m^{-1})$. Thus δ^a is a norm from $\mathbb{Q}_{p,m}$ to $\mathbb{Q}_p$. Therefore $\delta^a \equiv 1 \bmod p^{m+1}$.

Let $M = \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} = v_p(\delta - 1) - 1$. So $\delta \equiv 1 \bmod p^{M+1}$ but $\delta \not\equiv 1 \bmod p^{M+2}$. Since $\delta^a \equiv 1 \bmod p^{m+1}$, we must have $a \equiv 0 \bmod p^{m-M}$. Let $a = p^{m-M}b$. Then $\varepsilon = \delta^a N_m(\eta_m) = (\delta^b)^{p^{m-M}} N_m(\eta_m)$. Since $\widehat{H}^0(G_M, E_M) = 0$, $\delta^b = N_M(\eta_M)$ for some $\eta_M \in E_M$. Hence $\delta^a = (\delta^b)^{p^{m-M}} = N_m(\eta_M)$, and thus $\varepsilon = N_m(\eta_M\eta_m)$. Therefore $\varepsilon = N_m(\alpha) = N_m(\eta)$ for a unit $\eta = \eta_M\eta_m \in E_m$. So $\alpha = \eta\beta^{1-\sigma}$ for some $\beta \in k_m$. Hence $\mathfrak{a}^{\sigma-1} = (\alpha) = (\beta)^{1-\sigma}$, which means that $\mathfrak{a}(\beta)$ is fixed under G_m. Thus $\mathfrak{a}(\beta)$ is a product of ideals from k_0 and primes above p. Since primes above p capitulate, $j_m(C') = 0$. Therefore $j_n(C) = 0$. This completes the proof.

THEOREM 2. *Let $p \notin S$. Let M be the integer such that $v_p(\delta-1) = M+1$. If ${}^{\#}A_M^{G_M} = {}^{\#}B_M^{G_M}$, then $\lambda_p = 0$.*

Proof. By the remark after Theorem 1, it is enough to show that $v_p(\delta - 1) \le \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} + 1$. Thus it suffices to show that $\widehat{H}^0(G_M, E_M) = \{0\}$.

Since $\delta \equiv 1 \bmod p^{M+1}$, $\delta = N_{\mathbb{Q}_{p,M}/\mathbb{Q}_p}(\eta)$ for some unit η in $\mathbb{Q}_{p,M}$. Since primes of k above q ($q \ne p$) are unramified in k_M, δ is a local norm for all primes. Therefore δ is a global norm by the Hasse theorem. That is, $\widehat{H}^0(G_M, C_M) \to \widehat{H}^0(G_M, k_M^*)$ is a zero map. Since $\widehat{H}^0(G_M, C_M) \to \widehat{H}^0(G_M, E_M)$ is surjective by Lemma 3, $\widehat{H}^0(G_M, E_M) \to \widehat{H}^0(G_M, k_M^*)$ is a zero map. Hence $\ker(\widehat{H}^0(G_M, E_M) \to \widehat{H}^0(G_M, k_M^*)) = \widehat{H}^0(G_M, E_M)$. Thus from a sequence similar to (*) in Section 2, we have an exact sequence

$$0 \to H^1(G_M, E_M) \to I_M^{G_M}/I_0 \to A_M^{G_M} \to \widehat{H}^0(G_M, E_M) \to 0.$$

Since the Herbrand quotient for E_M is p^M, we have

$$^{\#}H^1(G_M, E_M)/^{\#}\widehat{H}^0(G_M, E_M) = p^M.$$

Since ${}^{\#}I_M^{G_M}/I_0 = p^{2M}$, we get ${}^{\#}A_M^{G_M} = p^M$.

By Lemma 3, we have another exact sequence

$$0 \to B_M^{G_M} \to H^1(G_M, C_M) \to H^1(G_M, E_M) \to 0.$$

Note that ${}^{\#}B_M^{G_M} = {}^{\#}A_M^{G_M} = p^M$ and that ${}^{\#}H^1(G_M, C_M) = p^{2M}$. Therefore ${}^{\#}H^1(G_M, E_M) = p^M$, and so ${}^{\#}\widehat{H}^0(G_M, E_M) = 0$.

To prove Theorem 3, we need a lemma. We let $\delta = \prod_{\tau\in\Delta}(1 - \zeta_d^\tau)^{\chi(\tau)(p-1)}$ as before.

LEMMA 4. *If $p \mid L_p(1, \chi)$, then $\widehat{H}^0(G_1, E_1) = \{0\}$.*

Proof. Consider the exact sequence $0 \to B_1^{G_1} \to H^1(G_1, C_1) \to H^1(G_1, E_1) \to 0$. Since $p \mid L_p(1, \chi)$, $B_1 \ne \{0\}$ ([6] or [8]) and so $B_1^{G_1} \ne \{0\}$.

Since $H^1(G_1, C_1) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, we see that $H^1(G_1, E_1)$ is either $\{0\}$ or $\mathbb{Z}/p\mathbb{Z}$. But $\pi^{\sigma-1}$ is a nontrivial element of $H^1(G_1, E_1)$, where π is a prime element of $\mathbb{Q}_1$, the subextension of $\mathbb{Q}(\zeta_{p^2})$ of degree p over $\mathbb{Q}$. Thus $H^1(G_1, E_1) \simeq \mathbb{Z}/p\mathbb{Z}$. Since the Herbrand quotient for E_1 is p, we have $\widehat{H}^0(G_1, E_1) = \{0\}$.

THEOREM 3. *Let $p \notin S$. If $v_p(L_p(1,\chi)) \le 1$, then $\lambda_p = 0$.*

Proof. If $v_p(L_p(1,\chi)) = 0$, then there is nothing to prove as was explained in the introduction. So we assume that $v_p(L_p(1,\chi)) = 1$. Let δ be as above. Note that

$$L_p(1,\chi) = -\left(1 - \frac{1}{p}\right)\frac{g(\chi)}{d}\sum_{\tau\in\Delta}\chi(\tau)\log_p(1-\zeta_d^\tau)$$
$$= -\frac{g(\chi)}{pd}\log_p\Big(\prod_{\tau\in\Delta}(1-\zeta_d^\tau)^{\chi(\tau)(p-1)}\Big) = -\frac{g(\chi)}{pd}\log_p\delta,$$

where $g(\chi) = \sum_{a=1}^{d}\chi(a)\zeta_d^a$ is the Gauss sum for χ. Thus $v_p(\log_p\delta) = v_p(L_p(1,\chi)) + 1 = 2$.

Since $\delta \equiv 1 \bmod p$, we have $v_p(\log_p\delta) = v_p(\delta-1)$. Therefore $v_p(\delta-1) = 2$. By Lemma 4, $\widehat{H}^0(G_1, E_1) = \{0\}$. But $\widehat{H}^0(G_2, E_2)$ is not trivial, for otherwise $v_p(\delta-1) \ge 2+1 = 3$ by the Remark after Theorem 1. Hence $v_p(\delta-1) = 2 = \max_t\{t \mid \widehat{H}^0(G_t, E_t) = 0\} + 1$, and thus $\lambda_p = 0$ by Theorem 1.

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