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A further note on "On the equation $aX^4 - bY^2 = 2$"

by

P. M. Voutier (London)

Let a and b be odd integers, (u_1, v_1) be the smallest solution in positive integers to $aX^2 - bY^2 = 2$ and define

$$\tau_{a,b} = \frac{u_1\sqrt{a} + v_1\sqrt{b}}{\sqrt{2}}.$$

In [1, 2], Akhtari, Togbé and Walsh show that the equation

$$aX^4 - bY^2 = 2 \tag{1}$$

has at most two solutions in positive integers X, Y if $\tau_{a,b} > 240$.

We show here that a few simple refinements of their use of the hypergeometric method, most importantly an improved numerator estimate, allow us to remove the condition $\tau_{a,b} > 240$.

Theorem 1. *For any two integers a and b, equation* (1) *has at most two solutions in positive integers X and Y.*

Proof. The authors of [2] have shown how to reduce consideration of (1) to an equation of the form $(t+2)X^4 - tY^2 = 2$.

Furthermore, in [1, Theorem 1.1], the same authors show that for all odd positive integers $t > 40\,000$, the equation $(t+2)X^4 - tY^2 = 2$ has at most two solutions in positive integers. They also show that there is exactly one solution in positive integers for $1 \le t < 1200$.

Here we show how to refine their argument in [1] to also include $1200 \le t \le 40\,000$. Thus the condition $t > 40\,000$ in Theorem 1.1 of [1] can be removed and no bound on $\tau_{a,b}$ is required on Theorem 1 of [2].

In Lemma 7.4 of [1], we can take $N_{4,r} = 4^r$ to be the greatest common divisor of the numerators of the coefficients of $X_{4,r}(1-4x)$, which will cause $(D_{4,r}/N_{4,r})X_{4,r}(1-4x)$ to have rational integer coefficients. This is Lemma 3.5(a) of [3].

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Similar to the expressions near the bottom of p. 164 of [1], we can write

$$(-2t)^{r/2}A_r(0) = \frac{-10t^{2r+1}N_{4,r}}{D_{4,r}}\left\{\frac{D_{4,r}}{N_{4,r}}\left[(-1)^r(2-\sqrt{-2t})^r X_r(1-4\overline{\eta}) + (2+\sqrt{-2t})^r X_r(1-4\eta)\right]\right\}$$

and

$$(-2t)^{r/2}B_r(0) = (-2t)^{1/2}\,\frac{-10t^{2r+1}N_{4,r}}{D_{4,r}}\left\{\frac{D_{4,r}}{N_{4,r}}\left[(-1)^r(2-\sqrt{-2t})^r X_r(1-4\overline{\eta}) - (2+\sqrt{-2t})^r X_r(1-4\eta)\right]\right\},$$

where $\eta = 1/(2+\sqrt{-2t})$ and $\overline{\eta} = 1/(2-\sqrt{-2t})$.

As in [1], the quantities in the braces can be expressed as $(-1)^r(e - f\sqrt{-2t}) \pm (e+f\sqrt{-2t})$, where e and f are integers. So for A_r, this expression is $2e$ for r even, and $2f\sqrt{-2t}$ for r odd. And for B_r, $(-2t)^{1/2}$ times this expression is $-4tf$ for r even, and $2e(-2t)^{1/2}$ for r odd. So we put

$$\begin{aligned} P_r &= \frac{D_{4,r}}{10t^{2r+1}N_{4,r}}\,\frac{1}{2(-2t)^{r/2-\lfloor r/2\rfloor}}\,(-2t)^{r/2}B_r(0) \\ &= \frac{-D_{4,r}}{10(-2t)^{2r+1}}\,\frac{1}{(-2t)^{r/2-\lfloor r/2\rfloor}}\,(-2t)^{r/2}B_r(0) = \frac{-D_{4,r}}{10(-2t)^{\lfloor 3r/2+3/2\rfloor}}\,B_r(0), \end{aligned}$$

recalling that $N_{4,r} = 4^r$. Notice that P_{2r} is divisible by $2t$.

With

$$Q_r = \frac{-D_{4,r}}{10(-2t)^{\lfloor 3r/2+3/2\rfloor}}\,A_r(0), \qquad S_r = \frac{-D_{4,r}}{10(-2t)^{\lfloor 3r/2+3/2\rfloor}}\,C_r(0),$$

we have

$$Q_r\beta^{(3)} - P_r = S_r.$$

And since $\beta^{(3)}(-\beta^{(4)}) = 2t$,

$$2tQ_r + \beta^{(4)}P_r = -\beta^4 S_r$$

and hence

$$Q_{2r} + \beta^{(4)}\,\frac{P_{2r}}{2t} = -\frac{\beta^{(4)}}{2t}\,S_{2r},$$

recalling our comment above that $(2t) \mid P_{2r}$.

So we define

$$Q'_r + \beta^{(4)}P'_r = S'_r,$$

where

$$P'_r = P_{2r}/(2t), \qquad Q'_r = -Q_{2r}, \qquad S'_r = -\beta^{(4)}S_{2r}/(2t).$$

We now obtain bounds for these quantities.

For $t \geq 300$, $(2+t)^{1/2} < 1.00334t^{1/2}$ and $1.998 < |1+\sqrt{w(0)}| < 2$.

Combining the definition of P_r with the inequalities in the middle of page 164 of [1] along with Lemma 7.4 there and the above inequalities, we find that

$$\begin{aligned} |P_r| &\le \frac{D_{4,r}}{10(2t)^{\lfloor 3r/2+3/2\rfloor}}\, 80\sqrt{2}\, t^{(3r+3)/2}(2+t)^{r/2}\, \frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\, |1+\sqrt{w(0)}|^{2r-2} \\ &= \frac{4(2t)^{(r+1)/2-\lfloor (r+1)/2\rfloor}}{|1+\sqrt{w(0)}|^2}\, \frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\, D_{4,r}\left(\frac{(2+t)^{1/2}}{2^{3/2}}\, |1+\sqrt{w(0)}|^2\right)^r \\ &< 1.69(2t)^{(r+1)/2-\lfloor (r+1)/2\rfloor}(7.58\sqrt{t})^r. \end{aligned}$$

So,

$$|P_r'| < \frac{1.2}{\sqrt{t}}\,(7.58\sqrt{t})^{2r} < \frac{1.2}{\sqrt{t}}\,(57.46t)^r.$$

Similarly, we get

$$\begin{aligned} |\beta^{(4)}S_r| &\le |\beta^{(4)}|\, \frac{D_{4,r}40t^{(3r+2)/2}}{10(2t)^{\lfloor 3r/2+3/2\rfloor}}\, |\varphi| \\ &\quad \times |\beta^{(3)} - \sqrt{-2t}|(2+t)^{r/2}\, \frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\, |1-\sqrt{w(0)}|^{2r} \\ &= \frac{2|\beta^{(4)}\varphi|\, |\beta^{(3)} - \sqrt{-2t}|}{(2t)^{r/2-\lfloor r/2\rfloor}} \\ &\quad \times \frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\, D_{4,r}\left(\frac{(2+t)^{1/2}}{2^{3/2}}\, |1-\sqrt{w(0)}|^2\right)^r \\ &< |\beta^{(4)}|\, \frac{2}{(2t)^{r/2-\lfloor r/2\rfloor}}\, 6.51\cdot 0.1924\left(\frac{3.79}{\sqrt{t}}\right)^r \\ &< 4.95(2t)^{1/2+(r+1)/2-\lfloor (r+1)/2\rfloor}\left(\frac{3.79}{\sqrt{t}}\right)^r, \end{aligned}$$

using the upper bound $|1-\sqrt{w(0)}|^2 < 2/t$ from the middle of p.165 of [1], along with the estimates $|\beta^{(4)}| < 4.009t$ and $|\varphi(\beta^{(3)} - \sqrt{-2t})| < 6.412$ for $t \ge 300$. So,

$$|S_r'| < 4.95\left(\frac{3.79}{\sqrt{t}}\right)^{2r} < 4.95\left(\frac{14.365}{t}\right)^r.$$

We can now proceed as in the proof of Lemma 7.7 of [1] with $k_0 = 1.2/\sqrt{t}$, $Q = 57.46t$, $l_0 = 4.95$ and $E = t/14.365$, to obtain

$$\left|\frac{x}{y} - \beta^{(4)}\right| > \frac{1}{c_4|y|^{\kappa+1}}, \tag{2}$$

for $y \ne 0$, where

$$\kappa = \frac{\log(57.46t)}{\log(t/14.365)} \quad \text{and} \quad c_4 = 2k_0Q(2l_0E)^\kappa < 138\sqrt{t}\,(0.7t)^\kappa,$$

provided $14.365/t < 1$.

Combining the upper bound for $|x/y - \beta^{(4)}|$ in Lemma 3.1 of [1] with inequality (2), we obtain

$$|y|^{3-\kappa} < \frac{c_4}{16t}.$$

Thus, from the lower bound for $|y|$ at the end of Section 4 of [1],

$$\frac{138\sqrt{t}\,(0.7t)^{\kappa}}{16t} > |y|^{3-\kappa} > (2^{10}t^{11})^{3-\kappa}.$$

We can write the "outer" inequality as

$$\frac{138}{16}\,(0.7)^3(0.7)^{-(3-\kappa)}t^{\kappa-0.5} > (2^{10}t^{11})^{3-\kappa}$$

or

$$2.96t^{\kappa-0.5} > (1.92^{10}t^{11})^{3-\kappa}, \tag{3}$$

since $0.7 > 0.96^{10}$.

For $t \geq 610$ we have $\kappa < 67/24$ and hence $11(3-\kappa) > \kappa - 0.5$. Furthermore, $10(3-\kappa) > 2.08$ in this case, so $1.92^{10(3-\kappa)} > 3.88$. Thus (3) cannot hold.

This shows that there are no solutions of $(t+2)X^4 - tY^2 = 2$ that come from $X^2 = V_{4k+1}$ for $t \geq 610$ and hence at most two solutions in positive integers of $(t+2)X^4 - tY^2 = 2$ for $t \geq 610$, as required.

NOTE. Using some quick continued-fraction calculations of $\beta^{(4)}$ for $415 \leq t \leq 609$, one can reduce the size of c_4 for such t and show that one need only consider $t \leq 414$. Combined with improved denominator estimates as in [4], this allows us to eliminate all $t \geq 310$.

References

[1] S. Akhtari, A. Togbé and P. G. Walsh, *On the equation $aX^4 - bY^2 = 2$*, Acta Arith. 131 (2008), 145–169.

[2] —, —, —, *Addendum on the equation $aX^4 - bY^2 = 2$*, ibid. 137 (2009), 199–202.

[3] P. M. Voutier, *Rational approximations to $\sqrt[3]{2}$ and other algebraic numbers revisited*, J. Théor. Nombres Bordeaux 19 (2007), 265–288.

[4] —, *Thue's Fundamentaltheorem, I: The general case*, submitted; http://arxiv.org/abs/0805.3176.

E-mail: paul.voutier@gmail.com

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