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# On Karatsuba Conjecture and the Lindelöf Hypothesis

by

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**1. Introduction.** Let $s = \sigma + it$ be a complex variable, and $\zeta(s)$ the Riemann zeta function. An interesting problem in the theory of the Riemann zeta function is to obtain lower estimates of the function

$$F(T; \Delta; \sigma) = \max_{T \le t \le T+\Delta} |\zeta(\sigma + it)|$$

for a fixed $\sigma$, $1/2 \le \sigma \le 1$, where $\Delta = \Delta(T)$ decreases with $T$. Many important results in this area are described in [2, 6].

The most interesting case is $\sigma = 1/2$. Set

$$F(T; \Delta) = F(T; \Delta; 1/2), \qquad G(T; \Delta) = \max_{|s - s_0| = \Delta} |\zeta(s)|,$$

where $s_0 = 1/2 + iT$. Karatsuba [3, 4] considered the behavior of $F(T; \Delta)$ and $G(T; \Delta)$ for $\Delta = \Delta(T) \to 0$ as $T \to \infty$. He stated the following conjectures:

Conjecture 1. *There exist a constant $A > 0$ and a function $\Delta = \Delta(T) \to 0$ as $T \to \infty$, such that for $T$ large enough,*

$$F(T; \Delta) \ge T^{-A}.$$

Conjecture 2. *Conjecture 1 is valid for $\Delta = (\log\log T)^{-1}$.*

Conjecture 3. *Conjecture 1 is valid for $\Delta = (\log T)^{-1}$.*

Conjecture 1′. *There exist a constant $A > 0$ and a function $\Delta = \Delta(T) \to 0$ as $T \to \infty$, such that for $T$ large enough,*

$$G(T; \Delta) \ge T^{-A}.$$

Conjecture 2′. *Conjecture 1′ is valid for $\Delta = (\log\log T)^{-1}$.*

Conjecture 3′. *Conjecture 1′ is valid for $\Delta = (\log T)^{-1}$.*

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Obviously for $N = 1, 2, 3$ Conjecture $N$ implies Conjecture $N'$, and Conjecture 3 implies all the other conjectures.

In [2], Garaev proved

THEOREM A. *Conjecture* $3'$ *is equivalent to Conjecture* 3.

It is known that the Riemann hypothesis

*all the complex zeros of* $\zeta(s)$ *lie on the critical line* $\sigma = 1/2$

implies Conjecture 3 and thus all the Karatsuba conjectures. This can be proved by using [6, Theorems 14.13 and 14.15] (see [2] for the details).

The *Lindelöf Hypothesis* is that

$$\zeta(1/2 + it) = O(t^{\varepsilon})$$

as $t \to \infty$ for every positive $\varepsilon$. There are many equivalent forms of the Lindelöf Hypothesis; one of them is the following.

THEOREM B ([6, Theorem 13.5]). *A necessary and sufficient condition for the truth of the Lindelöf Hypothesis is that*, *for every* $\sigma > 1/2$,

$$N(\sigma, T+1) - N(\sigma, T) = o(\log T), \tag{1.1}$$

*where* $N(\sigma, T)$ *is the number of zeros* $\beta + i\gamma$ *of the Riemann zeta function such that* $\beta > \sigma$, $0 < t \le T$.

By (1.1), we see that the Lindelöf Hypothesis is equivalent to a much less drastic hypothesis about the distribution of the zeros than the Riemann Hypothesis.

A natural question is: what is the relation between the Karatsuba Conjectures and Lindelöf Hypothesis? In this paper, we will prove

THEOREM 1. *On the Lindelöf Hypothesis*, *Conjecture* 1 *is valid for an arbitrary given constant* $A > 0$.

The proof of Theorem 1 is based on the following.

THEOREM 2. *On the Lindelöf Hypothesis*, *as* $T \to \infty$,

$$\max_{0 \le c \le 1} \left| \int_T^{T+c} \log|\zeta(1/2 + it)|\, dt \right| = o(\log T). \tag{1.2}$$

**2. Proof of Theorem 2.** To prove Theorem 2, we need the following lemmas.

LEMMA 1. *On the Lindelöf Hypothesis*, *as* $T \to \infty$,

$$\arg \zeta(\sigma + iT) = o(\log T)$$

*uniformly for* $\sigma \in [1/2, 2)$, *where*, *if* $T$ *is not the ordinate of a zero of* $\zeta(s)$, *the value of* $\arg \zeta(\sigma + iT)$ *is obtained by continuous variation along the* *straight lines joining* $2$, $2 + iT$, $\sigma + iT$, *starting with the value* $0$; *if* $T$ *is the ordinate of a zero of* $\zeta(s)$,

$$\arg \zeta(\sigma + iT) = \lim_{t \to T+0} \arg \zeta(\sigma + it).$$

*Proof.* We refer to Cramér [1], who proved the case $\sigma = 1/2$, and the proof also applies to the case $1/2 < \sigma < 2$. ∎

LEMMA 2. *On the Lindelöf Hypothesis*, *as* $T \to \infty$,

$$\int_{1/2}^{1} (N(\sigma, T+1) - N(\sigma, T))\, d\sigma = o(\log T).$$

*Proof.* Let

$$f(\sigma, T) = \frac{N(\sigma, T+1) - N(\sigma, T)}{\log T}.$$

By Theorem B, we have

$$\lim_{T \to \infty} f(\sigma, T) = 0, \qquad \forall \sigma \in (1/2, 1].$$

By [6, Theorem 9.2], we have

$$f(\sigma, T) \le f(1/2, T) \le M, \qquad \forall T > 0,\ \sigma \in (1/2, 1],$$

where $M$ is an absolute constant. Thus by the Lebesgue Theorem, we get

$$\lim_{T \to \infty} \int_{1/2}^{1} f(\sigma, T)\, d\sigma = \int_{1/2}^{1} \lim_{T \to \infty} f(\sigma, T)\, d\sigma = \int_{1/2}^{1} 0\, d\sigma = 0.$$

The proof is complete. ∎

We next need a general formula concerning the zeros of an analytic function in a rectangle, due to Littlewood.

LEMMA 3 (see [5, 6]). *Suppose that* $\phi(s)$ *is meromorphic in and upon the boundary of a rectangle bounded by the lines* $t = 0$, $t = T$, $\sigma = \alpha$, $\sigma = \beta$ $(\beta > \alpha)$, *and regular and not zero on* $\sigma = \beta$. *The function* $\log \phi(s)$ *is regular in the neighborhood of* $\sigma = \beta$, *and here*, *starting with any value of the logarithm*, *we define* $F(s) = \log \phi(s)$. *For other points* $s$ *of the rectangle*, *we define* $F(s)$ *to be the value obtained from* $\log \phi(\beta + it)$ *by continuous variation along* $t =$ *constant from* $\beta + it$ *to* $\alpha + it$, *provided that the path does not cross a zero or pole of* $\phi(s)$; *if it does*, *we put*

$$F(s) = \lim_{\varepsilon \to 0} F(\sigma + it + i\varepsilon).$$

*Let* $\nu(\sigma', T)$ *denote the excess of the number of zeros of* $\phi(s)$ *over the number of poles of* $\phi(s)$ *in the part of the rectangle for which* $\sigma > \sigma'$, *including zeros* *or poles on $t = T$, but not those on $t = 0$. Then*

$$\int F(s)\,ds = -2\pi i \int_{\alpha}^{\beta} \nu(\sigma, T)\,d\sigma, \tag{2.1}$$

*where the first integral is taken around the rectangle in the positive direction.*

*Proof of Theorem 2.* Applying Lemma 3 with $\phi(s) = \zeta(s)$, $\alpha = 1/2$, $\beta = 2$, and taking the imaginary part of (2.1), we get

$$\begin{aligned} \int_0^T \log|\zeta(1/2+it)|\,dt &= \int_0^T \log|\zeta(2+it)|\,dt \\ &\quad - \int_{1/2}^{2} \arg\zeta(\sigma+iT)\,d\sigma - 2\pi \int_{1/2}^{1} N(\sigma, T)\,d\sigma, \end{aligned} \tag{2.2}$$

where the value of $\arg\zeta(\sigma + iT)$ is defined in Lemma 1. Replacing $T$ with $T + c$ in (2.2), we obtain

$$\begin{aligned} \int_0^{T+c} \log|\zeta(1/2+it)|\,dt &= \int_0^{T+c} \log|\zeta(2+it)|\,dt \\ &\quad - \int_{1/2}^{2} \arg\zeta(\sigma+i(T+c))\,d\sigma - 2\pi \int_{1/2}^{1} N(\sigma, T+c)\,d\sigma. \end{aligned} \tag{2.3}$$

(2.3) minus (2.2) gives

$$\begin{aligned} \int_T^{T+c} \log|\zeta(1/2+it)|\,dt &= -\int_{1/2}^{2} (\arg\zeta(\sigma+i(T+c)) - \arg\zeta(\sigma+iT))\,d\sigma \\ &\quad + \int_T^{T+c} \log|\zeta(2+it)|\,dt - 2\pi \int_{1/2}^{1} (N(\sigma, T+c) - N(\sigma, T))\,d\sigma. \end{aligned}$$

Hence

$$\begin{aligned} &\max_{0\le c\le 1} \left| \int_T^{T+c} \log|\zeta(1/2+it)|\,dt \right| \\ &\quad \le \max_{0\le c\le 1} \left| \int_{1/2}^{2} (\arg\zeta(\sigma+i(T+c)) - \arg\zeta(\sigma+iT))\,d\sigma \right| \\ &\quad + \max_{0\le c\le 1} \left| \int_T^{T+c} \log|\zeta(2+it)|\,dt \right| \\ &\quad + \max_{0\le c\le 1} \left| 2\pi \int_{1/2}^{1} (N(\sigma, T+c) - N(\sigma, T))\,d\sigma \right|. \end{aligned} \tag{2.4}$$

By Lemma 1,

$$(2.5)\qquad \max_{0\le c\le 1}\Big|\int\limits_{1/2}^{2}(\arg\zeta(\sigma+i(T+c))-\arg\zeta(\sigma+iT))\,d\sigma\Big| \le \int\limits_{1/2}^{2}(\max_{0\le c\le 1}|\arg\zeta(\sigma+i(T+c))|+|\arg\zeta(\sigma+iT)|)\,d\sigma=o(\log T).$$

Let $\Lambda(n)=\log p$ if $n$ is $p$ or a power of $p$, and otherwise $\Lambda(n)=0$, and let $\Lambda_1(n)=\Lambda(n)/\log n$. We have

$$\log\zeta(2+it)=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^{2+it}}.$$

Hence

$$(2.6)\qquad \max_{0\le c\le 1}\Big|\int\limits_{T}^{T+c}\log|\zeta(2+it)|\,dt\Big| = \max_{0\le c\le 1}\Big|\mathrm{Re}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^2}\,\frac{n^{-i(T+c)}-n^{-iT}}{-i\log n}\Big|=O(1)=o(\log T).$$

By Lemma 2,

$$(2.7)\qquad \max_{0\le c\le 1}\Big|2\pi\int\limits_{1/2}^{1}(N(\sigma,T+c)-N(\sigma,T))\,d\sigma\Big| = 2\pi\int\limits_{1/2}^{1}(N(\sigma,T+1)-N(\sigma,T))\,d\sigma=o(\log T).$$

Combining (2.4)–(2.7), we get (1.2). ■

**3. Proof of Theorem 1.** By Theorem 2, there exists a function $h(T)$ such that

$$(3.1)\qquad h(T)>0,\quad \forall T>1,$$

$$(3.2)\qquad \lim_{T\to\infty}h(T)=0$$

and for all $T>1$,

$$(3.3)\qquad \max_{0\le c\le 1}\Big|\int\limits_{T}^{T+c}\log|\zeta(1/2+it)|\,dt\Big|=h(T)\log T.$$

Given arbitrary $A>0$, set

$$(3.4)\qquad h_1(T)=\sup_{t\ge T}h(t),\quad T>1,$$

$$\Delta = \Delta(T) = \frac{h_1(T)}{A}. \tag{3.5}$$

Then $\Delta(T)$ decreases, and $\lim_{T\to\infty}\Delta(T) = 0$. Hence there exists $T_0 > 1$ such that $\Delta(T) \le 1$ for $T > T_0$. Then by (3.3)–(3.5), for $T > T_0$ we have

$$\Bigl|\int_T^{T+\Delta(T)} \log|\zeta(1/2+it)|\,dt\Bigr| \le \Delta(T) A \log T.$$

Thus

$$\frac{1}{\Delta(T)}\int_T^{T+\Delta(T)} \log|\zeta(1/2+it)|\,dt \ge -A\log T.$$

That is, the mean value of $\log|\zeta(1/2+it)|$ on $[T, T+\Delta(T)]$ is not less than $-A\log T$, so there exists $t_0 \in [T, T+\Delta(T)]$ such that

$$\log|\zeta(1/2+it_0)| \ge -A\log T.$$

Hence $|\zeta(1/2+it_0)| \ge T^{-A}$. Then $F(T;\Delta) \ge T^{-A}$. The proof is complete. ∎

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