ACTA ARITHMETICA
105.2 (2002)

Cramér functions and Guinand equations

by

Georg Illies (Siegen)

In [Cr] Cramér introduced the function

$$V(t)=\sum_{\Im(\varrho)>0} e^{\varrho t},\qquad \Im(t)>0,$$

ϱ being the zeros of the Riemann zeta function in the upper half plane, and he examined its analytic behaviour. In [Gu] Guinand proved a functional equation for $V(t)$. We generalize these theorems to a large class of abstract L-series instead of the Riemann zeta function improving the results of [JL2]. At the end we treat as an example Hecke L-series.

1. Introduction. For $V(t)$ as defined above, Cramér [Cr] showed that the function

$$V(t)-\frac{1}{2\pi i}\cdot\frac{\log t}{1-e^{-t}}$$

has a meromorphic continuation to $\mathbb{C}$, determined the poles (which are all simple) and the corresponding residues and also found the constant term of the Laurent expansion at $t=0$. Guinand [Gu] proved the functional equation

$$e^{-t/2}(V(t)-1)+e^{t/2}(V(\exp(\pi i)t)-1)=-\frac{e^{-t/2}}{1-e^{-2t}},\qquad t\in\widetilde{\mathbb{C}}^{*}.$$

Summarizing one has the following three results:

(a) Meromorphic continuation of $V(t)$ to $\widetilde{\mathbb{C}}^{*}$ with explicit determination of poles and singular parts and explicit determination of $V(t)-V(\exp(2\pi i)t)$,

(b) Guinand's functional equation for $V(t)$,

(c) behaviour of $V(t)$ at $t=0$, i.e. the asymptotics for $|t|\to 0$ except for the coefficients of the terms t^n, $n\in\mathbb{N}$ (Cramér asymptotics).

There are several analogous results for other L-series instead of $\zeta(s)$, e.g. [CV] for the Selberg zeta function, [Ka] for Dirichlet L-series for $\mathbb{Q}$

2000 *Mathematics Subject Classification*: 11M26, 11M36, 14G10, 11M41, 11R42.

Key words and phrases: zeros of zeta and L-functions, Cramér functions, Guinand equations, regularized products, Hecke L-series.

and [DS] for motivic L-series ([1]). Jorgenson and Lang [JL1], [JL2] consider an abstract meromorphic function of finite order $L(s)$ which has a Dirichlet series representation in a half plane and satisfies a functional equation. They formulate a generalization of (a) and give a proof of the existence of a Cramér asymptotics. Guinand's functional equation and the *explicit* determination of the Cramér asymptotics are not given.

The aim of this article which reproduces Chapter 4 of my Ph.D. thesis [Il1] is to give a complete generalization of (a), (b) and (c) in the setting of Jorgenson and Lang, even slightly more general. For (a) and (b) we use the function $\widetilde{\theta}_D(t,s)$ defined in [Il1] and [Il2], which gives a more elegant access to the Cramér function than Cramér's contour integral method (also used in [JL2], [Ka], [CV], [Gu]) or the distributional approach of [DS] that does not apply to our general case. For (c) we also use results from [Il1] and [Il2]. The discussion of functional equations and ε-factors in terms of regularized determinants (Sections 7 and 9) should be interesting in its own right and in the context of [De].

For large classes of L-series from different branches of mathematics (compare the examples given in [JL2]) the concrete parameters for (a), (b) and (c) can be determined by trivial computations. One has just to follow the pattern given in Section 10 where the Theorem is applied to Hecke L-series.

Acknowledgements. I would like to thank C. Deninger for supervising my Ph.D. thesis as well as M. Schröter, I. Vardi, C. Bree, C. Soulé, A. Voros, J. B. Bost and J. Jorgenson for helpful discussions and improvements. Parts of the article were written during a visit to the IHES.

2. Some notations. A *meromorphic function* $\dot{L}(z)$ *of finite order* is a function that can be represented as the quotient of two entire functions of finite order. Recall ([Ti]) that there is a $\beta > 0$ such that

$$\sum_{\varrho\in\mathbb{C}} |\mathrm{ord}_{\dot{L},\varrho}| \cdot |\varrho|^{-\beta} < \infty. \tag{2.1}$$

The *exponent* $\dot{r}$ of $\dot{L}$ is the infimum of all $\beta > 0$ satisfying (2.1); the *genus* $\dot{g}$ of $\dot{L}$ is the smallest $n \in \mathbb{N}_0$ such that (2.1) is satisfied for $\beta = n+1$; note that $g + 1 \geq r \geq g$.

In what follows, to meromorphic functions of finite order $\dot{L}(z)$ we will associate auxiliary functions ([2]) such as $\xi_{\dot{D}}(s)$ or $\theta_{\dot{D}}(t)$. The symbol $\dot{L}$ is meant as a "wildcard": Instead of $\dot{L}(z)$ we could for example have the function $L_\infty(z)$ which would have genus g_∞ and an auxiliary function $\xi_{D_\infty}(s)$

([1]) The Cramér function plays a role in certain speculative considerations in arithmetic algebraic geometry [De].

([2]) The strange symbol $\dot{D}$ for the "divisor" of $\dot{L}$ has no deeper meaning here, it is used just for compatibility with [Il2].

or a function $L(z)$ with exponent r and so on. We also use the abbreviated notation

$$\sum_{\varrho\in\dot{D}}\varphi(\varrho) := \sum_{\varrho\in\mathbb{C}}\mathrm{ord}_{\dot{L},\varrho}\cdot\varphi(\varrho).$$

3. The Theorem

GS (General situation) *$L(z)$ and $L^*(z)$ are meromorphic functions of finite order such that for a suitable branch of the logarithm one has absolutely convergent Dirichlet series representations*

$$\log L(z) = \sum_{n=1}^{\infty}\frac{c_n}{q_n^z}, \qquad \log L^*(z) = \sum_{n=1}^{\infty}\frac{c_n^*}{q_n^{*z}} \tag{3.1}$$

for $\Re(z) > \sigma_0$ with a certain $\sigma_0 \in \mathbb{R}$, with ([3]) $q_n^{(*)} > 1$, $\lim_{n\to\infty} q_n^{(*)} = \infty$ *and with complex $c_n^{(*)}$. $L_\infty(z)$ and $L^*_\infty(z)$ are meromorphic functions of finite order such that almost all zeros and poles ϱ satisfy $\varphi < \arg(\varrho) < 2\pi - \varphi$ for some fixed $0 < \varphi < 2\pi$. The functions*

$$\widehat{L}(z) := L(z)L_\infty(z), \qquad \widehat{L}^*(z) := L^*(z)L^*_\infty(z), \tag{3.2}$$

satisfy the functional equation

$$\widehat{L}(z) = e^{Q(z)}\widehat{L}^*(-z), \tag{3.3}$$

with a polynomial $Q(z)$.

Definition 3.1. In the situation **GS** one defines the *Cramér function* of $\widehat{L}(z)$ by ([4])

$$\theta_{\widehat{D}_+}(t) := \sum_{\varrho\in\mathbb{C}}\mathrm{ord}^+_{\widehat{L},\varrho}\cdot e^{\varrho t} \tag{3.4}$$

where $\mathrm{ord}^+_{\widehat{L},\varrho}$ is defined by

$$\mathrm{ord}^+_{\widehat{L},\varrho} := \begin{cases} \mathrm{ord}_{\widehat{L},\varrho} & \text{for } \Im(\varrho) \geq 0 \text{ with } \varrho \notin \mathbb{R}^{\geq 0},\\ \frac{1}{2}\,\mathrm{ord}_{\widehat{L},0} & \text{for } \varrho = 0,\\ 0 & \text{else.}\end{cases}$$

The series (3.1) is absolutely convergent for $\pi/2-\varphi < \arg(t) < \pi/2+\varphi$, uniformly in compact sets, so $\theta_{\widehat{D}_+}(t)$ is holomorphic. Correspondingly one has the Cramér function $\theta_{\widehat{D}^*_+}(t)$ of $\widehat{L}^*(z)$.

In the following Theorem we use the universal covering pr: $\widetilde{\mathbb{C}}^* \to \mathbb{C}^*$ with the canonical isomorphism log: $\widetilde{\mathbb{C}}^* \to \mathbb{C}$ such that pr $= \exp\circ\log$. For $t \in \widetilde{\mathbb{C}}^*$

([3]) Here and in what follows, a term or an equation with $(*)$ (e.g. $q_n^{(*)}$) is meant as a simultaneous expression, one with and one without $*$ (i.e. q_n and q_n^* in the example).

([4]) This choice is arbitrary insofar as the results remain valid for series that differ only by finitely many summands.

one has $\arg(t) := \Im(\log t)$. For $t \in \widetilde{\mathbb{C}}^*$ and $\alpha \in \mathbb{C}$ we denote by $\exp(\alpha) \cdot t$ that $t' \in \widetilde{\mathbb{C}}^*$ with $\log t' = \alpha + \log t$.

THEOREM. *In addition to* **GS** *assume that there are* $\vartheta, \vartheta^* \in [-\pi/2, \pi/2]$ *and* $\varepsilon > 0$ *such that* $\pi/2 + \vartheta + \varepsilon < \arg(\varrho) < 3\pi/2 + \vartheta - \varepsilon$ *for almost all zeros and poles of* $L_\infty(z)$ *and* $\pi/2 + \vartheta^* + \varepsilon < \arg(\varrho) < 3\pi/2 + \vartheta^* - \varepsilon$ *for almost all zeros and poles of* $L_\infty^*(z)$. *Also assume that the partition functions* $\theta_{D_\infty}(t)$ *and* $\theta_{D_\infty^*}(t)$ *which are defined by* (4.3) *have meromorphic continuations to* $\mathbb{C}$. *Identify the domain of definition for* (3.4) *with those* $t \in \widetilde{\mathbb{C}}^*$ *with* $\pi/2 - \varphi < \arg(t) < \pi/2 + \varphi$ *in the natural way. Then*:

(a) *The function* $\theta_{\widehat{D}_+}(t)$ *has a meromorphic continuation to* $\widetilde{\mathbb{C}}^*$. *The function*

$$\theta_{\widehat{D}_+}(t) + \frac{1}{2\pi i}\,\theta_{D_\infty}(t)\log t - \frac{1}{2\pi i}\,\theta_{D_\infty^*}(-t)\log t \tag{3.5}$$

is single-valued, *i.e. is meromorphic in* $\mathbb{C}$. *Except for the pole at* $t = 0$ *which is described in* (c), *the only poles are at* $t = \log q_n$ *and* $t = -\log q_n^*$ *with residues* $\frac{1}{2\pi i} c_n \log q_n$ *and* $\frac{1}{2\pi i} c_n^* \log q_n^*$ *respectively*, *and some poles arising from* $\theta_{D_\infty}(t)$ *and* $\theta_{D_\infty^*}(t)$. *The latter can be characterized as follows*: $\theta_{\widehat{D}_+}(t)$ *has no singularities arising from* $\theta_{D_\infty}(t)$ *in* $-\vartheta - \varepsilon < \arg(t) < 2\pi - \vartheta + \varepsilon$, *and* $\theta_{\widehat{D}_+}(t)$ *has no singularities arising from* $\theta_{D_\infty^*}(t)$ *in* $-\pi - \vartheta^* - \varepsilon < \arg(t) < \pi - \vartheta^* + \varepsilon$.

(b) (Guinand's functional equation) *For all* $t \in \widetilde{\mathbb{C}}^*$,

$$\theta_{\widehat{D}_+}(t) + \theta_{\widehat{D}_+^*}(\exp(\pi i)t) = \theta_{D_\infty}(t). \tag{3.6}$$

(c) *For the* ε-*polynomial of the* ξ-*regularization* (see Definitions 6.1 and 7.1)

$$P(z) = \sum_{l=0}^{\widehat{g}} p_l z^l \tag{3.7}$$

and for the Laurent expansion ([5]) *at* $t = 0$

$$\theta_{D_\infty}(t) = \sum_{l=0}^{\infty} \alpha_l t^{-l} + O(t) \tag{3.8}$$

one has the following Laurent expansion at $t = 0$:

$$\begin{aligned}\theta_{\widehat{D}_+}(t) &+ \frac{1}{2\pi i}\,\theta_{D_\infty}(t)\log t - \frac{1}{2\pi i}\,\theta_{D_\infty^*}(-t)\log t \\ &= \sum_{l=0}^{\widehat{g}} \left(\frac{\alpha_l}{2} - \frac{(-1)^l l! p_l}{2\pi i}\right) t^{-l} + k + O(t)\end{aligned} \tag{3.9}$$

([5]) From Proposition 6.4 it follows that $\alpha_l = 0$ for all $l > g_\infty$.

with a certain $k \in \mathbb{Z}$ or $k \in \mathbb{Z} + 1/2$ depending on whether the multiplicity of $\varrho = 0$ for $\widehat{L}$ is even or odd, respectively, and with $\alpha_l = 0$ for all $l > \widehat{g}$. In the symmetric case $\widehat{L} = \widehat{L}^*$ the expansion (3.9) *is valid with $p_l = 0$ for all even* (6) *l and with $k = 0$.*

The analogous properties of the dual function $\theta_{\widehat{D}_+^*}(t)$ can be derived by substituting all parameters by their star duals (using $** = \mathrm{id}$). For example one has

$$\theta_{\widehat{D}_+^*}(t) + \theta_{\widehat{D}_+}(\exp(\pi i)t) = \theta_{D_\infty^*}(t).$$

The dual ε-polynomial $P^*(z)$ is just $-P(-z)$ as one can see from the functional equation (7.1); so $\widehat{g}^* = \widehat{g}$. The dual k^* of k in (3.9) is $k^* = k$.

Remark. From the proof of the Theorem one can see at once that everything remains valid analogously for L_∞ and L_∞^* being finite products of functions described in the Theorem with different ϑ and ϑ^* (see [Il1]). One also rediscovers the (corrected forms) of the results in [JL2]: Their proofs do not work unless one demands $\alpha_j \in \mathbb{R}^{\geq 0}$ in the definition of "regularized product type" instead of just $\Re(\alpha_j) \geq 0$. Theorem 1.2 of [JL2] which states the existence of a meromorphic continuation to $\mathbb{C} \setminus \mathbb{R}^{\geq 0}$ then also fails as for the G^*-terms the cut has to be made on the other side. Our method applies to the "polynomial Bessel class" introduced in [JL3] as well.

Sections 4–8 are devoted to the proof of the Theorem. For this proof we assume in addition:

AC (Additional condition) *All zeros and poles of the functions $\dot{L} = \widehat{L}^{(*)}, L^{(*)}, L_\infty^{(*)}$ satisfy $\varrho \neq 0$ and $\varphi < \arg(\varrho) < 2\pi - \varphi$.*

This is necessary in order to define the $\widetilde{\theta}$-functions (Definition 4.1). The general case can easily be reduced to this case by multiplicative combination with the trivial special case $\widehat{L}(z) = (1 - e^{-(z-\varrho)})^{\pm 1}$ and $\widehat{L}^*(z) = (1 - e^{-(z-\varrho)})^{\pm 1}$ with $e^{P(z)} = (-e^{-(z-\varrho)})^{\pm 1}$ (as both $\widehat{L}(z)$ and $\widehat{L}^*(z)$ are ξ-regularized by Proposition 6.3). The trivial calculations for this case are left to the reader.

4. The functions $\xi_D(s,z)$, $\widetilde{\theta}_D(s,t)$ and $\theta_D(t)$

Definition 4.1. If all zeros and poles ϱ of a meromorphic function of finite order $\dot{L}$ satisfy $\varrho \neq 0$ and $\varphi < \arg(\varrho) < 2\pi - \varphi$ for a fixed $0 < \varphi < \pi$ its *Hurwitz ξ function* is defined by

$$\xi_{\dot{D}}(s,z) := \Gamma(s) \sum_{\varrho \in \dot{D}} (z - \varrho)^{-s}, \qquad \Re(s) > \dot{r},\ |z| \text{ small.} \tag{4.1}$$

(6) In this case the expansion is also valid for negative even l when we define $p_l := 0$.

In this definition for sufficiently small $|z|$ we choose $-\pi < \arg(z-\varrho) < \pi$. $\xi_{\dot{D}}(s) := \xi_{\dot{D}}(s,0)$ is called the ξ *function* of $\dot{L}$.

For $t \in \widetilde{\mathbb{C}}^*$ with $\pi/2 - \varphi < \arg(t) < 3\pi/2 + \varphi$ and for $\Re(s) > \dot{r}$ one defines

$$\widetilde{\theta}_{\dot{D}}(t,s) := \frac{t^{-(s-1)}}{2\pi i} \int_0^{e^{i\alpha}\infty} e^{wt}\xi_{\dot{D}}(s,w)\,dw \tag{4.2}$$

for every $\alpha \in\,]-\varphi,\varphi[$ satisfying $\pi/2 < \arg(t) + \alpha < 3\pi/2$.

If, in addition, $\pi/2 < \varphi$ then the *partition function* is defined for $t \in \mathbb{C}^*$ with $-(\varphi - \pi/2) < \arg(t) < \varphi - \pi/2$ by

$$\theta_{\dot{D}}(t) := \sum_{\varrho \in \dot{D}} e^{\varrho t}. \tag{4.3}$$

In (4.2), wt is just the complex number with $|wt| = |w|\cdot|t|$ and $\arg(wt) = \arg(w)+\arg(t)$. Convergence of the series (4.1) and (4.3) is absolute and uniform and the functions are holomorphic in their variables. $\xi_{\dot{D}}(s,z)$, $\Re(s) > \dot{r}$, has a holomorphic continuation for $z \in \mathbb{C}^*$ with $-\varphi < \arg(z) < \varphi$: just regard finitely many $(z-\varrho)^{-s}$ separately. By the elementary estimate

$$\sum_{\varrho\in\mathbb{C}} |\mathrm{ord}_{\dot{L},\varrho} \cdot (z-\varrho)^{-s}| = o(|z|^{\dot{r}-\Re(s)+\delta}), \qquad |z| \to \infty, \tag{4.4}$$

valid for $-\varphi + \varepsilon < \arg(z) < \varphi - \varepsilon$ and for all fixed $\delta, \varepsilon > 0$ (cf. [Il2, Lemma 12.1]) the Laplace integral (4.2) converges absolutely, is well defined and holomorphic in the two variables.

PROPOSITION 4.2. *For fixed $m \in \mathbb{N}$, $m > \dot{r}$ and $\varepsilon > 0$ the estimate*

$$\widetilde{\theta}_{\dot{D}}(t,m) = O(|t|^{-m}), \qquad |t| \to 0, \tag{4.5}$$

is valid for $\pi/2 - \varphi + \varepsilon < \arg(t) < 3\pi/2 + \varphi - \varepsilon$. In the case $\pi/2 < \varphi$ the following functional equation is valid:

$$\widetilde{\theta}_{\dot{D}}(t,m) - \widetilde{\theta}_{\dot{D}}(\exp(2\pi i)t,m) = \theta_{\dot{D}}(t), \tag{4.6}$$

where the overlap in the domain of definition for $\widetilde{\theta}_{\dot{D}}(t,m)$ is identified with the domain of definition for the partition function $\theta_{\dot{D}}(t)$ in the natural way.

Proof. Lemma 13.1 and Proposition 9.2 in [Il2]. ∎

REMARK. Further properties of $\widetilde{\theta}_{\dot{D}}(t,s)$ can be found in Section 9 of [Il2] and Section 2.6 of [Il1]. For the following proof of the Theorem it is sufficient to have the estimate (4.5) with a weaker exponent than $-m$, and such a weaker estimate follows trivially from (4.2) and (4.4). (4.6) can be immediately seen by the residue theorem using majorized convergence because of (4.4).

5. Proof of parts (a) and (b). Proposition 5.1 below, a consequence of (4.6), is the principle behind parts (a) and (b) of the Theorem: The function $\theta_{\widehat{D}_+}(t)$ is represented in terms of the $\widetilde{\theta}_{\widehat{D}^{(*)}}(t,m)$, so by (3.2), i.e.

$$\widetilde{\theta}_{\widehat{D}^{(*)}}(t,m) = \widetilde{\theta}_{D^{(*)}}(t,m) + \widetilde{\theta}_{D_\infty^{(*)}}(t,m), \tag{5.1}$$

everything is reduced to the properties of the functions $\widetilde{\theta}_{D^{(*)}}$ and $\widetilde{\theta}_{D_\infty^{(*)}}$ which are given by Proposition 5.2 and the assumption of the Theorem, respectively.

PROPOSITION 5.1. *In the situation* **GS** + **AC** *with* $m \in \mathbb{N}$, $m > \widehat{r}, \widehat{r}^*$ *the following is valid*:

(a) *If the* $t \in \mathbb{C}^*$ *with* $\pi/2 - \varphi < \arg(t) < \pi/2 + \varphi$ (*the domain of definition for* $\theta_{\widehat{D}_+}(t)$) *are identified with those* $t \in \widetilde{\mathbb{C}}^*$ *with the same arguments then*

$$\theta_{\widehat{D}_+}(t) = \widetilde{\theta}_{\widehat{D}}(t,m) - \widetilde{\theta}_{\widehat{D}^*}(\exp(\pi i)t,m). \tag{5.2}$$

(b) *If* $\widetilde{\theta}_{\widehat{D}}(t,m)$ *and* $\widetilde{\theta}_{\widehat{D}^*}(t,m)$ *have meromorphic continuations in* t *to all of* $\widetilde{\mathbb{C}}^*$, *then so also do* $\theta_{\widehat{D}_+}(t)$ *and* $\theta_{\widehat{D}_+^*}(t)$, *and we have Guinand's functional equation*

$$\theta_{\widehat{D}_+}(t) + \theta_{\widehat{D}_+^*}(\exp(\pi i)t) = \widetilde{\theta}_{\widehat{D}}(t,m) - \widetilde{\theta}_{\widehat{D}}(\exp(2\pi i)t,m). \tag{5.3}$$

Proof. Using the multiplicities $\mathrm{ord}^+_{\widehat{L}}$ from (3.4) we can define the function $\widetilde{\theta}_{\widehat{D}_+}(t)$. (To be precise: Choose a meromorphic function $\widehat{L}^+(z)$ of finite order with $\mathrm{ord}_{\widehat{L}^+,\varrho} = \mathrm{ord}^+_{\widehat{L},\varrho}$ for all $\varrho \in \mathbb{C}$ and use Definition 4.1.) Correspondingly we have the function $\widetilde{\theta}_{\widehat{D}_+^*}(t)$. By the functional equation (3.3) a $\varrho \in \mathbb{C}$ is a zero or pole of $\widehat{L}(z)$ if and only if $-\varrho$ is a zero or pole of $\widehat{L}^*(z)$, respectively. So by (3.4) and Definition 4.1 we have, for $\pi/2 - \varphi < \arg(t) < 3\pi/2 + \varphi$,

$$\widetilde{\theta}_{\widehat{D}}(t,m) = \widetilde{\theta}_{\widehat{D}_+}(t,m) + \widetilde{\theta}_{\widehat{D}_+^*}(\exp(\pi i)t,m),$$
$$\widetilde{\theta}_{\widehat{D}^*}(t,m) = \widetilde{\theta}_{\widehat{D}_+}(\exp(\pi i)t,m) + \widetilde{\theta}_{\widehat{D}_+^*}(t,m).$$

From (4.6) (and a trivial rotation) everything follows. ∎

PROPOSITION 5.2. *In the situation* **GS** + **AC** *with sufficiently large* $m \in \mathbb{N}$ *the function* $\widetilde{\theta}_D(t,m)$ *is meromorphic in* $\mathbb{C}$. *Except for* $t = 0$ *the only poles are at* $t = \log q_n$, *they are of first order with residues* $\frac{1}{2\pi i} c_n \log q_n$, *respectively. The corresponding statement is true for* $\widetilde{\theta}_{D^*}(t,m)$.

Proof. As $\Delta_D^{\mathrm{Wei}}(z)$ (defined by (6.3) below) and $L(z)$ are meromorphic of finite order with the same zeros and poles one gets for the mth logarithmic derivatives $\log^{(m)} \Delta_D^{\mathrm{Wei}}(z) = \log^{(m)} L(z)$ for sufficiently large m and comparing (6.3) and (4.1) also $\xi_D(m,z) = (-1)^{m-1}\log^{(m)} \Delta_D^{\mathrm{Wei}}(z)$, thus by (3.1),

$$\xi_D(m,w) = (-1)^{m-1}\log^{(m)} L(w) = -\sum_{n=1}^{\infty} \frac{c_n \log^m q_n}{q_n^w}, \qquad \Re(w) > \sigma_0.$$

Equation (4.2) then yields, for $\pi/2 < \arg(t) < 3\pi/2$ and a $\sigma_1 > \sigma_0$,

$$\begin{aligned}\widetilde{\theta}_D(t,m) &= \frac{t^{-(m-1)}}{2\pi i}\int_0^\infty e^{wt}\xi_D(m,w)\,dw \\ &= -\frac{t^{-(m-1)}}{2\pi i}\int_{\sigma_1}^\infty e^{wt}\Big(\sum_{n=1}^{\infty}\frac{c_n\log^m q_n}{q_n^w}\Big)\,dw \\ &\quad + \underbrace{\frac{t^{-(m-1)}}{2\pi i}\int_0^{\sigma_1} e^{wt}\xi_D(m,w)\,dw}_{=:\,h_{D,m,\sigma_1}(t)} \\ &= \frac{t^{-(m-1)}}{2\pi i}\, e^{\sigma_1 t}\sum_{n=1}^{\infty}\frac{c_n\log^m q_n}{q_n^{\sigma_1}(t-\log q_n)} + h_{D,m,\sigma_1}(t),\end{aligned}$$

by majorized convergence. The parameter-dependent integral $h_{D,m,\sigma_1}(t)$ is meromorphic in $\mathbb{C}$ with at most one pole, at $t = 0$, of order not exceeding $m-1$. The absolutely convergent series represents a function that is meromorphic in $\mathbb{C}$ with the needed poles and residues. ∎

Proof of (a) and (b) of the Theorem. Let $m \in \mathbb{N}$ be sufficiently large. By Definition 4.1, $\widetilde{\theta}_{D_\infty}(t,m)$ is holomorphic for $-\vartheta-\varepsilon < \arg(t) < 2\pi-\vartheta+\varepsilon$ and according to (4.6) the function

$$\widetilde{\theta}_{D_\infty}(t,m) + \frac{1}{2\pi i}\,\theta_{D_\infty}(t)\log t$$

is meromorphic in $\mathbb{C}^*$ because $\theta_{D_\infty}(t)$ is meromorphic in $\mathbb{C}$. By (4.5) this function increases for $t \to 0$ not faster than a pole, so it is in fact meromorphic in $\mathbb{C}$. $\widetilde{\theta}_{D_\infty^*}(t,m)$ is treated in the same way. Then (a) and (b) follow at once from Proposition 5.1, equation (5.1) and Proposition 5.2. ∎

6. Regularization. In Sections 7 and 8 we will apply the following facts about regularized products to get information about the poles of $\xi_{\widehat{D}_+}(s)$, which by Proposition 6.4 below will give the behaviour of $\theta_{\widehat{D}_+}(t)$ at $t = 0$.

DEFINITION 6.1. A meromorphic function $\dot{L}(z)$ of finite order as in Definition 4.1 is called *regularizable* if $\xi_{\dot{D}}(s)$ is meromorphic in a half plane $\Re(s) > -\varepsilon$ with $\varepsilon > 0$. For sufficiently small $|z|$,

(6.1) $$\Delta_{\dot{D}}(z) := \exp(-\,\mathrm{CT}_{s=0}(\xi_{\dot{D}}(s,z)))$$

is called the *ξ-regularized determinant* of $\dot{L}$.

REMARK 1. By the Taylor series expansion of $(z-\varrho)^{-s}$ in the variable z one finds that if $\dot{L}(z)$ is regularizable then $\xi_{\dot{D}}(s,z)$ is meromorphic at $s=0$ for $|z|$ sufficiently small (cf. [Il2, Prop. 1]); $\mathrm{CT}_{s=0}$ denotes the constant term in the Laurent expansion. The definition obviously extends to functions $\dot{L}$ satisfying **AC** only up to finitely many zeros and poles.

The following propositions are consequences of Theorem 1, Corollary 1 and Theorem 3 of [Il2].

PROPOSITION 6.2. *$\Delta_{\dot{D}}(z)$ is a meromorphic function of finite order with the same zeros and poles as $\dot{L}(z)$. The polynomial $P_{\dot{D}}(z)$ such that*

$$\Delta_{\dot{D}}(z) = e^{P_{\dot{D}}(z)}\Delta_{\dot{D}}^{\mathrm{Wei}}(z) \tag{6.2}$$

with the (*absolutely convergent*) canonical Weierstrass product

$$\Delta_{\dot{D}}^{\mathrm{Wei}}(z) := \prod_{\varrho\in\mathbb{C}}\left\{\left(1-\frac{z}{\varrho}\right)\exp\left(\sum_{k=0}^{\dot{g}}\frac{1}{k}\left(\frac{z}{\varrho}\right)^{k}\right)\right\}^{\mathrm{ord}_{\dot{L},\varrho}} \tag{6.3}$$

is explicitly given by

$$P_{\dot{D}}(z) = \sum_{l=0}^{\dot{g}}\frac{z^l}{l!}\log^{(l)}\Delta_{\dot{D}}(0),$$

$$\log^{(l)}\Delta_{\dot{D}}(z) = (-1)^{l+1}\,\mathrm{CT}_{s=0}(\xi_{\dot{D}}(s+l,z)),\qquad l=0,1,\ldots$$

PROPOSITION 6.3. *In the situation* **GS** + **AC**, *$\xi_D(s)$ and $\xi_{D^*}(s)$, the ξ functions of $L(z)$ and $L^*(z)$, are holomorphic. For the regularized determinants one has*

$$L(z) = \Delta_D(z),\qquad L^*(z) = \Delta_{D^*}(z). \tag{6.4}$$

PROPOSITION 6.4. *Let $p_n\in\mathbb{C}$ $(n=0,1,\ldots)$ with $\Re(p_0)\le\Re(p_1)\le\ldots\le\Re(p_n)\le\ldots$, let $p:=\lim_{n\to\infty}\Re(p_n)\in\mathbb{R}\cup\{\infty\}$ and $B_n(z)$ $(n=0,1,\ldots)$ be complex polynomials. If in the situation of Definition* 4.1 *with $\pi/2<\varphi<\pi$ for every $q'<q<p$ the asymptotics*

$$\theta_{\dot{D}}(t) - \sum_{\Re(p_n)<q} t^{p_n}B_n(\log t) = O(|t|^{q'})\quad \textit{for } |t|\to 0 \tag{6.5}$$

is valid for $-(\varphi-\pi/2)<\arg(t)<\varphi-\pi/2$ then $\xi_{\dot{D}}(s)$ is meromorphic for $\Re(s)>-p$ with poles only at $s=-p_n$, $n=0,1,\ldots$, with singular part

$$B_n(\partial_s)\left[\frac{1}{s+p_n}\right].$$

REMARK 2. Proposition 6.4 is an easy consequence of the Mellin integral representation $\xi_{\dot{D}}(s)=\int_0^\infty\theta_{\dot{D}}(s)t^{s-1}\,dt$ while for the proof of Proposition 6.3 a sort of Hankel integral is used.

7. Functional equations and ε-factors. In the situation of the Theorem, $L(z)$ and $L^*(z)$ are regularizable by Proposition 6.3 and $L_\infty(z)$ and $L^*_\infty(z)$ are regularizable by Proposition 6.4. Thus also $\widehat{L}(z)$ and $\widehat{L}^*(z)$ are regularizable and so by Proposition 6.2 there is a polynomial $P(z)$ such that the following functional equation for the ξ-regularized determinants is valid:

$$\varDelta_{\widehat{D}}(z) = e^{P(z)} \varDelta_{\widehat{D}^*}(-z), \tag{7.1}$$

as by (3.3) both sides represent meromorphic functions of finite order with the same zeros and poles.

Definition 7.1. $e^{P(z)}$ is called the *ε-factor* and $P(z)$ (of course only determined up to a summand $2\pi ik$, $k \in \mathbb{Z}$) is called the *ε-polynomial* of the ξ-regularization.

Again by (3.3) one has $\varDelta^{\mathrm{Wei}}_{\widehat{D}}(z) = \varDelta^{\mathrm{Wei}}_{\widehat{D}^*}(-z)$ for the canonical Weierstrass product (6.3) and comparison with (7.1) using Proposition 6.2 shows:

Proposition 7.2. *The ε-polynomial is explicitly given with a $k \in \mathbb{Z}$ by*

$$P(z) = 2\pi ik + \sum_{l=0}^{\widehat{g}} \frac{z^l}{l!} \operatorname{CT}_{s=0}(\xi_{\widehat{D}^*}(s+l) - (-1)^l \xi_{\widehat{D}}(s+l)). \tag{7.2}$$

The ξ function of $\widehat{L}^+(z)$ (compare the proof of Proposition 5.1) can obviously be expressed by the $\xi_{\widehat{D}^{(*)}}(s)$:

$$\xi_{\widehat{D}_+}(s) = \frac{e^{\pi is}\xi_{\widehat{D}}(s) - \xi_{\widehat{D}^*}(s)}{e^{\pi is} - e^{-\pi is}}. \tag{7.3}$$

Conclusion. *By* (7.3) *the singular parts of $\xi_{\widehat{D}_+}(s)$ are determined by those of the $\xi_{\widehat{D}^{(*)}}(s)$ except for the residues at $s = l$, $l \in \mathbb{Z}$, as the denominators of* (7.3) *become* 0.

Knowing the ε-polynomial $P(z)$ the missing coefficients can be derived from (7.2) *for all $l \in \mathbb{N}$, and it can be derived up to a summand $k \in \mathbb{Z}$ for $l = 0$.*

Special case symmetry $\widehat{L} = \widehat{L}^*$. To completely derive the singular parts of $\xi_{\widehat{D}_+}(s) = (1 + e^{-\pi is})^{-1}\xi_{\widehat{D}}(s)$ one only needs (the singular parts of $\xi_{\widehat{D}}(s)$ and) the values $\operatorname{CT}_{s=2n+1}(\xi_{\widehat{D}}(s))$, $n \in \mathbb{Z}$. For $n \geq 0$ these values can be derived from the ε-factor by (7.2):

$$P(z) = 2\pi ik + \sum_{n=0}^{[(\widehat{g}-1)/2]} 2 \operatorname{CT}_{s=0} \xi_{\widehat{D}}(s + 2n + 1) \frac{z^{2n+1}}{(2n+1)!} \tag{7.4}$$

with $k \in \mathbb{Z}$. In particular the ε-polynomial is *uneven* up to the unknown $2\pi ik$. The singular parts of $\xi_{\widehat{D}_+}(s)$ at $s \neq 2n + 1$ (in particular for $s = 0$!) are completely determined by those of $\xi_{\widehat{D}}(s)$.

8. Proof of part (c). The idea is to get the asymptotics of $\theta_{\widehat{D}_+}(t)$, $t \to 0$, from the singular parts of $\xi_{\widehat{D}_+}(s)$ by applying Proposition 6.4. And these are derived from (7.3) and (7.2), having as input the information about $\xi_{\widehat{D}^{(*)}}(s)$ available from **GS** and Propositions 6.3 and 6.4.

By Propositions 6.3 and 6.4 the only singular parts of $\xi_{\widehat{D}^{(*)}}(s) = \xi_{D^{(*)}}(s) + \xi_{D_\infty^{(*)}}$ in the half plane $\Re(s) > -1$ are

$$\frac{\alpha_l^{(*)}}{s-l}, \qquad l = 0, 1, \ldots, g_\infty^{(*)}.$$

$\xi_{\widehat{D}}(s)$ and $\xi_{\widehat{D}^*}(s)$ are holomorphic for $\Re(s) \geq \widehat{g} + 1$ by the absolute convergence of the defining series, thus $\alpha_l^{(*)} = 0$ for $l > \widehat{g}$. Using the Laurent expansion at $s = l$,

$$\frac{1}{e^{\pi is} - e^{-\pi is}} = \frac{(-1)^l}{2\pi i} \cdot \frac{1}{s-l} + O(s-l),$$

one gets from (7.3), (7.2) and the described singular parts of $\xi_{\widehat{D}}(s)$ and $\xi_{\widehat{D}^*}(s)$ the singular parts of $\xi_{\widehat{D}_+}(s)$:

$$\frac{\alpha_l - (-1)^l \alpha_l^*}{2\pi i(s-l)^2} + \frac{\alpha_l}{2(s-l)} - \frac{(-1)^l l! \widetilde{p}_l}{2\pi i(s-l)}, \qquad l = 0, \ldots, \widehat{g},$$

with $\widetilde{p}_l = p_l$ for $l \neq 0$ and $\widetilde{p}_0 = p_0 - 2\pi ik$. By part (a) of the Theorem, $\theta_{\widehat{D}_+}(t)$ satisfies an asymptotics of the form (6.5) in the domain of convergence $\pi - \varphi' < \arg(t) < \pi + \varphi'$ with a suitable $0 < \varphi' < \pi/2$. From the just determined singular parts one can derive this asymptotics up to order 1 by Proposition 6.4 (and a trivial rotation):

$$\theta_{\widehat{D}_+}(t) \sim \sum_{l=0}^{\widehat{g}} \left(-\frac{\alpha_l - (-1)^l \alpha_l^*}{2\pi i}\, t^{-l} \log t + \left(\frac{\alpha_l}{2} - \frac{(-1)^l l! \widetilde{p}_l}{2\pi i} \right) t^{-l} \right), \qquad |t| \to 0,$$

which yields the Laurent expansion. For the symmetric case compare the discussion in Section 7. ∎

9. Other regularizations. Definition 6.1 can be generalized: Let $\delta = (\delta_n)_{n=0,1,\ldots}$ be any sequence of complex numbers with $\delta_0 = 1$. Then in the situation of Definition 6.1 the *δ-regularized determinant* is defined by

$$\text{(9.1)} \qquad \Delta^\delta_{\dot{D}}(z) := \exp(-\operatorname{CT}_{s=0}(\delta(s)\xi_{\dot{D}}(s,z)))$$

with the formal power series $\delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \ldots$ So $\delta(s) = 1$ corresponds to ξ-regularization. $\delta(s) = \Gamma^{-1}(s+1)$ is called the *zeta-regularization* and is the most natural one and most frequently used (see [Il2]).

As generalization of Proposition 6.2 one has ([Il2, Theorem 1])

$$\Delta^{\delta}_{\dot{D}}(z) = e^{P_{\dot{D},\delta}(z)} \Delta^{\mathrm{Wei}}_{\dot{D}}(z),$$

$$P_{\dot{D},\delta}(z) = \sum_{l=0}^{g} (-1)^{l+1} \operatorname{CT}_{s=0}(\delta(s)\xi_D(s+l)) \frac{z^l}{l!}.$$

In the situation of the Theorem one can define an ε-polynomial $P^{\delta}(z)$ for the δ-regularization analogously by (7.1). Since by Proposition 6.3, $\xi_D(s,z)$ is holomorphic at $s=0$, we have $L(z) = \Delta^{\delta}_{D}(z)$ independently of δ, and the corresponding fact is true for $L^*(z)$. Using this the following proposition is easily extracted from the above proof of part (c) of the Theorem.

PROPOSITION 9.1. *In the situation of the Theorem the ε-polynomial P^{δ} of the δ-regularization is given by*

$$P^{\delta}(z) = P(z) + 2\pi i k' + \sum_{l=0}^{\widehat{g}} \delta_1(\alpha^*_l - (-1)^l \alpha_l) \frac{z^l}{l!} \tag{9.2}$$

with the $\alpha^{(*)}_l$ from (3.8) *and its "dual", $k' \in \mathbb{Z}$ and $P(z)$ being the ε-polynomial for the ξ-regularization.*

10. Example: Hecke L-series. We now apply the Theorem to Hecke L-series, thus, in particular, rediscover the results of [Cr], [Gu], [Ka]. To do this the functions $\theta_{D^{(*)}_{\infty}}(t)$ have to be found, shown to be meromorphic in $\mathbb{C}$ and the Laurent expansion (3.8) and the ε-polynomial $P(z)$ of the ξ-regularization have to be explicitly determined.

In this case the functions $L^{(*)}_{\infty}(z)$ are products of the usual Γ-functions whose regularizations are obtained from Proposition 10.1. In [Il2, Proposition 8.3] also the regularizations of the *higher* Γ-functions are given, so for any kind of L-series with functional equation including only higher Γ-functions (in particular Selberg-type zeta functions) the application of the Theorem is reduced to trivial calculations.

10.1. *Review of groessencharacters and Hecke L-series.* Let k be an algebraic number field with $n = [k : \mathbb{Q}]$ and for an integral ideal $\mathfrak{m}$ let $J^{\mathfrak{m}}$ be the group of fractional ideals coprime to $\mathfrak{m}$. For every infinite place $\mathfrak{p}$ we have the absolute value $|a|_{\mathfrak{p}} = |i_{\mathfrak{p}}(a)|$ for a corresponding embedding $i_{\mathfrak{p}} : k \hookrightarrow \mathbb{C}$. Let $n_{\mathfrak{p}} = 1$ or 2 if $\mathfrak{p}$ is real or complex, respectively, and for $\mathfrak{a} \in J^{\mathfrak{m}}$ let $N(\mathfrak{a})$ be the ideal norm.

A quasi-character $\chi : J^{\mathfrak{m}} \to \mathbb{C}^*$ such that there exist parameters $m_{\mathfrak{p}} \in \mathbb{Z}$, $\varphi_{\mathfrak{p}} \in \mathbb{R}$ for all $\mathfrak{p} \mid \infty$ and $s_0 \in \mathbb{C}$ with $m_{\mathfrak{p}} \in \{0,1\}$ for real $\mathfrak{p}$ and $\sum_{\mathfrak{p}|\infty} n_{\mathfrak{p}} \varphi_{\mathfrak{p}}$

$= 0$ and

$$\chi((a)) = (N((a)))^{-s_0} \prod_{\mathfrak{p}|\infty} |a|_{\mathfrak{p}}^{-i\varphi_{\mathfrak{p}}} \left(\frac{a_{\mathfrak{p}}}{|a|_{\mathfrak{p}}}\right)^{m_{\mathfrak{p}}} \qquad \text{for } a \in \mathfrak{o}_k,\ a \equiv 1 \bmod \mathfrak{m}$$

is called a *groessencharacter modulo* $\mathfrak{m}$ over k. The parameters $s_0, m_{\mathfrak{p}}, \varphi_{\mathfrak{p}}$ are uniquely determined by χ. From now on we assume that χ is *primitive*, i.e. it is not the restriction of a groessencharacter modulo $\mathfrak{m}'$ with $\mathfrak{m}' \mid \mathfrak{m}$, and that it is *normalized*, i.e. $s_0 = 0$. The results for the general case can easily be derived from this case. The Hecke L-series of χ is defined by the absolutely convergent Euler product

$$L(\chi, s) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}, \qquad \Re(s) > 1, \tag{10.1}$$

over all prime ideals $\mathfrak{p}$ with the convention $\chi(\mathfrak{p}) := 0$ for $\mathfrak{p} \notin J^{\mathfrak{m}}$. With $\Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$ the completed Hecke L-series is defined by

$$\widehat{L}(\chi, s) := L(\chi, s) L_\infty(\chi, s),$$

$$L_\infty(\chi, s) := \prod_{\mathfrak{p}|\infty} \Gamma_{\mathfrak{p}}\left(s + \frac{|m_{\mathfrak{p}}|}{n_{\mathfrak{p}}} + i\varphi_{\mathfrak{p}}\right) \tag{10.2}$$

with $\Gamma_{\mathfrak{p}} = \Gamma_{\mathbb{R}}$ or $\Gamma_{\mathbb{C}}$ if $\mathfrak{p}$ is real or complex, respectively. Let ε_0 be 1 for the trivial groessencharacter $\chi_0 = 1$, else $\varepsilon_0 = 0$ and let d_k denote the discriminant of k. Then $(s(s-1))^{\varepsilon_0}\widehat{L}(\chi, s)$ is an entire function of order 1 with exponent 1 and genus 1 and satisfies the functional equation

$$\widehat{L}(\chi, s) = W(\chi)(N(\mathfrak{m})|d_k|)^{1/2-s}\widehat{L}(\overline{\chi}, 1-s) \tag{10.3}$$

with $|W(\chi)| = 1 = W(\chi_0)$. For the trivial character $\chi = \chi_0$, $L(\chi, s) = \zeta_k(s)$ is the Dedekind zeta function which has a simple pole at $s = 1$. Compare [Ne, Kap. VII, §8] for proofs and [We], [Ba] for the notation.

10.2. *The Theorem for Hecke L-series.* Let χ be a primitive normalized groessencharacter modulo $\mathfrak{m}$ over k with the above notations. We apply the Theorem to $L(z) := L(\chi, z + 1/2)$, $L^*(z) := L(\overline{\chi}, z + 1/2)$, $L_\infty(z) := L_\infty(\chi, z + 1/2)$ and $L_\infty^*(z) := L_\infty(\overline{\chi}, z + 1/2)$. Then we have the absolutely convergent Dirichlet series representation

$$\log L(z) = \sum_{\mathfrak{p}<\infty} \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p})}{m(N(\mathfrak{p}))^{m(z+1/2)}}, \qquad \Re(z) > 1/2,$$

as well as the corresponding one for $\log L^*(z)$, and from (10.2) we get

$$\theta_{D_\infty}(t) = -\sum_{\mathfrak{p}\text{ real}} \frac{e^{(-|m_{\mathfrak{p}}| - i\varphi_{\mathfrak{p}} - 1/2)t}}{1 - e^{-2t}} - \sum_{\mathfrak{p}\text{ comp.}} \frac{e^{(-|m_{\mathfrak{p}}|/2 - i\varphi_{\mathfrak{p}} - 1/2)t}}{1 - e^{-t}}, \tag{10.4}$$

$$(10.5)\quad \theta_{D_\infty^*}(t) = -\sum_{\mathfrak{p}\text{ real}} \frac{e^{(-|m_\mathfrak{p}|+i\varphi_\mathfrak{p}-1/2)t}}{1-e^{-2t}} - \sum_{\mathfrak{p}\text{ comp.}} \frac{e^{(-|m_\mathfrak{p}|/2+i\varphi_\mathfrak{p}-1/2)t}}{1-e^{-t}}.$$

Both functions are meromorphic in $\mathbb{C}$ and keeping in mind that $\sum_{\mathfrak{p}|\infty} n_\mathfrak{p}\varphi_\mathfrak{p} = 0$ we obtain for the Laurent coefficients (3.8):

$$(10.6)\qquad \alpha_1 = \alpha_1^* = -\frac{n}{2}, \qquad \alpha_0 = \alpha_0^* = -\frac{n}{4} + \frac{r_2}{2} + \frac{1}{2}\sum_{\mathfrak{p}|\infty} |m_\mathfrak{p}|,$$

with r_2 being the number of complex places of k.

Now to calculate $P(z)$ we need to determine the ξ-regularized determinant $\Delta_{\widehat{D}^{(*)}}(z) = \Delta_{D^{(*)}}(z)\Delta_{D_\infty^{(*)}}(z)$ of $\widehat{L}^{(*)}(z)$. But we have $\Delta_{D^{(*)}}(z) = L^{(*)}(z)$ by Proposition 6.3 and the $\Delta_{D_\infty^{(*)}}(z)$ can be evaluated with the following application of [Il2, Corollary 8.4].

Proposition 10.1. *The ξ-regularized determinant of the function $\dot{L}(z) = \Gamma(az+b)$, $a>0$, $b \in \mathbb{C}$, is given with the Euler–Mascheroni constant γ by*

$$\Delta_{\dot{D}}(z) = \frac{\Gamma(az+b)}{\sqrt{2\pi}}\, e^{-(\log a-\gamma)(az+b-1/2)}.$$

From (10.2), (7.1) and (10.3) one finally gets (3.7) explicitly: $P(z) = p_1 z + p_0$, where

$$(10.7)\qquad p_1 = \gamma n + \log\left(\frac{(2\pi)^n}{N(\mathfrak{m})|d_k|}\right), \qquad p_0 = \log W(\chi) + 2\pi i k',\ k' \in \mathbb{Z}.$$

Conclusion. *For the functions $\theta_{\widehat{D}_+}(z)$ and $\theta_{\widehat{D}_+^*}(z)$ the statements of the Theorem are true with the parameters given by* (10.4)–(10.7). *In particular*, *equation* (3.6) *is satisfied. For $-\pi-\varepsilon < \arg(t) < \pi+\varepsilon$ the only poles of $\theta_{\widehat{D}_+}(t)$ are the following simple poles*:

- *for all $m \in \mathbb{N}$ and $\mathfrak{p} < \infty$: at $t = m\log N(\mathfrak{p})$ with residue $\frac{\chi(\mathfrak{p})\log N(\mathfrak{p})}{2\pi i(N(\mathfrak{p}))^{m/2}}$ and at $t = -m\log N(\mathfrak{p})$ with residue $\frac{\overline{\chi(\mathfrak{p})}\log N(\mathfrak{p})}{2\pi i(N(\mathfrak{p}))^{m/2}}$*,
- *for all $m \in \mathbb{N}$: at $t = -2\pi i m$ with residue $-n/2$ (with $n = [k:\mathbb{Q}]$)*,
- *for all $m \in \mathbb{N}$: at $t = -\pi i(2m+1)$ with residue $-r_1/2$, where r_1 is the number of real embeddings of k*

(*which can cancel partially*). *Define*

$$w_\mathfrak{p}(t) := \begin{cases} 1, & \mathfrak{p} \textit{ real}, \\ e^{\frac{|m_\mathfrak{p}|}{2}t} + e^{-\frac{|m_\mathfrak{p}|}{2}t}, & \mathfrak{p} \textit{ complex}. \end{cases}$$

Then there is a $k \in \mathbb{Z}$ such that with the Euler–Mascheroni constant γ and with $0 < \arg(t) < \pi$ for $\Im(t) > 0$,

$$\theta_{\widehat{D}_+}(t) - \frac{1}{2\pi i}\Bigg(\sum_{\mathfrak{p}|\infty} e^{-(1/2+i\varphi_{\mathfrak{p}})t}\frac{w_{\mathfrak{p}}(t)}{1-e^{-t}}\Bigg)\log t$$
$$- \frac{1}{2\pi i}\bigg(\log\bigg(\frac{(2\pi)^n}{N(\mathfrak{m})|d_k|}\bigg) + n\bigg(\gamma - \frac{\pi i}{2}\bigg)\bigg)\frac{1}{t}$$
$$+ \frac{n}{8} - \frac{r_2}{4} - \frac{1}{4}\sum_{\mathfrak{p}|\infty}|m_{\mathfrak{p}}| + \frac{\log W(\chi)}{2\pi i} + k$$

is a function which is meromorphic for $t \in \mathbb{C}$ *with a zero at* $t = 0$*. For* $\chi = \overline{\chi}$ *one can choose* $\log W(\chi) = k = 0$*.*

(For checking the result keep in mind $m_{\mathfrak{p}} \in \{0, 1\}$ for $\mathfrak{p}$ real.)

Remark. For the function

$$\theta^+_{\widehat{D}_+}(t) := \sum_{\Im(\varrho)>0} e^{\varrho t}, \qquad \Im(t) > 0,$$

where the summation with multiplicities is over all zeros ϱ of $\widehat{L}(z)$ with positive imaginary part, one also has the functional equation (independent of Guinand's)

$$\overline{\theta^+_{\widehat{D}_+}(\overline{t})} = \theta^+_{\widehat{D}_+}(\exp(\pi i)t), \tag{10.8}$$

if one defines complex conjugation in $\widetilde{\mathbb{C}}^*$ by $|\overline{t}| := t$, $\arg(\overline{t}) := -\arg(t)$. This follows immediately from the fact that $\varrho \mapsto -\overline{\varrho}$ maps the set of zeros ϱ of $\widehat{L}(z) = \widehat{L}(\chi, z + 1/2)$ with $\Im(\varrho) > 0$ onto itself.

References

[Ba] K. Barner, *On A. Weil's explicit formula*, J. Reine Angew. Math. 323 (1981), 139–152.

[CV] P. Cartier et A. Voros, *Une nouvelle interprétation de la formule des traces de Selberg*, in: The Grothendieck Festschrift, Vol. 2, Birkhäuser, 1991, 1–67.

[Cr] H. Cramér, *Studien über die Nullstellen der Riemannschen Zetafunktion*, Math. Z. 4 (1919), 104–130.

[De] C. Deninger, *Evidence for a cohomological approach to analytic number theory*, in: Proc. First European Congr. of Math., Birkhäuser, 1994, 491–510.

[DS] C. Deninger and M. Schröter, *A distribution theoretic proof of Guinand's functional equation for Cramér's V-function and generalizations*, J. London Math. Soc. (2) 52 (1995), 48–60.

[Gu] A. D. Guinand, *Fourier reciprocities and the Riemann zeta function*, Proc. London Math. Soc. (2) 51 (1950), 401–414.

[Il1] G. Illies, *Regularisierte Produkte, Spurformeln und Cramér-Funktionen*, Ph.D. thesis, Schriftenreihe Math. Inst. Univ. Münster 3. Ser. 22, 1998.

[Il2] —, *Regularized products and determinants*, Comm. Math. Phys. 220 (2001), 69–94.

[JL1] J. Jorgenson and S. Lang, *Basic Analysis of Regularized Series and Products*, Lecture Notes in Math. 1564, Springer, Berlin, 1994.

[JL2] —, —, *On Cramér's theorem for general Euler products with functional equation*, Math. Ann. 297 (1993), 383–416.

[JL3] —, —, *Extension of analytic number theory and the theory of regularized harmonic series from Dirichlet series to Bessel series*, ibid. 306 (1996), 75–124.

[Ka] J. Kaczorowski, *The k-functions in multiplicative number theory I*, Acta Arith. 56 (1990), 195–211.

[Ne] J. Neukirch, *Algebraic Number Theory*, Springer, Berlin, 1999.

[Ti] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed., Oxford Univ. Press, Oxford, 1939.

[We] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund 1952, Tome Supplémentaire, 252–265.

Fachbereich Mathematik
Universität-Gesamthochschule Siegen
Friedrich-Ebert-Str. 3
57068 Siegen, Germany
E-mail: illies@math.uni-siegen.de

Received on 29.4.1999
and in revised form on 26.6.2001 (3592)