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The Romanoff theorem revisited

by

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For a subset A of positive integers, define $A(x) = |\{1 \leq a \leq x : a \in A\}|$. Let $\mathcal{P}$ denote the set of all primes and $2^{\mathbb{N}} = \{2^n : n \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. A classical result of Romanoff [6] asserts that the sumset

$$2^{\mathbb{N}} + \mathcal{P} = \{2^n + p : n \in \mathbb{N},\, p \in \mathcal{P}\}$$

has a positive lower density, i.e., there exists a positive constant C_R such that $(2^{\mathbb{N}} + \mathcal{P})(x) \geq C_R x$ for sufficiently large x. Recently, the lower bound of C_R has been calculated in [2, 3, 5]. Now let

$$\mathcal{P}_2 = \{q : q \text{ is a prime or the product of two primes}\}.$$

Motivated by Romanoff's theorem, in this short note we shall show that:

THEOREM 1. *The sumset*

$$2^{\mathcal{P}} + \mathcal{P}_2 = \{2^p + q : p \in \mathcal{P},\, q \in \mathcal{P}_2\}$$

has a positive lower density.

Proof. In our proof, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ will be always absolute.

For $q \in \mathcal{P}_2 \setminus \mathcal{P}$, let $\psi(q)$ be the least prime factor of q. Let

$$\mathcal{P}_2^* = \{q \in \mathcal{P}_2 \setminus \mathcal{P} : \psi(q) < q^{1/3}\}.$$

It suffices to show that $2^{\mathcal{P}} + \mathcal{P}_2^*$ has a positive lower density.

By the Chebyshev theorem, we have

$$\frac{x}{5 \log x} \leq \mathcal{P}(x) \leq \frac{5x}{\log x}.$$

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Hence for $x \geq e^{750}$,

$$\begin{aligned}\mathcal{P}_2^*(x) &= |\{(p_1, p_2) : p_1, p_2 \in \mathcal{P},\ p_1^2 < p_2 \leq x/p_1\}| \\ &\geq \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \leq x^{1/3}}} \left(\frac{x/p_1}{5\log(x/p_1)} - \frac{5p_1^2}{\log(p_1^2)} \right) \\ &\geq \frac{x}{5\log x} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \leq x^{1/3}}} \frac{1}{p_1} - \frac{5x^{1/3}}{\log(x^{1/3})} \cdot \frac{5x^{2/3}}{\log(x^{2/3})} \\ &\geq \frac{x \log\log x}{10 \log x},\end{aligned}$$

since (cf. [1, Theorem 8.8.5])

$$\log\log x \leq \sum_{p \in \mathcal{P} \cap [1,x]} \frac{1}{p} \leq \log\log x + C,$$

where C is an absolute constant.

Similarly it is not difficult to deduce that $\mathcal{P}_2^*(x) \ll x \log\log x / \log x$. Let

$$r(n) = |\{(p,q) : n = 2^p + q,\ p \in \mathcal{P},\ q \in \mathcal{P}_2^*\}|.$$

Clearly we have

$$\begin{aligned}\sum_{n \leq x} r(n) &= |\{(p,q) : p \in \mathcal{P},\ q \in \mathcal{P}_2^*,\ 2^p + q \leq x\}| \\ &\geq 2^{\mathcal{P}}(x/2)\mathcal{P}_2^*(x/2) \\ &\gg \frac{\log x}{\log\log x} \cdot \frac{x \log\log x}{\log x} = x.\end{aligned}$$

And by Cauchy–Schwarz's inequality,

$$\Big(\sum_{n \leq x} r(n) \Big)^2 \leq (2^{\mathcal{P}} + \mathcal{P}_2^*)(x) \sum_{n \leq x} r(n)^2.$$

Therefore we only need to prove that

$$\begin{aligned}(1)\quad &\sum_{n \leq x} r(n)^2 \\ &= |\{(p_1, p_2, q_1, q_2) : p_1, p_2 \in \mathcal{P},\ q_1, q_2 \in \mathcal{P}_2^*,\ 2^{p_1} + q_1 = 2^{p_2} + q_2 \leq x\}|\end{aligned}$$

is $O(x)$.

Below we shall show that

$$(2)\qquad |\{q \leq x - N : q, q + N \in \mathcal{P}_2^*\}| \ll \frac{x(\log\log x)^2}{(\log x)^2} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)$$

for each positive even integer N. Define

$$\mathfrak{S}(n)=\prod_{p\mid n}\left(1+\frac{1}{p}\right).$$

Suppose that k_1, k_2, l_1, l_2 are positive integers such that $(k_i, l_i) = 1$ and $2 \mid k_2 l_1 - k_1 l_2$. Let

$$\mathscr{A} = \{(k_1 n + l_1)(k_2 n + l_2) : 1 \le n \le x\}$$

and $\mathscr{A}_d = \{a \in \mathscr{A} : d \mid a\}$. Then for any square-free d,

$$\mathscr{A}_d = \frac{\omega(d)}{d}\, x + O(\omega(d)),$$

where $\omega(d)$ is a multiplicative function such that for a prime p,

$$\omega(p)=\begin{cases} 2 & \text{if } p\nmid k_1k_2(k_2l_1-k_1l_2),\\ 1 & \text{if } p\nmid k_1k_2 \text{ and } p\mid (k_2l_1-k_1l_2),\\ & \text{or } p\mid k_1 \text{ and } p\nmid k_2, \text{ or } p\mid k_2 \text{ and } p\nmid k_1,\\ 0 & \text{if } p\mid k_1 \text{ and } p\mid k_2.\end{cases}$$

As an application of Selberg's sieve method (cf. [4, Sections 7.2 and 7.3]), we know that

$$|\{1\le n\le x : k_1n+l_1, k_2n+l_2\in\mathcal{P}\}| \ll \frac{x}{(\log x)^2}\,\mathfrak{S}(k_1k_2)\mathfrak{S}(k_2l_1-k_1l_2). \tag{3}$$

Observe that $n, n+N \in \mathcal{P}_2 \setminus \mathcal{P}$ if and only if there exist $p_1, p_2 \in \mathcal{P}$ such that $n/p_1, (n+N)/p_2 \in \mathcal{P}$. Assume that $n/p_1 = p_2 m + l$ where $1 \le l \le p_2$. Then

$$(n+N)/p_2 = (p_1p_2m + p_1l + N)/p_2 = p_1m + (p_1l+N)/p_2,$$

whence $p_1 l \equiv -N \pmod{p_2}$. Note that l is uniquely determined by p_1 and p_2 unless $p_1 = p_2$. Thus

$$\begin{aligned}
&|\{n\le x : n, n+N\in\mathcal{P}_2^*,\ p_1\mid n,\ p_2\mid (n+N)\}|\\
&\qquad\le\begin{cases} |\{m\le x/p_1 : m, m+N/p_1\in\mathcal{P}\}| & \text{if } p_1=p_2,\\ |\{m\le x/p_1p_2 : p_2m+l, p_1m+(p_1l+N)/p_2\in\mathcal{P}\}| & \text{otherwise},\end{cases}\\
&\qquad\ll\begin{cases} \dfrac{x/p_1}{(\log(x/p_1))^2}\,\mathfrak{S}(N/p_1) & \text{if } p_1=p_2\mid N,\\ \dfrac{x/p_1p_2}{(\log(x/p_1p_2))^2}\,\mathfrak{S}(p_1p_2)\mathfrak{S}(N) & \text{otherwise}.\end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\{q\le x-N : q, q+N\in\mathcal{P}_2^*\}|\\
&\ll \sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_1,p_2\le x^{1/3}}} \frac{x/p_1p_2}{(\log(x/p_1p_2))^2}\,\mathfrak{S}(p_1p_2)\mathfrak{S}(N) + \sum_{\substack{p\in\mathcal{P}\\ p\mid N,\, p\le x^{1/3}}} \frac{x/p}{(\log(x/p))^2}\,\mathfrak{S}(N/p).
\end{aligned}$$

Now

$$\sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_1,p_2\le x^{1/3}}} \frac{x/p_1p_2}{(\log(x/p_1p_2))^2}\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)$$

$$\le \frac{36x}{(\log x)^2}\sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_1,p_2\le x^{1/3}}}\frac{1}{p_1p_2} \ll \frac{x(\log\log x)^2}{(\log x)^2}.$$

And

$$\sum_{\substack{p\in\mathcal{P}\\ p\mid N,\, p\le x^{1/3}}} \frac{x/p}{(\log(x/p))^2} \le \sum_{\substack{p\in\mathcal{P}\\ p\le x^{1/3}}} \frac{x/p}{(\log(x/p))^2} \ll \frac{x\log\log x}{(\log x)^2}.$$

This concludes the proof of (2).

Let us return to the proof of (1). Clearly

$$\sum_{n\le x} r(n)^2 \le 2 \sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_2\le p_1\le \log x/\log 2}} |\{q_1\in\mathcal{P}_2^* : 2^{p_1}-2^{p_2}+q_1\in\mathcal{P}_2^*\cap[1,x]\}|.$$

If $p_1=p_2$, then

$$\sum_{q_1\in\mathcal{P}_2^*\cap[1,x]} |\{q_2\in\mathcal{P}_2^*\cap[1,x] : q_2 = 2^{p_1}-2^{p_2}+q_1\}| = \mathcal{P}_2^*(x) \ll \frac{x\log\log x}{\log x}.$$

And if $p_1>p_2$, then

$$\sum_{q_1\in\mathcal{P}_2^*\cap[1,x]} |\{q_2\in\mathcal{P}_2^*\cap[1,x] : q_2 = 2^{p_1}-2^{p_2}+q_1\}|$$

$$\ll \frac{x(\log\log x)^2}{(\log x)^2}\prod_{p\mid(2^{p_1-p_2}-1)}\left(1+\frac{1}{p}\right).$$

Hence we have

$$\sum_{n\le x} r(n)^2 \ll \mathcal{P}\left(\frac{\log x}{\log 2}\right)\frac{x\log\log x}{\log x} + \frac{x(\log\log x)^2}{(\log x)^2}\sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_2<p_1\le\frac{\log x}{\log 2}}}\prod_{p\mid(2^{p_1-p_2}-1)}\left(1+\frac{1}{p}\right)$$

$$\ll \frac{\log x}{\log\log x}\cdot\frac{x\log\log x}{\log x} + \frac{x(\log\log x)^2}{(\log x)^2}\sum_{0<k\le\frac{\log x}{\log 2}} 2\prod_{p\mid(2^k-1)}\left(1+\frac{1}{p}\right)\sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_2<p_1\le\frac{\log x}{\log 2}\\ p_1-p_2=k}} 1$$

$$\ll x + \frac{x(\log\log x)^2}{(\log x)^2}\cdot\frac{2\log x}{(\log\log x)^2}\sum_{0<k\le\frac{\log x}{\log 2}}\prod_{p\mid(2^k-1)}\left(1+\frac{1}{p}\right)\prod_{p\mid k}\left(1+\frac{1}{p}\right).$$

For any positive odd integer d, let $e(d)$ denote the least positive integer such that $2^{e(d)} \equiv 1 \pmod{d}$. Then $2^k \equiv 1 \pmod{d}$ if and only if $e(d) \mid k$. Now

$$\begin{aligned}\sum_{n\le x} r(n)^2 &\ll x + \frac{2x}{\log x} \sum_{0<k\le\frac{\log x}{\log 2}} \prod_{p\mid k}\left(1+\frac{1}{p}\right) \sum_{\substack{d\mid(2^k-1)\\ d \text{ square-free}}} \frac{1}{d} \\ &= x + \frac{2x}{\log x} \sum_{\substack{d \text{ square-free}\\ 2\nmid d}} \frac{1}{d} \sum_{\substack{0<k\le\frac{\log x}{\log 2}\\ e(d)\mid k}} \prod_{p\mid k}\left(1+\frac{1}{p}\right) \\ &= x + \frac{2x}{\log x} \sum_{\substack{d \text{ square-free}\\ 2\nmid d}} \frac{1}{d} \sum_{d' \text{ square-free}} \frac{1}{d'} \sum_{\substack{0<k\le\frac{\log x}{\log 2}\\ e(d)\mid k,\, d'\mid k}} 1 \\ &\le x + \frac{2x}{\log x}\cdot\frac{\log x}{\log 2} \sum_{\substack{d,d' \text{ square-free}\\ 2\nmid d}} \frac{1}{dd'[e(d),d']}.\end{aligned}$$

Our final task is to show that the series

$$\sum_{\substack{d,d' \text{ square-free}\\ 2\nmid d}} \frac{1}{dd'[e(d),d']}$$

converges. Clearly

$$\sum_{\substack{d,d' \text{ square-free}\\ 2\nmid d}} \frac{1}{dd'[e(d),d']} = \sum_{k>0} \sum_{d' \text{ square-free}} \frac{1}{d'[k,d']} \sum_{\substack{d \text{ square-free}\\ e(d)=k}} \frac{1}{d}.$$

Let

$$W(x) = \sum_{0<k\le x} \sum_{\substack{d \text{ square-free}\\ e(d)=k}} \frac{1}{d}.$$

With the help of the arguments of Romanoff (cf. [6], [4, p. 201]), we know that $W(x) \ll \log x$. And

$$\begin{aligned}\sum_{d' \text{ square-free}} \frac{1}{d'[k,d']} &= \frac{1}{k} \prod_{p\in\mathcal{P},\, p\mid k}\left(1+\frac{1}{p}\right) \prod_{p\in\mathcal{P},\, p\nmid k}\left(1+\frac{1}{p^2}\right) \\ &\ll \frac{1}{k} \prod_{p\in\mathcal{P},\, p\mid k}\left(1+\frac{1}{p}\right) \le \frac{1}{\phi(k)} \ll k^{-2/3}.\end{aligned}$$

Hence

$$\sum_{\substack{d,d' \text{ square-free} \\ 2\nmid d}} \frac{1}{dd'[e(d),d']} \ll \int_{1/2}^{\infty} x^{-2/3}\,dW(x) = \int_{1/2}^{\infty} \frac{2W(x)}{3x^{5/3}}\,dx + O(1)$$

$$\ll \int_{1/2}^{\infty} \frac{\log x}{x^{5/3}}\,dx + O(1) \ll 1.$$

This completes the proof. ■

REMARK. Professor Y.-G. Chen communicated to the second author the following two conjectures:

CONJECTURE 1. *Let A and B be two sets of positive integers. If there exists a constant $c > 0$ such that $A(\log x/\log 2)B(x) > cx$ for all sufficiently large x, then the set $\{2^a + b : a \in A,\ b \in B\}$ has a positive lower asymptotic density.*

CONJECTURE 2. *Let A and B be two sets of positive integers. If there exists a constant $c > 0$ such that $A(\log x/\log 2)B(x) > cx$ for infinitely many positive integers x, then the set $\{2^a + b : a \in A,\ b \in B\}$ has a positive upper asymptotic density.*

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