On Karatsuba Conjecture and the Lindelöf Hypothesis

by

Shao-Ji Feng (Beijing)

1. Introduction. Let \( s = \sigma + it \) be a complex variable, and \( \zeta(s) \) the Riemann zeta function. An interesting problem in the theory of the Riemann zeta function is to obtain lower estimates of the function

\[
F(T; \Delta; \sigma) = \max_{T \leq t \leq T + \Delta} |\zeta(\sigma + it)|
\]

for a fixed \( \sigma \), \( 1/2 \leq \sigma \leq 1 \), where \( \Delta = \Delta(T) \) decreases with \( T \). Many important results in this area are described in [2, 6].

The most interesting case is \( \sigma = 1/2 \). Set

\[
F(T; \Delta) = F(T; \Delta; 1/2), \quad G(T; \Delta) = \max_{|s-s_0|=\Delta} |\zeta(s)|,
\]

where \( s_0 = 1/2+iT \). Karatsuba [3, 4] considered the behavior of \( F(T; \Delta) \) and \( G(T; \Delta) \) for \( \Delta = \Delta(T) \to 0 \) as \( T \to \infty \). He stated the following conjectures:

**Conjecture 1.** There exist a constant \( A > 0 \) and a function \( \Delta = \Delta(T) \to 0 \) as \( T \to \infty \), such that for \( T \) large enough,

\[
F(T; \Delta) \geq T^{-A}.
\]

**Conjecture 2.** Conjecture 1 is valid for \( \Delta = (\log \log T)^{-1} \).

**Conjecture 3.** Conjecture 1 is valid for \( \Delta = (\log T)^{-1} \).

**Conjecture 1’.** There exist a constant \( A > 0 \) and a function \( \Delta = \Delta(T) \to 0 \) as \( T \to \infty \), such that for \( T \) large enough,

\[
G(T; \Delta) \geq T^{-A}.
\]

**Conjecture 2’.** Conjecture 1’ is valid for \( \Delta = (\log \log T)^{-1} \).

**Conjecture 3’.** Conjecture 1’ is valid for \( \Delta = (\log T)^{-1} \).

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Obviously for $N = 1, 2, 3$ Conjecture $N$ implies Conjecture $N'$, and Conjecture 3 implies all the other conjectures.

In [2], Garaev proved

**Theorem A.** *Conjecture 3' is equivalent to Conjecture 3.*

It is known that the Riemann hypothesis

all the complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$

implies Conjecture 3 and thus all the Karatsuba conjectures. This can be proved by using [6, Theorems 14.13 and 14.15] (see [2] for the details).

The *Lindelöf Hypothesis* is that

$$\zeta(1/2 + it) = O(t^{\varepsilon})$$

as $t \to \infty$ for every positive $\varepsilon$. There are many equivalent forms of the Lindelöf Hypothesis; one of them is the following.

**Theorem B** ([6, Theorem 13.5]). *A necessary and sufficient condition for the truth of the Lindelöf Hypothesis is that, for every $\sigma > 1/2$,

\begin{equation}
N(\sigma, T + 1) - N(\sigma, T) = o(\log T),
\end{equation}

where $N(\sigma, T)$ is the number of zeros $\beta + i\gamma$ of the Riemann zeta function such that $\beta > \sigma$, $0 < t \leq T$.

By (1.1), we see that the Lindelöf Hypothesis is equivalent to a much less drastic hypothesis about the distribution of the zeros than the Riemann Hypothesis.

A natural question is: what is the relation between the Karatsuba Conjectures and Lindelöf Hypothesis? In this paper, we will prove

**Theorem 1.** *On the Lindelöf Hypothesis, Conjecture 1 is valid for an arbitrary given constant $A > 0.*

The proof of Theorem 1 is based on the following.

**Theorem 2.** *On the Lindelöf Hypothesis, as $T \to \infty$,

\begin{equation}
\max_{0 \leq c \leq 1} \left| \int_{T}^{T+c} \log |\zeta(1/2 + it)| \, dt \right| = o(\log T).
\end{equation}

**2. Proof of Theorem 2.** To prove Theorem 2, we need the following lemmas.

**Lemma 1.** *On the Lindelöf Hypothesis, as $T \to \infty$,

$$\arg \zeta(\sigma + iT) = o(\log T)$$

uniformly for $\sigma \in [1/2, 2]$, where, if $T$ is not the ordinate of a zero of $\zeta(s)$, the value of $\arg \zeta(\sigma + iT)$ is obtained by continuous variation along the
straight lines joining $2, 2 + iT, \sigma + iT$, starting with the value 0; if $T$ is the ordinate of a zero of $\zeta(s)$,

$$\arg \zeta(\sigma + iT) = \lim_{t \to T+0} \arg \zeta(\sigma + it).$$

Proof. We refer to Cramér [1], who proved the case $\sigma = 1/2$, and the proof also applies to the case $1/2 < \sigma < 2$.

**Lemma 2.** On the Lindelöf Hypothesis, as $T \to \infty$,

$$\int_{1/2}^{1} (N(\sigma, T + 1) - N(\sigma, T)) d\sigma = o(\log T).$$

Proof. Let

$$f(\sigma, T) = \frac{N(\sigma, T + 1) - N(\sigma, T)}{\log T}.$$ 

By Theorem B, we have

$$\lim_{T \to \infty} f(\sigma, T) = 0, \quad \forall \sigma \in (1/2, 1].$$

By [6, Theorem 9.2], we have

$$f(\sigma, T) \leq f(1/2, T) \leq M, \quad \forall T > 0, \quad \sigma \in (1/2, 1],$$

where $M$ is an absolute constant. Thus by the Lebesgue Theorem, we get

$$\lim_{T \to \infty} \int_{1/2}^{1} f(\sigma, T) d\sigma = \int_{1/2}^{1} \lim_{T \to \infty} f(\sigma, T) d\sigma = \int_{1/2}^{1} 0 d\sigma = 0.$$

The proof is complete.

We next need a general formula concerning the zeros of an analytic function in a rectangle, due to Littlewood.

**Lemma 3** (see [5, 6]). Suppose that $\phi(s)$ is meromorphic in and upon the boundary of a rectangle bounded by the lines $t = 0, \ t = T, \ \sigma = \alpha, \ \sigma = \beta$ ($\beta > \alpha$), and regular and not zero on $\sigma = \beta$. The function $\log \phi(s)$ is regular in the neighborhood of $\sigma = \beta$, and here, starting with any value of the logarithm, we define $F(s) = \log \phi(s)$. For other points $s$ of the rectangle, we define $F(s)$ to be the value obtained from $\log \phi(\beta + it)$ by continuous variation along $t = \text{constant}$ from $\beta + it$ to $\alpha + it$, provided that the path does not cross a zero or pole of $\phi(s)$; if it does, we put

$$F(s) = \lim_{\varepsilon \to 0} F(\sigma + it + i\varepsilon).$$

Let $\nu(\sigma', T)$ denote the excess of the number of zeros of $\phi(s)$ over the number of poles of $\phi(s)$ in the part of the rectangle for which $\sigma > \sigma'$, including zeros
or poles on \( t = T \), but not those on \( t = 0 \). Then

\[
(2.1) \quad \int F(s) \, ds = -2\pi i \int_{\alpha}^{\beta} \nu(\sigma, T) \, d\sigma,
\]

where the first integral is taken around the rectangle in the positive direction.

Proof of Theorem 2. Applying Lemma 3 with \( \phi(s) = \zeta(s) \), \( \alpha = 1/2 \), \( \beta = 2 \), and taking the imaginary part of (2.1), we get

\[
(2.2) \quad \int_{0}^{T} \log |\zeta(1/2 + it)| \, dt = \int_{0}^{T} \log |\zeta(2 + it)| \, dt
- 2 \int_{1/2}^{2} \arg \zeta(\sigma + iT) \, d\sigma - 2\pi \int_{1/2}^{1} N(\sigma, T) \, d\sigma,
\]

where the value of \( \arg \zeta(\sigma + iT) \) is defined in Lemma 1. Replacing \( T \) with \( T + c \) in (2.2), we obtain

\[
(2.3) \quad \int_{0}^{T+c} \log |\zeta(1/2 + it)| \, dt = \int_{0}^{T+c} \log |\zeta(2 + it)| \, dt
- 2 \int_{1/2}^{2} \arg \zeta(\sigma + iT + c) \, d\sigma - 2\pi \int_{1/2}^{1} N(\sigma, T + c) \, d\sigma.
\]

(2.3) minus (2.2) gives

\[
\int_{T}^{T+c} \log |\zeta(1/2 + it)| \, dt = - 2 \int_{1/2}^{2} (\arg \zeta(\sigma + iT + c) - \arg \zeta(\sigma + iT)) \, d\sigma
+ \int_{T}^{T+c} \log |\zeta(2 + it)| \, dt - 2\pi \int_{1/2}^{1} (N(\sigma, T + c) - N(\sigma, T)) \, d\sigma.
\]

Hence

\[
(2.4) \quad \max_{0 \leq c \leq 1} \left| \int_{T}^{T+c} \log |\zeta(1/2 + it)| \, dt \right|
\leq \max_{0 \leq c \leq 1} \left| \int_{1/2}^{2} (\arg \zeta(\sigma + iT + c) - \arg \zeta(\sigma + iT)) \, d\sigma \right|
+ \max_{0 \leq c \leq 1} \left| \int_{T}^{T+c} \log |\zeta(2 + it)| \, dt \right|
+ \max_{0 \leq c \leq 1} \left| 2\pi \int_{1/2}^{1} (N(\sigma, T + c) - N(\sigma, T)) \, d\sigma \right|.
\]
By Lemma 1,
\begin{align}
\max_{0 \leq c \leq 1} \left| \int_{1/2}^{2} \left( \arg \zeta(\sigma + i(T + c)) - \arg \zeta(\sigma + iT) \right) d\sigma \right| \\
\leq \int_{1/2}^{2} \left( \max_{0 \leq c \leq 1} \left| \arg \zeta(\sigma + i(T + c)) \right| + \left| \arg \zeta(\sigma + iT) \right| \right) d\sigma = o(\log T).
\end{align}

Let \( A(n) = \log p \) if \( n \) is \( p \) or a power of \( p \), and otherwise \( A(n) = 0 \), and let \( A_1(n) = A(n)/\log n \). We have
\[ \log \zeta(2 + it) = \sum_{n=2}^{\infty} \frac{A_1(n)}{n^{2+it}}. \]

Hence
\begin{align}
\max_{0 \leq c \leq 1} \left| \int_{T}^{T+c} \log |\zeta(2 + it)| dt \right| \\
= \max_{0 \leq c \leq 1} \left| \Re \sum_{n=2}^{\infty} \frac{A_1(n)}{n^2} \frac{n^{-i(T+c)} - n^{-iT}}{-i \log n} \right| = O(1) = o(\log T).
\end{align}

By Lemma 2,
\begin{align}
\max_{0 \leq c \leq 1} \left| 2\pi \int_{1/2}^{1} \left( N(\sigma, T + c) - N(\sigma, T) \right) d\sigma \right| \\
= 2\pi \int_{1/2}^{1} \left( N(\sigma, T + 1) - N(\sigma, T) \right) d\sigma = o(\log T).
\end{align}

Combining (2.4)–(2.7), we get (1.2). \( \blacksquare \)

3. Proof of Theorem 1. By Theorem 2, there exists a function \( h(T) \) such that
\begin{align}
(3.1) & \quad h(T) > 0, \quad \forall T > 1, \\
(3.2) & \quad \lim_{T \to \infty} h(T) = 0
\end{align}

and for all \( T > 1 \),
\begin{align}
(3.3) & \quad \max_{0 \leq c \leq 1} \left| \int_{T}^{T+c} \log |\zeta(1/2 + it)| dt \right| = h(T) \log T.
\end{align}

Given arbitrary \( A > 0 \), set
\begin{align}
(3.4) & \quad h_1(T) = \sup_{t \geq T} h(t), \quad T > 1,
\end{align}
\[ \Delta = \Delta(T) = \frac{h_1(T)}{A}. \]

Then \( \Delta(T) \) decreases, and \( \lim_{T \to \infty} \Delta(T) = 0 \). Hence there exists \( T_0 > 1 \) such that \( \Delta(T) \leq 1 \) for \( T > T_0 \). Then by (3.3)–(3.5), for \( T > T_0 \) we have

\[
\left| \int_T^{T+\Delta(T)} \log |\zeta(1/2 + it)| \, dt \right| \leq \Delta(T) A \log T.
\]

Thus

\[
\frac{1}{\Delta(T)} \int_T^{T+\Delta(T)} \log |\zeta(1/2 + it)| \, dt \geq -A \log T.
\]

That is, the mean value of \( \log |\zeta(1/2 + it)| \) on \([T, T + \Delta(T)]\) is not less than \(-A \log T\), so there exists \( t_0 \in [T, T + \Delta(T)] \) such that

\[
\log |\zeta(1/2 + it_0)| \geq -A \log T.
\]

Hence \( |\zeta(1/2 + it_0)| \geq T^{-A} \). Then \( F(T; \Delta) \geq T^{-A} \). The proof is complete. \( \blacksquare \)

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**References**


Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100080, P.R. China
E-mail: jxfsj@hotmail.com

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