Corrigendum to Theorem 5 of the paper
“Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$”

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by

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In the proof of Theorem 5 in [2], step 3 is incorrect. We want to thank S. V. Konyagin who has pointed it out. The wrong Theorem 5 asserts that for every increasing sequence of positive integers $x_n$, $n = 1, 2, \ldots$, with a positive lower asymptotic density, if there exists an interval $(u, v)$ containing no limit points of the ratio sequence $x_m/x_n$, $m, n = 1, 2, \ldots$, where $u, v$ are limit points, then there are infinitely many such intervals. In the new form of Theorem 5 we replace intervals $(u, v)$ containing no limit points of $x_m/x_n$ with intervals having some zero asymptotic density of $x_m/x_n$ and we reformulate it in terms of distribution functions of $x_m/x_n$. We prove that if there exists an interval $(u, v)$, containing no limit points of $x_m/x_n$, then every distribution function of $x_m/x_n$ has infinitely many intervals with constant values, assuming positive lower asymptotic density of $x_n$. For an illustration, we give two examples. In Example 1, $x_m/x_n$ has only one such interval $(u, v)$, and in Example 2 it has infinitely many, and in both cases every distribution function of $x_m/x_n$ has infinitely many intervals with constant values. Finally, we discuss via Examples 1 and 2 a possibility of adding a proposition contained in the incorrect step 3 as an assumption of Theorem 5.

To do this we need the following concept used in [3] (see [1] for a general account).

A function $g : [0, 1] \rightarrow [0, 1]$ will be called a distribution function (abbreviated d.f.) if $g(0) = 0$, $g(1) = 1$, and $g$ is nondecreasing. We will identify any two distribution functions coinciding a.e. on $[0, 1]$. A point $\beta \in [0, 1]$
is called a point of increase (or a point of the spectrum) of the d.f. \( g(x) \) if either \( g(x) > g(\beta) \) for every \( x > \beta \) or \( g(x) < g(\beta) \) for every \( x < \beta \), \( x \in [0, 1] \). Now, for \( x_n \) we define the sequence of blocks

\[
X_n = \left( \frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n} \right)
\]

and consider a step d.f.

\[
F(X_n, x) = \frac{\# \{ i \leq n : x_i/x_n < x \}}{n}
\]

for \( x \in [0, 1) \) and \( F(X_n, 1) = 1 \). A d.f. \( g \) is a d.f. of the block sequence \( X_n \) if there exists a sequence of positive integers \( n_1 < n_2 < \ldots \) such that

\[
\lim_{k \to \infty} F(X_{n_k}, x) = g(x)
\]

a.e. on \([0, 1]\). The set of all d.f. of the sequence of blocks \( X_n \) is denoted by \( G(X_n) \). Finally, denote the counting function by \( A(t) = \# \{ n \in \mathbb{N} : x_n < t \} \) and define the lower asymptotic density \( \underline{d} \) and upper asymptotic density \( \overline{d} \) of \( x_n \) by

\[
\underline{d} = \liminf_{t \to \infty} \frac{A(t)}{t} = \liminf_{n \to \infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t \to \infty} \frac{A(t)}{t} = \limsup_{n \to \infty} \frac{n}{x_n}.
\]

A corrected form of Theorem 5 of [2] is as follows:

**Theorem.** Assume that \( \underline{d} > 0 \). If there exists an interval \((u, v) \subset [0, 1]\) such that every \( g \in G(X_n) \) has a constant value on \((u, v)\) (maybe different), then every \( g \in G(X_n) \) has infinitely many intervals with constant values such that \( g \) increases at their endpoints.

**Proof.** Since \( x_i < xx_m \Leftrightarrow x_i < \left( \frac{x_m}{x_n} \right) x_n \), we have

\[
F(X_m, x) = \frac{n}{m} F\left( X_n, \frac{x_m}{x_n} \right)
\]

for every \( m \leq n \) and \( x \in [0, 1) \). Using the Helly selection principle, we can select a subsequence \((m_k, n_k)\) of the sequence \((m, n)\) such that \( F(X_{n_k}) \to g(x) \) and \( F(X_{m_k}) \to \tilde{g}(x) \) as \( k \to \infty \); furthermore \( x_{m_k}/x_{n_k} \to \beta \) and \( n_k/m_k \to \alpha \), but \( \alpha \) may be infinity. Assuming \( \beta > 0 \) and \( g(\beta - 0) > 0 \), we have \( \alpha < \infty \) and

\[
\tilde{g}(x) = \alpha g(x/\beta) \quad \text{a.e. on } [0, 1].
\]

Thus, if \( \tilde{g}(x) \) has a constant value on \((u, v)\), then \( g(x) \) must be constant on the interval \((u\beta, v\beta)\). Furthermore, if \( \underline{d} > 0 \), then for every \( g \in G(X_n) \) we have

\[
(x/\underline{d}) \leq g(x) \leq (\overline{d}/d)x
\]
for every $x \in [0, 1]$. Thus, there exists a sequence $\beta_k \in (0, 1)$ such that $\beta_k \searrow 0$ and $g(x)$ increases at $\beta_k$, $g(\beta_k) > 0$, $k = 1, 2, \ldots$. For such $\beta_k$ and $g(x)$, applying the Helly principle, we can find sequences $\alpha_k$ and $\tilde{g}_k(x) \in G(X_n)$ such that

$$
\tilde{g}_k(x) = \alpha_k g(x\beta_k)
$$
a.e. on $[0, 1]$. Every $\tilde{g}_k(x)$ has a constant value on the interval $(u, v)$, hence, $g(x)$ must be constant on the intervals $(u\beta_k, v\beta_k)$ for $k = 1, 2, \ldots$.

For completeness we provide

Proof of (2). First, we prove

(4) \[ \lim_{k \to \infty} F \left( X_{n_k}, x \frac{x_{m_k}}{x_{n_k}} \right) = g(x\beta). \]

Setting, for abbreviation, $\beta_k = x_{m_k}/x_{n_k}$ and substituting $u = x\beta_k$ we find

$$
0 \leq \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \, dx
\leq \frac{1}{\beta_k} \int_0^{\beta_k} (F(X_{n_k}, u) - g(u))^2 \, du \leq \frac{1}{\beta_k} \int_0^1 (F(X_{n_k}, u) - g(u))^2 \, du \to 0,
$$

which leads to $F(X_{n_k}, x\beta_k) - g(x\beta_k) \to 0$ as $k \to \infty$ (here, necessarily, $\beta > 0$). Furthermore,

$$
\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta))^2 \, dx
= \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k) + g(x\beta_k) - g(x\beta))^2 \, dx
\leq 2 \left( \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2 \, dx + \int_0^1 (g(x\beta_k) - g(x\beta))^2 \, dx \right).
$$

Since $g(x)$ is continuous a.e. on $[0, 1]$, $g(x\beta_k) - g(x\beta) \to 0$ a.e. and applying the Lebesgue dominant convergence theorem, we have $\int_0^1 (g(x\beta_k) - g(x\beta))^2 \, dx \to 0$, which gives (4) and implies (2). Further, $\alpha < \infty$ follows from (1) and $g(\beta - 0) > 0$.

Proof of (3). Since

$$
\# \{ i \leq n : x_i/x_n < x \} = \# \{ i = 1, 2, \ldots : x_i < xx_n \},
$$

we have

$$
\frac{F(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n} \quad \text{for every } x \in [0, 1].
$$
Whenever \( g \in G(X_n) \), there exists \( n_k \) such that \( F(X_{n_k}, x) \to g(x) \) a.e. and \( n_k/x_{n_k} \to d_1 \). Then for some \( d_2(x) \) with \( \lim_{k \to \infty} A(x_{n_k})/(xx_{n_k}) = d_2(x) \) we get

\[
g(x)/x \to d_1 = d_2(x)
\]
a.e. on \([0, 1]\). Using the fact that \( d \leq d_1 \leq \bar{d} \) and \( d \leq d_2 \leq \bar{d} \), we have \((g(x)/x)d \leq \bar{d} \) and \((g(x)/x)\bar{d} \geq d \) a.e. If \( d > 0 \), these inequalities are valid for every \( x \in (0, 1] \). ■

Further properties of \( G(X_n) \) can be found in [3], e.g. if \( d > 0 \), then each \( g \in G(X_n) \) is everywhere continuous on \([0, 1]\).

The basic idea of the following type of sequences \( x_n \) is also due to Konyagin.

**Example 1.** Let \( k_0 < k_1 < k_2 < \ldots \) be an increasing sequence of positive integers, \( n_0 \) and \( m_0 \) be two integers and \( \gamma, \delta \) and \( a \) be real numbers satisfying

(i) \( k_s - k_{s-1} \to \infty \) as \( s \to \infty \),
(ii) \( 0 < \gamma < \delta, a > 1, n_0 \leq m_0 \) and \( 1/a^{n_0} \leq \gamma/\delta \).

(In what follows, we will abbreviate the interval \((\gamma \lambda, \delta \lambda)\) as \((\gamma, \delta)\lambda\).) Let \( x_n \) be an increasing sequence of all integer points lying in the intervals

\[
(\gamma, \delta)a^{k_1m_0n_0+1+2}, \quad 0 \leq j < (k_{s+1} - k_s)m_0, \quad s = 0, 2, 4, \ldots,
\]

\[
(\gamma, \delta)a^{k_1m_0n_0+1+2}, \quad 0 \leq j < (k_{s+1} - k_s)n_0, \quad s = 1, 3, 5, \ldots,
\]
i.e. we have a sequence of intervals of the form \((\gamma, \delta)(a^{n_0})^j\) and \((\gamma, \delta)(a^{m_0})^j\), where these forms alternate on common \((\gamma, \delta)(a^{m_0n_0})^{k_s}\).

**Complement of limit points.** Let \( X \) be the complement in \([0, 1]\) of the limit points of \( x_m/x_n \). Define

\[
I(n_0) = \left( \frac{\delta \gamma}{\gamma a^{n_0}}, \frac{\gamma}{\delta} \right), \quad I(m_0) = \left( \frac{\delta \gamma}{\gamma a^{m_0}}, \frac{\gamma}{\delta} \right),
\]

\[
B(n_0, j) = I(n_0) \cup \frac{I(n_0)}{a^{n_0}} \cup \ldots \cup \frac{I(n_0)}{(a^{n_0})^{j-1}}
\]

\[
\cup \frac{1}{(a^{n_0})^j} \left( I(m_0) \cup \frac{I(m_0)}{a^{m_0}} \cup \frac{I(m_0)}{(a^{m_0})^2} \cup \frac{I(m_0)}{(a^{m_0})^3} \cup \ldots \right),
\]

\[
B(m_0, j) = I(m_0) \cup \frac{I(m_0)}{a^{m_0}} \cup \ldots \cup \frac{I(m_0)}{(a^{m_0})^{j-1}}
\]

\[
\cup \frac{1}{(a^{m_0})^j} \left( I(n_0) \cup \frac{I(n_0)}{a^{n_0}} \cup \frac{I(n_0)}{(a^{n_0})^2} \cup \frac{I(n_0)}{(a^{n_0})^3} \cup \ldots \right).
\]
Then

\[ X = \left( \bigcap_{j=0}^{\infty} B(n_0, j) \right) \cap \left( \bigcap_{j=0}^{\infty} B(m_0, j) \right). \]

Thus, in all cases \( X \supset I(n_0) \) and assuming additionally

(iii) \( 1 < n_0 < m_0, \gcd(n_0, m_0) = 1, \)

(iv) \( \frac{1}{a^{n_0}} < \left( \frac{\gamma}{\delta} \right)^2, \)

(v) \( \left( \frac{\gamma}{\delta} \right)^2 \leq \frac{a^{n_0}}{a^{m_0}}, \left( \frac{\gamma}{\delta} \right)^2 \leq \frac{a^{m_0}}{a^{2n_0}}, \)

(vi) \( \left( \frac{\gamma}{\delta} \right)^2 \leq \frac{(a^{n_0})^{[m_0k/n_0] + 1}}{(a^{m_0})^{k + 1}}, \left( \frac{\gamma}{\delta} \right)^2 \leq \frac{(a^{m_0})^k}{(a^{n_0})^{[m_0k/n_0] + 1}}, k = 1, \ldots, n_0 - 2, \)

we have

\[ X = I(n_0) \neq \emptyset. \]

The assumptions (i)–(vi) hold for \( k_s = s^2, \gamma = 1, \delta = 2, a = 2, n_0 = 3 \)

and \( m_0 = 4. \) Here \( X = (1/2^2, 1/2). \)

**Proof of (5) and (6).** We briefly mention the following steps.

1. For terms \( x_n \in (\gamma, \delta)a^{k_s n_0 + j n_0}, n \to \infty, \) we have two possibilities:
   (a) \( s \) even \( \to \infty, j \) fixed;
   (b) \( s \) even \( \to \infty, j \to \infty. \)

   Similarly, for \( x_n \in (\gamma, \delta)a^{k_s m_0 + j m_0} \) we have
   (c) \( s \) odd \( \to \infty, j \) fixed;
   (d) \( s \) odd \( \to \infty, j \to \infty. \)

   By direct computation we find that \( B(n_0, j) \) is the complement of the limit points of \( x_n/x_n \) having \( x_n \) of type (a), \( B(m_0, j) \) of type (c), \( B(m_0, 0) \) of type (b) and \( B(n_0, 0) \) of type (d).

2. Define

\[
A(n_0) = I(n_0) \cup \frac{I(n_0)}{(a^{n_0})^1} \cup \ldots \cup \frac{I(n_0)}{(a^{n_0})^{m_0-2}} \cup \frac{I(n_0)}{(a^{n_0})^{m_0-1}},
\]

\[
A(m_0) = I(m_0) \cup \frac{I(m_0)}{(a^{m_0})^1} \cup \ldots \cup \frac{I(m_0)}{(a^{m_0})^{n_0-2}} \cup \frac{I(m_0)}{(a^{m_0})^{n_0-1}}.
\]

Since \( A(n_0) \) and \( A(m_0) \) lie in \( I = (\delta/(\gamma a^{m_0 n_0}), \gamma/\delta) \) we have

\[
B(n_0, 0) \cap B(m_0, 0) = \left( A(n_0) \cap A(m_0) \right) \cup \frac{A(n_0) \cap A(m_0)}{a^{m_0 n_0}} \cup \frac{A(n_0) \cap A(m_0)}{a^{2m_0 n_0}} \cup \ldots
\]

\[
\cup \frac{A(n_0) \cap A(m_0)}{a^{3m_0 n_0}} \cup \ldots
\]
3. Assumptions (iii) and (vi) imply

$$A(n_0) \cap A(m_0) = I(n_0) \cup \frac{I(n_0)}{(a_n)^{m_0 - 1}}.$$  

4. Applying (v) we have

$$a^{m_0} A(n_0) \cap A(n_0) = \frac{I(n_0)}{(a_n)^{m_0 - 1}}.$$

which gives

$$B(n_0, 0) \cap B(m_0, 0) \cap B(m_0, n_0 - 1) = I(n_0).$$

**Distribution functions.** Here we assume only (i) and (ii). Define

$$I(n_0, t) = \frac{1}{t \gamma + (1 - t) \delta} \left( \delta \frac{\gamma}{a_n}, \gamma \right), \quad I(m_0, t) = \frac{1}{t \gamma + (1 - t) \delta} \left( \delta \frac{\gamma}{a_m}, \gamma \right),$$

$$I(t) = \frac{1}{t \gamma + (1 - t) \delta} (\gamma, \delta).$$

The set $G(X_n)$ of all d.f. of $X_n$ has the structure

$$G(X_n) = \{g_{n_0, j, t}(x) : j = 0, 1, \ldots, t \in [0, 1]\} \cup \{g_{m_0, j, t}(x) : j = 0, 1, \ldots, t \in [0, 1]\},$$

where the d.f. $g_{n_0, j, t}(x)$ has constant values on the intervals

$$I(n_0, t), \frac{I(n_0, t)}{(a_n)^j}, \ldots, \frac{I(n_0, t)}{(a_n)^{j - 1}}, \frac{I(n_0, t)}{(a_n)^j}, \frac{I(m_0, t)}{(a_m)^j}, \frac{I(m_0, t)}{(a_m)^{j - 1}}, \frac{I(m_0, t)}{(a_m)^j}, \ldots,$$

while on the complement intervals in $[0, 1]$ (7)

$$I(t) = \frac{1}{(a_n)^j (a_m)^{j - 1}}, \frac{I(t)}{(a_n)^j}, \frac{I(t)}{(a_n)^{j - 1}}, \frac{I(t)}{(a_n)^j (a_m)^{j - 1}}, \frac{I(t)}{(a_n)^j (a_m)^{j - 1}}, \ldots$$

it has a constant derivative

$$g'_{n_0, j, t}(x) = 1/d,$$

where $d \leq d \leq \bar{d}$ and

$$d = \frac{\delta - \gamma}{t \gamma + (1 - t) \delta} \left( 1 - t + \frac{1}{(a_n)^{m_0 - 1}} - \frac{1}{(a_n)^j} \left( \frac{1}{(a_n)^{m_0 - 1}} - \frac{1}{(a_m)^{m_0 - 1}} \right) \right).$$

Here

$$d = \frac{\delta - \gamma}{\gamma} \cdot \frac{1}{a_n^{m_0 - 1}}, \quad \bar{d} = \frac{\delta - \gamma}{\delta} \cdot \frac{a_n^{m_0}}{a_n^{m_0} - 1}.$$
These assertions characterize the d.f. \( g_{n_0,j,t}(x) \). Similarly we define d.f. \( g_{m_0,j,t}(x) \), exchanging \( n_0 \) with \( m_0 \) in the intervals and derivatives defined above.

**Proof of (8).** 1. If \( F(X_n,x) \to g(x) \) for some \( n \to \infty \), then we can select a subsequence of \( n \) such that \( n/x_n \to d \) and, for some \( t \in [0,1] \),

\[
x_n = (t\gamma + (1-t)\delta)a_{k_{m_0}n_0+j_{m_0}} + o(a_{k_{m_0}n_0+j_{m_0}}), \quad s \text{ even } \to \infty,
\]

\[
x_n = (t\gamma + (1-t)\delta)a_{k_{m_0}n_0+j_{m_0}} + o(a_{k_{m_0}n_0+j_{m_0}}), \quad s \text{ odd } \to \infty,
\]

and vice versa for any \( t \in [0,1] \) and any \( x_n \) of these forms we have \( n/x_n \to d \), which implies \( F(X_n,x) \to g(x) \) for some d.f. \( g(x) \), since we have

\[
\frac{\Delta F(X_n,x)}{\Delta x} = \frac{1/n}{(i+1)/x_n - i/x_n} = \frac{x_n}{n}
\]
on intervals (7). For such \( x_n \), the complement of (7) contains no \( x_m/x_n \).

2. We directly compute the limit \( d \) for cases (a)–(d) specified in step 1 of the above proof. ■

**Example 2.** In Example 1 we put \( k_s = s \) for \( s = 0,1,2,\ldots \), i.e. \( x_n \) is a sequence of all integer points lying in the intervals

\[
(\gamma, \delta)(a_{n_0}^0), (\gamma, \delta)(a_{n_0}^1), \ldots, (\gamma, \delta)(a_{n_0}^{m_0-1}),
\]

\[
(\gamma, \delta)(a_{n_0}^{m_0}), (\gamma, \delta)(a_{n_0}^{m_0+1}), \ldots, (\gamma, \delta)(a_{n_0}^{2m_0-1}),
\]

\[
(\gamma, \delta)(a_{n_0}^{2m_0}), (\gamma, \delta)(a_{n_0}^{2m_0+1}), \ldots, (\gamma, \delta)(a_{n_0}^{3m_0-1}),
\]

\[
(\gamma, \delta)(a_{n_0}^{3m_0}), (\gamma, \delta)(a_{n_0}^{3m_0+1}), \ldots
\]

**Complement of limit points.** Define

\[
B(n_0,j) = I(n_0) \cup \frac{I(n_0)}{a_{n_0}} \cup \ldots \cup \frac{I(n_0)}{(a_{n_0})_{j-1}}
\]

\[
\cup \frac{1}{(a_{n_0})^j} \left( A(m_0) \cup \frac{A(n_0)}{a_{m_0}n_0} \cup \frac{A(m_0)}{a^{2m_0}n_0} \cup \frac{A(n_0)}{a^{3m_0}n_0} \cup \ldots \right),
\]

\[
B(m_0,j) = I(m_0) \cup \frac{I(m_0)}{a_{m_0}} \cup \ldots \cup \frac{I(m_0)}{(a_{m_0})_{j-1}}
\]

\[
\cup \frac{1}{(a_{m_0})^j} \left( A(n_0) \cup \frac{A(m_0)}{a_{n_0}m_0} \cup \frac{A(n_0)}{a^{2n_0}m_0} \cup \frac{A(m_0)}{a^{3n_0}m_0} \cup \ldots \right).
\]

Then

\[
X = \left( \bigcap_{j=0}^{m_0-1} B(n_0,j) \right) \cap \left( \bigcap_{j=0}^{n_0-1} B(m_0,j) \right).
\]
For $n_0 = m_0$ this gives (cf. [2, Ex. 1])

$$X = \bigcup_{i=0}^{\infty} \frac{I(n_0)}{(a^{n_0})^i}. \quad (10)$$

Assuming (i)–(vi) we have

$$X = I(n_0) \cup \frac{I(n_0)}{a^{2m_0n_0}} \cup \frac{I(n_0)}{a^{4m_0n_0}} \cup \frac{I(n_0)}{a^{6m_0n_0}} \cup \ldots \cup a^{n_0}\left( \frac{I(n_0)}{a^{2m_0n_0}} \cup \frac{I(n_0)}{a^{4m_0n_0}} \cup \frac{I(n_0)}{a^{6m_0n_0}} \cup \ldots \right).$$

Proof of (9) and (10). Similarly to proof of (5) and (6) in Example 1, but the step 4 can only be used for odd $s$, since here $B(m_0, n_0 - 1)$ contains only $a^{m_0}A(n_0)/a^{(2t+1)m_0n_0}$. ■

Distribution functions. As in Example 1,

$$I(n_0, t) = \frac{1}{t\gamma + (1 - t)\delta} \left( \frac{\delta}{a^{n_0}}, \gamma \right), \quad I(m_0, t) = \frac{1}{t\gamma + (1 - t)\delta} \left( \frac{\delta}{a^{m_0}}, \gamma \right), \quad I(t) = \frac{1}{t\gamma + (1 - t)\delta} (\gamma, \delta).$$

The set $G(X_n)$ of all d.f. of $X_n$ has the structure

$$G(X_n) = \{g_{n_0, j, t}(x) : j = 0, 1, \ldots, m_0 - 1, \ t \in [0, 1]\} \cup \{g_{m_0, j, t}(x) : j = 0, 1, \ldots, n_0 - 1, \ t \in [0, 1]\},$$

where the d.f. $g_{n_0, j, t}(x)$ has constant values on the intervals

$$I(n_0, t), \quad \frac{I(n_0, t)}{a^{n_0}}, \ldots, \frac{I(n_0, t)}{(a^{n_0})^{j-1}}, \quad \frac{I(m_0, t)}{(a^{n_0})^j}, \quad \frac{I(m_0, t)}{(a^{m_0})^j a^{n_0}}, \ldots, \frac{I(m_0, t)}{(a^{n_0})^j (a^{m_0})^{n_0-1}}, \frac{I(n_0, t)}{(a^{m_0})^j (a^{m_0 n_0})}, \ldots, \frac{I(n_0, t)}{(a^{n_0})^j (a^{m_0 n_0}) (a^{n_0})^{m_0-1}}, \frac{I(m_0, t)}{(a^{m_0})^j (a^{2m_0 n_0})}, \ldots,$$

while on the complements intervals in $[0, 1]$

$$\left( \frac{\gamma}{t\gamma + (1 - t)\delta}, 1 \right), \quad \frac{I(t)}{a^{n_0}}, \frac{I(t)}{(a^{n_0})^2}, \ldots, \frac{I(t)}{(a^{n_0})^j}, \quad \frac{I(t)}{(a^{n_0})^j (a^{m_0})}, \frac{I(t)}{(a^{n_0})^j (a^{m_0})^2}, \ldots, \frac{I(t)}{(a^{n_0})^j a^{m_0 n_0}}, \quad \frac{I(t)}{(a^{n_0})^j a^{m_0 n_0} (a^{n_0})}, \frac{I(t)}{(a^{n_0})^j a^{m_0 n_0} (a^{n_0})^2}, \ldots,$$

it has a constant derivative

$$g'_{n_0, j, t}(x) = 1/d, \quad (11)$$
Thus we take exists no of positive integers 0 and let
\[ a^0_0; \ldots; a^1_0 = 0 \]

In Example 2 we can construct
\[ x = \left[ \left( t\gamma + (1-t)\delta \right) a^{2s+m_0+j_0} \right], \quad s = 0, 1, \ldots, j = 0, 1, \ldots, m_0 - 1, \]
we take \( x^{g(n)} = \left[ \left( t\gamma + (1-t)\delta \right) a^{(2s-2)m_0+j_0} \right] \) and similarly for 2s + 1. Thus \( \lambda = 1/a^{2m_0+n_0} \).

These assertions characterize d.f. \( g_{n_0,j,t}(x) \). Similarly we define d.f. \( g_{m_0,j,t}(x) \), exchanging \( n_0 \) with \( m_0 \) in the intervals and derivatives defined above.

**Proof of (11).** As the proof of (8) in Example 1.

**Concluding remarks.** Theorem 5 in [2] can also be amended by adding the assertion of the incorrect step 3 to the assumptions of this theorem. This gives the following second correct form: Assume that there exists a sequence of positive integers \( g(n) \) such that \( \lim_{n \to \infty} x^{g(n)}/x_n = \lambda \) and \( 0 < \lambda < 1 \) and let \( \delta > 0 \). If there exists an interval \((u, v)\) containing no limit points of \( x_m/x_n \), then there are infinitely many such intervals, e.g. \((u, v)\lambda^j, j = 0, 1, 2, \ldots \) All possible limits \( \lambda \) form a cyclic group.

By this theorem, for \( x_n \) in Example 1, there exists no such \( \lambda \). We can see this directly, since such \( \lambda \) must be a common term of the following sequences:

\[
\begin{align*}
\frac{1}{a^n_0}, \frac{1}{(a^n_0)^2}, \ldots, \frac{1}{(a^n_0)^j}, \frac{1}{(a^n_0)^j(a^m_0)}, \frac{1}{(a^n_0)^j(a^m_0)^2}, \ldots, \\
\frac{1}{a^m_0}, \frac{1}{(a^m_0)^2}, \ldots, \frac{1}{(a^m_0)^j(a^n_0)}, \frac{1}{(a^m_0)^j(a^n_0)^2}, \ldots,
\end{align*}
\]

For \( j = 0 \) we see that \( \lambda \) must have a form \( 1/a^{k_{m_0+n_0}} \), but for \( j = 1 \) there exists no \( i \) such that \( 1/a^{k_{m_0+n_0}} = 1/a^{n_0}(a^m_0)^i \). Here we use only (i)–(iv).

In Example 2 we can construct \( \lambda \) directly: For
\[ x_n = \left[ \left( t\gamma + (1-t)\delta \right) a^{2sm_0+n_0+j_0} \right], \quad s = 0, 1, \ldots, j = 0, 1, \ldots, m_0 - 1, \]
we take \( x^{g(n)} = \left[ \left( t\gamma + (1-t)\delta \right) a^{(2s-2)m_0+n_0+j_0} \right] \) and similarly for 2s + 1.

Thus \( \lambda = 1/a^{2m_0+n_0} \).
References

