The narrow class groups of some $\mathbb{Z}_p$-extensions over the rationals

by

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1. Introduction. Let $p$ be an odd prime number. Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers, and $\mathbb{B}_\infty$ the $\mathbb{Z}_p$-extension over the rational field $\mathbb{Q}$, that is, the unique abelian extension over $\mathbb{Q}$ in the complex field $\mathbb{C}$ such that the Galois group $\text{Gal}(\mathbb{B}_\infty/\mathbb{Q})$ is topologically isomorphic to the additive group of $\mathbb{Z}_p$. As is well known, the $p$-class group of $\mathbb{B}_\infty$ is trivial (cf. Iwasawa [I]).

Let us choose a prime number $l$ which is a primitive root modulo $p^2$. It is shown in [H1], through the study of circular units in $\mathbb{B}_\infty$, that the $l$-class group of $\mathbb{B}_\infty$ is trivial if

$$p = 3 \text{ or } l \geq \frac{3}{2 \log 2} (p - 1) \varphi(p - 1) \log(p \log p),$$

where $\varphi$ denotes the Euler function. Furthermore, in case $p = 5$ or $p = 7$, the triviality of the $l$-class group of $\mathbb{B}_\infty$ is proved in [H2] by arguments similar to and more precise than those of [H1]. In this paper, using some results of [H1, H2], we shall first prove the following result with the help of a personal computer.

**Theorem 1.** Let $l$ be, as above, a prime number which is a primitive root modulo $p^2$. If $p = 11$ or $p = 13$, then the $l$-class group of $\mathbb{B}_\infty$ is trivial.

Now, for any algebraic extension $K$ of $\mathbb{Q}$ in $\mathbb{C}$, we let $\mathfrak{O}$ denote the ring of algebraic integers in $K$, $\mathcal{I}$ the ideal group of $K$, and $C_+$ the ideal class group of $K$ in the narrow sense, that is, the quotient group of $\mathcal{I}$ modulo the group of principal ideals $\alpha \mathfrak{O}$ in $\mathcal{I}$ for all totally positive elements $\alpha$ of $K$; $C_+$ is also called the narrow class group of $K$. The natural homomorphism of $C_+$ onto the ideal class group of $K$ induces, for every odd prime $q$, an isomorphism of the $q$-primary component of $C_+$ onto the $q$-class group of $K$. The 2-primary component of $C_+$ is called the 2-class group of $K$ in the narrow sense or, simply, the narrow 2-class group of $K$. After discussing the parity of certain

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kinds of class numbers together with a basic criterion by Washington [W1] for such study, we shall secondly prove the following result, still with the help of a (personal) computer in the case \( p = 13 \).

**Theorem 2.** When \( p \leq 13 \), the 2-class group of \( \mathbb{B}_\infty \) in the narrow sense is trivial.

The proof of the above theorem, as well as that of Theorem 1, is essentially based upon the analytic class number formula. Most of our computations are done with Mathematica.

**Remark 1.** A classical theorem of Weber implies the triviality of the narrow 2-class group of the \( \mathbb{Z}_2 \)-extension over \( \mathbb{Q} \), \( \mathbb{Z}_2 \) being the ring of 2-adic integers.

**Remark 2.** Apart from our proof of Theorem 2, when \( p = 11 \) or \( p = 13 \) so that 2 is a primitive root modulo \( p \), assertion IV of Armitage and Fröhlich [AF] shows that Theorem 1 implies Theorem 2.

At the end of the paper, we shall briefly explain how to show, for \( p \leq 487 \), the triviality of the narrow 2-class group of the subfield of \( \mathbb{B}_\infty \) with degree \( p \).

**2. Proof of Theorem 1.** For each integer \( u \geq 0 \), let \( \mathbb{B}_u \) denote the subfield of \( \mathbb{B}_\infty \) with degree \( p^u \), and \( h_u \) the class number of \( \mathbb{B}_u \). Let \( n \) be any positive integer, which will be fixed throughout the paper. Since the prime ideal of \( \mathbb{B}_{n-1} \) dividing \( p \) is totally ramified in \( \mathbb{B}_n \), we know from class field theory that \( h_{n-1} \) divides \( h_n \), i.e., \( h_n / h_{n-1} \) is an integer. Now, for each positive integer \( a \), we denote by \( \mathbb{K}_a \) the cyclotomic field of \( a \)th roots of unity:

\[
\mathbb{K}_a = \mathbb{Q}(e^{2\pi i/a}).
\]

Let \( \nu \) be the number of distinct prime divisors of \( (p-1)/2 \), and let \( g_1, \ldots, g_\nu \) be the prime powers > 1 pairwise relatively prime such that

\[
\frac{p-1}{2} = g_1 \cdots g_\nu.
\]

Let \( V \) denote the subset of the cyclic group \( \langle e^{2\pi i/(p-1)} \rangle \) consisting of

\[
e^{\pi i u_1 / g_1} \cdots e^{\pi i u_\nu / g_\nu}
\]

for all \( \nu \)-tuples \((u_1, \ldots, u_\nu)\) of integers with \( 0 \leq u_1 < g_1, \ldots, 0 \leq u_\nu < g_\nu \). It is naturally understood that \( V = \{1\} \) if \( p = 3 \). Taking the ring \( \mathbb{Z} \) of (rational) integers, let \( \Psi \) denote the set of maps

\[
z : V \to \{ u \in \mathbb{Z} \mid 0 \leq u \leq 2l \}
\]

such that, for some \( \xi \in V \),

\[
l \nmid z(\xi) \quad \text{or} \quad z(\xi) > 0
\]
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according to whether $l > 2$ or $l = 2$, and that

\[ l \mid z(\xi') \quad \text{for all } \xi' \in V \setminus \{\xi\}. \]

We put

\[ M = \max_{z \in \Psi} \Re \left( \sum_{\xi \in V} z(\xi)\xi - 1 \right), \]

where $\Re$ denotes the norm map from $\mathbb{K}_{p-1} = \mathbb{Q}(e^{2\pi i/(p-1)})$ to $\mathbb{Q}$. We easily see that $M$ is a positive integer.

**Lemma 1.** If $l$ divides $h_n/h_{n-1}$, then

\[ p^n \leq M, \quad l < \frac{p-1}{2} \log 2 \log \left( \frac{p^{n+1}}{\pi \sin \frac{\pi}{p}} \right). \]

**Proof.** This follows from [H1, Lemma 4] and [H2, Lemma 2].

Let $p$ be a prime ideal of $\mathbb{K}_{p-1}$ dividing $p$. Let $I$ be the set of positive integers $a < p^{n+1}$ for which $a \equiv \xi \pmod{p^{n+1}}$ with some $\xi \in V$. Let $\mathfrak{F}$ denote the family of all maps from $I$ to $\{0, l\}$ and, for each $a \in I$, let $\mathfrak{G}_a$ denote the family of maps $j : I \to \mathbb{Z}$ such that $\min(l-2, 1) \leq j(a) < l$ and that $j(b) = 0$ or $j(b) = l$ for every $b \in I \setminus \{a\}$. Given any $u \in \mathbb{Z}$, we then let

\[ \mathcal{P}_a(u) = \left\{ (j, y) \in \mathfrak{G}_a \times \mathfrak{F} \left| \sum_{b \in I} ((p^n + 1)j(b) + y(b))b \equiv u \pmod{p^{n+1}} \right. \right\}, \]

\[ \mathcal{Q}_a(u) = \left\{ (j, y) \in \mathfrak{F} \times \mathfrak{G}_a \left| \sum_{b \in I} ((p^n + 1)j(b) + y(b))b \equiv u \pmod{p^{n+1}} \right. \right\}. \]

In the case $l > 2$, we put

\[ s(u) = \sum_{a \in I} \left( \sum_{(j, y) \in \mathcal{Q}_a(u)} (-1)^{\sum_{b \in I} (j(b) + y(b)) + y(a)} y(a) \right. \]

\[ \left. - \sum_{(j, y) \in \mathcal{P}_a(u)} (-1)^{\sum_{b \in I} (j(b) + y(b)) + j(a)} j(a) \right), \]

where, for each integer $r$ relatively prime to $l$, $\tilde{r}$ denotes the positive integer smaller than $l$ such that $rr \equiv 1 \pmod{l}$; in the case $l = 2$, we put

\[ s(u) = \sum_{a \in I} (|\mathcal{Q}_a(u)| - |\mathcal{P}_a(u)|), \]

where, for each finite set $W$, $|W|$ denotes the number of elements of $W$. Lemma 3 of [H2] can now be restated as follows:

**Lemma 2.** If there exist integers $c$ and $d$ satisfying

\[ c \equiv d \pmod{p^n}, \quad s(c) \not\equiv s(d) \pmod{l}, \]

then $l$ does not divide $h_n/h_{n-1}$. 
To prove Theorem 1, let us first consider the case \( p = 11 \). Take any \( z \in \Psi \) and put

\[
\alpha = \sum_{\xi \in V} z(\xi)\xi - 1, \quad \alpha' = \sum_{\xi \in V} z(\xi)\xi^3 - 1.
\]

Let \( \varrho = e^{\pi i/5} \), so that \( V = \{1, \varrho, \varrho^2, \varrho^3, \varrho^4\} \). It follows that

\[
\alpha = z(1) - 1 - z(\varrho^4) + \sum_{u=1}^{3} (z(\varrho^u) + (-1)^{u-1}z(\varrho^4))\varrho^u,
\]

\[
|\alpha|^2 \in \mathbb{Z}[\varrho + \varrho^{-1}], \quad \varrho + \varrho^{-1} = \frac{1 + \sqrt{5}}{2}.
\]

Let \( A \) and \( B \) be the integers determined by

\[
|\alpha|^2 = A + B(\varrho + \varrho^{-1}).
\]

Then

\[
A = (z(1) - 1)(z(1) - 1 + z(\varrho^3) - z(\varrho^2)) + z(\varrho)z(\varrho^4) + z(\varrho)^2 - z(\varrho)z(\varrho^3) + z(\varrho^3)^2 + z(\varrho^2)^2 - z(\varrho^2)z(\varrho^4) + z(\varrho^4)^2,
\]

\[
\mathfrak{N}(\alpha) = |\alpha\alpha'|^2 = A^2 + AB - B^2 = \frac{5A^2}{4} - \left( \frac{A}{2} - B \right)^2.
\]

In particular, the former equation above implies \( A \geq 0 \), because

\[
A \geq (z(1) - 1)^2 - \frac{(z(1) - 1)^2 + z(\varrho^2)^2}{2} - \frac{(z(1) - 1)^2 + z(\varrho^2)^2}{2}
\]

\[
+ z(\varrho)^2 + z(\varrho^3)^2 + z(\varrho^3)^2 + z(\varrho^4)^2 - \frac{z(\varrho^2)^2 + z(\varrho^4)^2}{2} + z(\varrho^4)^2
\]

\[
= \frac{z(\varrho)^2 + z(\varrho^4)^2}{2}.
\]

Hence, noting that

\[
z(\varrho)^2 - z(\varrho)z(\varrho^3) + z(\varrho^3)^2 \leq 4l^2,
\]

\[
z(\varrho^2)^2 - z(\varrho^2)z(\varrho^4) + z(\varrho^4)^2 \leq 4l^2,
\]

we have

\[
\mathfrak{N}(\alpha) < \frac{5}{4} (5(4l^2))^2 = 500l^4.
\]

This gives

\[
M < 500l^4.
\]
Let $S$ be the set of the following pairs of integers:

\[
(1, 2), \quad (1, 7), \quad (1, 13), \quad (1, 17), \quad (2, 2), \quad (2, 7), \\
(2, 13), \quad (2, 17), \quad (2, 19), \quad (2, 29), \quad (3, 2), \quad (3, 7), \\
(3, 13), \quad (3, 17), \quad (3, 19), \quad (3, 29), \quad (3, 41), \quad (4, 7), \\
(4, 13), \quad (4, 17), \quad (4, 19), \quad (4, 29), \quad (4, 41), \quad (4, 61), \\
(5, 7), \quad (5, 13), \quad (5, 17), \quad (5, 19), \quad (5, 29), \quad (5, 41), \\
(5, 61), \quad (5, 73), \quad (5, 79), \quad (5, 83), \quad (6, 13), \quad (6, 17), \\
(6, 19), \quad (6, 29), \quad (6, 41), \quad (6, 61), \quad (6, 73), \quad (6, 79), \\
(6, 83), \quad (6, 101), \quad (7, 17), \quad (7, 19), \quad (7, 29), \quad (7, 41), \\
(7, 61), \quad (7, 73), \quad (7, 79), \quad (7, 83), \quad (7, 101), \quad (7, 107), \\
(8, 29), \quad (8, 41), \quad (8, 61), \quad (8, 73), \quad (8, 79), \quad (8, 83), \\
(8, 101), \quad (8, 107), \quad (8, 127), \quad (9, 61), \quad (9, 73), \quad (9, 79), \\
(9, 83), \quad (9, 101), \quad (9, 107), \quad (9, 127), \quad (9, 139), \quad (9, 149), \\
(9, 151), \quad (10, 101), \quad (10, 107), \quad (10, 127), \quad (10, 139), \quad (10, 149), \\
(10, 151), \quad (10, 167), \quad (11, 167), \quad (11, 173).
\]

By simple calculations, we find that the inequalities

\[
11^n < 500l^4, \quad l < \frac{5}{\log 2} \log \left( \frac{11^{n+1}}{\pi} \sin \frac{\pi}{11} \right)
\]

hold if and only if $(n, l) \in S$. Hence Lemma 1 implies that $(n, l) \in S$ if $l$ divides $h_n/h_{n-1}$.

Assume now that $(n, l) \in S$. For each $r$ in $\{1, 2, 3, 4\}$, let $b_r$ denote the integer such that

\[
b_r \equiv 2^{11^n r} \pmod{11^{n+1}}, \quad 0 < b_r < 11^{n+1}.
\]

Since 2 is a primitive root modulo $11^{n+1}$, we can take as $\mathfrak{p}$ the prime ideal of $\mathbb{K}_{10} = \mathbb{Q}(\varrho)$ generated by 11 and $b_1 - \varrho$. We then have

\[
I = \{1, b_1, b_2, b_3, b_4\}.
\]

In view of Lemma 2 and [H2, Lemma 1], it suffices for our proof to find integers $c$ and $d$ which satisfy

\[
c \equiv d \pmod{11^n}, \quad s(c) \neq s(d) \pmod{l}.
\]

By using a (personal) computer, we have computed $s(u)$ for suitable integers $u$ after the determination of $\mathcal{P}_a(u)$ and $\mathcal{Q}_a(u)$ for all $a \in I$. When $n \geq 4$ but $(n, l) \neq (4, 61)$, the computations show that

\[
s(1) = 1, \quad s(1 + 11^n) = -1.
\]

Furthermore,

\[
s(1) = 0, \quad s(1 + 3 \cdot 11^3) = -1, \quad \text{if } (n, l) = (3, 2); \\
s(1) = 1, \quad s(1 + 3 \cdot 11^3) = -2, \quad \text{if } (n, l) = (2, 2).
\]
Results for the other cases of \((n, l)\) are given in the following table:

<table>
<thead>
<tr>
<th>((n, l))</th>
<th>(s(1))</th>
<th>(s(1 + 11^n))</th>
<th>((n, l))</th>
<th>(s(1))</th>
<th>(s(1 + 11^n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, 61)</td>
<td>-21</td>
<td>-48</td>
<td>(2, 19)</td>
<td>69</td>
<td>61</td>
</tr>
<tr>
<td>(3, 41)</td>
<td>-31</td>
<td>-36</td>
<td>(2, 17)</td>
<td>21</td>
<td>-3</td>
</tr>
<tr>
<td>(3, 29)</td>
<td>24</td>
<td>17</td>
<td>(2, 13)</td>
<td>-45</td>
<td>21</td>
</tr>
<tr>
<td>(3, 19)</td>
<td>5</td>
<td>52</td>
<td>(2, 7)</td>
<td>25</td>
<td>-22</td>
</tr>
<tr>
<td>(3, 17)</td>
<td>5</td>
<td>-5</td>
<td>(1, 17)</td>
<td>-257</td>
<td>203</td>
</tr>
<tr>
<td>(3, 13)</td>
<td>-5</td>
<td>-29</td>
<td>(1, 13)</td>
<td>-56</td>
<td>12</td>
</tr>
<tr>
<td>(3, 7)</td>
<td>1</td>
<td>-2</td>
<td>(1, 7)</td>
<td>-23</td>
<td>-32</td>
</tr>
<tr>
<td>(2, 29)</td>
<td>-36</td>
<td>103</td>
<td>(1, 2)</td>
<td>-2</td>
<td>-9</td>
</tr>
</tbody>
</table>

We thus obtain the conclusion for \(p = 11\).

Let us next deal with the case \(p = 13\). Take any \(z \in \Psi\) and put

\[
\alpha = \sum_{\xi \in V} z(\xi)\xi - 1, \quad \alpha' = \sum_{\xi \in V} z(\xi)\xi^7 - 1.
\]

Clearly, \(\Re(\alpha) = |\alpha\alpha'|^2\). Let \(\varrho = i e^{2\pi i/3}\), so that

\[
V = \{1, \varrho, \varrho^2, -\varrho^3, \varrho^4, -\varrho^5\}, \quad -\varrho^3 = i, \quad \varrho^4 = e^{2\pi i/3}, \quad i\varrho^4 = \varrho.
\]

Let further

\[
\begin{align*}
a_1 &= z(1) - 1 + z(\varrho^2), \quad a_2 = z(\varrho^2) + z(\varrho^4), \\
a_3 &= z(i) + z(-\varrho^5), \quad a_4 = z(\varrho) + z(-\varrho^5).
\end{align*}
\]

Then

\[
\begin{align*}
\alpha &= a_1 + a_2 \varrho^4 + a_3 i + a_4 i\varrho^4, \\
\alpha' &= a_1 + a_2 \varrho^4 - a_3 i - a_4 i\varrho^4.
\end{align*}
\]

We therefore see that

\[
\Re(\alpha) = |(a_1 + a_2 \varrho^4)^2 + (a_3 + a_4 \varrho^4)^2|^2 \\
\leq |a_1 + a_2 \varrho^4|^2 |a_3 + a_4 \varrho^4|^2 \\
= (a_1^2 - a_1 a_2 + a_2^2 + a_3^2 - a_3 a_4 + a_4^2)^2.
\]

Since \(a_1^2 - a_1 a_2 + a_2^2 < 16l^2\) and \(a_3^2 - a_3 a_4 + a_4^2 \leq 16l^2\), it follows that

\[
\Re(\alpha) < (32l^2)^2 = 2^{10}l^4.
\]

Hence we have

\[
M < 2^{10}l^4.
\]
On the other hand, let $S$ be the set of the following pairs of integers:

$$(1, 2), (1, 7), (1, 11), (2, 2), (2, 7), (2, 11),
(2, 37), (3, 2), (3, 7), (3, 11), (3, 37), (3, 41),
(3, 59), (4, 7), (4, 11), (4, 37), (4, 41), (4, 59),
(4, 67), (4, 71), (5, 7), (5, 11), (5, 37), (5, 41),
(5, 59), (5, 67), (5, 71), (5, 97), (6, 11), (6, 37),
(6, 41), (6, 59), (6, 67), (6, 71), (6, 97), (7, 37),
(7, 41), (7, 59), (7, 67), (7, 71), (7, 97), (7, 137),
(7, 149), (8, 37), (8, 41), (8, 59), (8, 67), (8, 71),
(8, 97), (8, 137), (8, 149), (8, 163), (8, 167), (9, 59),
(9, 67), (9, 71), (9, 97), (9, 137), (9, 149), (9, 163),
(9, 167), (9, 193), (9, 197), (10, 137), (10, 149), (10, 163),
(10, 167), (10, 193), (10, 197), (11, 223), (11, 227), (11, 241).$$

Simple calculations show that $\ (n, l) \in S$ if and only if

$$13^n < 2^{10}l^4, \quad l < \frac{6}{\log 2} \log \left(\frac{13^{n+1}}{\pi \sin \frac{\pi}{13}}\right).$$

Lemma 1 therefore implies that $\ (n, l) \in S$ if $l$ divides $h_n/h_{n-1}$.

Suppose $\ (n, l) \in S$ if $l$ divides $h_n/h_{n-1}$. Let $b_1$ be the integer such that

$$b_1 \equiv 2^{13^n} \pmod{13^{n+1}}, \quad 0 < b_1 < 13^{n+1}.$$

For each $r \in \{2, 3, 4, 5\}$, let $b_r$ denote the integer such that

$$b_r \equiv (-b_1)^r \pmod{13^{n+1}}, \quad 0 < b_r < 13^{n+1}.$$

Since 2 is a primitive root modulo $13^{n+1}$, we take as $p$ the prime ideal of

$$K_{12} = \mathbb{Q}(\varrho)$$

generated by 13 and $b_1 - \varrho$. We then see that

$$I = \{1, b_1, b_2, b_3, b_4, b_5\}.$$

By Lemma 2 and [H2, Lemma 1], it suffices to find integers $c$ and $d$ which satisfy

$$c \equiv d \pmod{13^n}, \quad s(c) \not\equiv s(d) \pmod{l}.$$

Using a computer as in the case $p = 11$, we have computed $s(u)$ for suitable integers $u$. When $n \geq 4$ but $(n, l) \neq (5, 71), (5, 67), (4, 71), (4, 41)$, the computations yield

$$s(1) = 1, \quad s(1 + 13^n) = -1.$$
Results for the other cases of \((n, l)\) are given in the following tables:

\[
\begin{array}{ccc}
(n, l) & s(1) & s(1 + 13^n) \\
(5, 71) & -48 & -1 \\
(5, 67) & -31 & -1 \\
(4, 71) & 181 & -8 \\
(4, 41) & 12 & -1 \\
(3, 59) & 84 & -133 \\
(3, 41) & 13 & -80 \\
(3, 37) & 143 & -9 \\
(3, 11) & -9 & 3 \\
(2, 37) & 14 & 266 \\
(2, 11) & -107 & 55 \\
(2, 7) & 6 & -34 \\
(1, 11) & 16 & 101 \\
(1, 7) & 48 & 4 \\
\end{array}
\]

\[
\begin{array}{ccc}
(n, l) & s(1) & s(1 + 5 \cdot 13^n) \\
(3, 7) & 1 & 0 \\
(3, 2) & 0 & -5 \\
(2, 2) & 1 & 2 \\
(1, 2) & 13 & 26 \\
\end{array}
\]

Thus our proof is completed.

3. Some lemmas. In this section we give some preliminary results for the proof of Theorem 2. Let \(t\) be the positive integer such that \(2^t\) is the highest power of 2 dividing \(p - 1\). Let \(k\) be the subfield of \(K_p\) with degree \(2^t\), whence \(k\) is an imaginary abelian extension over \(\mathbb{Q}\). For each integer \(u \geq 0\), we denote by \(h_u^-\) the relative class number of the composite \(k\mathbb{B}_u\).

**Lemma 3.** The class number of \(\mathbb{B}_n\) in the narrow sense is odd if and only if \(h_n^-\) is odd.

**Proof.** Let \(p\) be the unique prime ideal of \(\mathbb{B}_n\) dividing \(p\). Since \(k\mathbb{B}_n\) is an abelian 2-extension over \(\mathbb{B}_n\) and since no prime ideal of \(\mathbb{B}_n\) other than \(p\) is ramified in \(k\mathbb{B}_n\) but \(p\) is fully ramified in \(k\mathbb{B}_n\), a well-known argument of [I] tells us that the class number of \(k\mathbb{B}_n\) is odd if and only if the class number of \(\mathbb{B}_n\) in the narrow sense is odd (cf. the proof of Washington [W2, Theorem 10.4]). On the other hand, \(k\mathbb{B}_n\) is a cyclic extension over \(\mathbb{Q}\) so that, by Hasse [H, Satz 45], the indivisibility \(2 \nmid h_n^-\) means that the class number of \(k\mathbb{B}_n\) is odd. Therefore, the lemma follows. \(\blacksquare\)

**Remark 3.** As is seen from the above proof, Lemma 3 still holds even if one replaces \(\mathbb{B}_n\) by any intermediate field of the extension \(k\mathbb{B}_n/\mathbb{B}_n\).

Let \(\mathcal{X}\) be the set of primitive Dirichlet characters of order \(2^t\) with conductor \(p\), so that all Dirichlet characters in \(\mathcal{X}\) are odd. Let \(\mathcal{Y}\) be the set of primitive Dirichlet characters of order \(p^n\) with conductor \(p^{n+1}\). Since \(k\mathbb{B}_n\) does not contain \(i\) and since the unit indices of \(k\mathbb{B}_n\) and \(k\mathbb{B}_{n-1}\) are equal
to 1, the analytic class number formula implies that

\[ \frac{h_{n-1}^-}{h_n^-} = \tilde{p} \prod_{\chi \in \mathcal{X}} \prod_{\psi \in \mathcal{Y}} \left( -\frac{1}{2p^{n+1}} \sum_{a=1}^{p^{n+1}} \chi \psi(a) a \right), \]

where \( \tilde{p} = p \) or \( \tilde{p} = 1 \) according to whether \( p - 1 \) is a power of 2 or not (cf. [H, §36, (3)]). Furthermore, the right hand side of (1) is known to be an integer: \( h_{n-1}^- \mid h_n^- \) (cf. [H, Satz 32]).

Let \( R \) be a set of positive integers smaller than \( p \) such that
\[ R \cap \{ p - a \mid a \in R \} = \emptyset, \quad R \cup \{ p - a \mid a \in R \} = \{ 1, \ldots, p - 1 \}. \]

Given any integer \( u \geq 0 \), let \( R_u \) denote the set of integers \( b \) for which \( b^{p-1} \equiv 1 \pmod{p^{u+1}} \), \( 0 < b < p^{u+1} \), and \( b \equiv a \pmod{p} \) with some \( a \in R \). It then follows that \( |R_u| = |R| = (p - 1)/2 \), because, for each \( a \in R \), there exists a unique \( b \in R_u \) with \( b \equiv a \pmod{p} \). Obviously, \( R_0 = R \). Take any positive integer \( m \leq (n + 1)/2 \):
\[ n \geq 2m - 1 \geq 1, \quad \text{i.e.,} \quad n - m + 1 \geq m \geq 1. \]

For each integer \( a \) not divisible by \( p \), let \( a_* \) denote the integer such that
\[ a_* \equiv a \pmod{p^{n-m+1}}, \quad 0 < a_* < p^{n-m+1}, \]
and let \( a^* \) denote the integer such that
\[ aa^* \equiv 1 \pmod{p^m}, \quad 0 < a^* < p^m. \]

To state the following lemma, we note that, for any \( \psi \in \mathcal{Y} \), \( \psi(1 + p^{n-m+1}) \) is a primitive \( p^m \)th root of unity.

**Lemma 4.** Let \( \psi \) be a Dirichlet character in \( \mathcal{Y} \). Assume that \( 2^{p-1} \equiv 1 \pmod{p^{m+1}} \), i.e., \( \mathbb{K}_{p^m} \) contains the decomposition field of 2 for the abelian extension \( \mathbb{K}_p \mathbb{B}_\infty / \mathbb{Q} \), and that, for some integer \( c \) not divisible by \( p \), the algebraic number
\[ \sum_{b \in R_{n-m}} \frac{\psi(1 + p^{n-m+1})(bc)^*}{\psi(c)^{-1}(bc)_* - 1} \]

is relatively prime to 2. Then the integer \( h_n^- / h_{n-1}^- \) is odd.

**Proof.** Let \( \chi \) be any Dirichlet character in \( \mathcal{X} \). We put
\[ \Theta = -\frac{1}{2p^{n+1}} \sum_{a=1}^{p^{n+1}} \chi \psi(a) a, \quad \omega = \psi(1 + p^{n-m+1}). \]

The field in \( \mathbb{C} \) generated by the images of \( \chi \) and \( \psi \) over \( \mathbb{Q} \) is \( \mathbb{K}_{2^t p^n} \) and, by [H, Satz 32], \( (\omega - 1)\Theta \) is an algebraic integer. Let \( \mathfrak{T} \) denote the trace map from \( \mathbb{K}_{2^t p^n} \) to \( \mathbb{K}_{2^t p^m} \). The argument in the first part of [W1, §IV] then shows
that
$$
\mathcal{T}(\psi(c)^{-1}\Theta) = p^{n-m} \sum_{b \in R_{n-m}} \psi(c)^{-1} \chi_{\psi((bc)*)} \frac{\psi((bc)*)}{\omega^{(bc)*)-1} - 1}
$$
(cf., in particular, [W1, (**)]). Let \(i\) be the integral ideal of \(K_{2t_p^m}\) generated by \(e^{\pi i/2^t-1} - 1\), so that, in \(K_{2t_p^n}\),
$$
\chi_{\psi((bc)*)} \equiv \psi((bc)*) \pmod{i}
$$
for each \(b \in R_{n-m}\). Therefore,
$$
\mathcal{T}(\psi(c)^{-1}(\omega - 1)\Theta) \equiv p^{n-m}(\omega - 1) \sum_{b \in R_{n-m}} \psi(c)^{-1} \psi((bc)*) \frac{\psi((bc)*)}{\omega^{(bc)*)-1} - 1} \pmod{i}.
$$
We see as well that \(i\) is the product of all prime ideals of \(K_{2t_p^m}\) dividing \(2\). Hence, by the assumption, any prime ideal of \(K_{2t_p^m}\) dividing \(2\) does not divide \((\omega - 1)\Theta\); indeed, it remains prime in \(K_{2t_p^n}\). Thus the norm of \((\omega - 1)\Theta\) for \(K_{2t_p^n}/\mathbb{Q}\) is an odd integer. We can now deduce from (1) that \(h_{n}/h_{n-1}^{-1}\) is odd.

We may omit \(\psi(c)^{-1}\) in the statement of Lemma 4, while we should note that \(\psi(c')^{-1}\psi((bc')*)\) is a \(p^m\)th root of unity for any \((b, c') \in R_{n-m} \times \mathbb{Z}\) with \(p \nmid c'\). Further, not only do we have \(\{a^{*} \mid a \in R_{m-1}\} = \{b^{*} \mid b \in R_{n-m}\}\) but also \(R\) may be replaced, from the start, by the set of positive integers \(a' < p\) such that \(a'a \equiv 1 \pmod{p}\) for some \(a \in R\). Thus Lemma 4 gives us the following.

**Lemma 5.** Let \(\zeta\) be any primitive \(p^m\)th root of unity. Assume that \(2^{p-1} \not\equiv 1 \pmod{p^{m+1}}\) and that, for each map \(f\) from \(R_{m-1}\) to the set of non-negative integers smaller than \(p^m\),
$$
\sum_{a \in R_{m-1}} \frac{\zeta^{f(a)}}{\zeta^a - 1}
$$
is relatively prime to \(2\). Then \(h_n^{-1}/h_{n-1}^{-1}\) is odd.

Let us give one more result.

**Lemma 6.** Let \(\zeta\) be any primitive \(p^m\)th root of unity, \(u\) any integer, \(N\) any positive integer, and \(\mu\) any map from \(\{1, \ldots, N\}\) to \(\mathbb{Z}\). Assume that \(2^{p-1} \not\equiv 1 \pmod{p^{r+1}}\) with a positive integer \(r < m\) and that
$$
\sum_{c=1}^{N} \zeta^{\mu(c)} \equiv 0 \pmod{1}
$$
with a prime ideal \(I\) of \(K_{pr}\) dividing \(2\). Then
$$
\sum_{c'} \zeta^{\mu(c')} \equiv 0 \pmod{1},
$$
where \( c' \) ranges over all positive integers not exceeding \( N \) such that \( \mu(c') \equiv u \) (mod \( p^{m-r} \)).

**Proof.** Let \( T \) be the trace map from \( \mathbb{K}_p^m \) to \( \mathbb{K}_p^r \). By the hypothesis, \( l \) remains prime in \( \mathbb{K}_p^m \) and so

\[
T\left( \sum_{c=1}^{N} \zeta^{\mu(c)-u} \right) \zeta^u \equiv 0 \pmod{l}.
\]

However, for each positive integer \( c \leq N \),

\[
T(\zeta^{\mu(c)-u}) = p^{m-r} \zeta^{\mu(c)-u} \quad \text{or} \quad T(\zeta^{\mu(c)-u}) = 0
\]

according to whether \( \mu(c) \equiv u \) (mod \( p^{m-r} \)) or not. Thus the lemma is proved.

**4. Proof of Theorem 2.** To prove Theorem 2, we suppose that \( p \leq 13 \), and hence \( 2p^{p-1} \not\equiv 1 \pmod{p^2} \). The integer \( m \) of the preceding section will still be used. For each integer \( u \geq 0 \), let \( C_u \) denote the ideal class group of \( \mathbb{B}_u \) in the narrow sense. Then the ideal class group of \( \mathbb{B}_\infty \) in the narrow sense is canonically isomorphic to the direct limit of \( C_u \) for all integers \( u \geq 0 \) with respect to the natural homomorphisms \( C_u \to C_{u'} \) for all \((u, u') \in \mathbb{Z} \times \mathbb{Z}\) with \( 0 \leq u \leq u' \). On the other hand, \( k \) is an abelian 2-extension over \( \mathbb{Q} \) in which no prime number other than \( p \) is ramified. This fact implies by [W2, Theorem 10.4] that the class number of \( k \) is odd, whence \( h^-_0 \), the relative class number of \( k \), is odd. Therefore, by Lemma 3, it suffices to prove that \( h^-_n/h^-_{n-1} \) is always odd.

When \( p = 3 \) or \( p = 5 \), the assertion \( 2 \nmid h^-_n/h^-_{n-1} \) is part of more general results in [W1] but is proved very simply as follows. For \( p = 3 \), letting \( m = 1 \) and \( R = R_0 = \{1\} \), we obtain the assertion immediately from Lemma 5 (cf. [W1, §IV]). For \( p = 5 \), we let \( m = 1 \) and \( R = \{1, 2\} \). Let \( \zeta \) be any primitive 5th root of unity. Then, for any map \( f \) from \( R = R_0 \) to \( \{0, 1, 2, 3, 4\} \),

\[
(\zeta - 1) \sum_{a \in R} \frac{\zeta^f(a)}{\zeta^a - 1} = \zeta^{f(1)}(1 + \zeta^{f(2)-f(1)}(\zeta^4 + \zeta^2 + 1)).
\]

Since 2 remains prime in \( \mathbb{K}_5 \), we easily see that the above algebraic integer is relatively prime to 2. Hence Lemma 5 implies that \( h^-_n/h^-_{n-1} \) is odd (cf. [W1, Proposition 3]).

Let us deal with the case \( p = 7 \). Take \( \{1, 2, 4\} \) as \( R \) so that \( R_1 = \{1, 18, 30\} \). We let \( m = 2 \) first, and remark that \( \mathbb{Q}(\sqrt{-7}) \) is the decomposition field of 2 for \( \mathbb{K}_7/\mathbb{Q} \). Let \( \zeta \) be any primitive 49th root of unity. Then

\[
\sum_{a \in R_1} \frac{\zeta^f(a)}{\zeta^a - 1} = \zeta^{f(1)} \left( \frac{1}{\zeta - 1} + \frac{\zeta^{f(18)-f(1)}}{\zeta^{18} - 1} + \frac{\zeta^{f(30)-f(1)}}{\zeta^{30} - 1} \right)
\]
for any map \( f : R_1 \to \{0, \ldots, 48\} \). Assume now that
\[
\frac{1}{\zeta - 1} + \frac{\zeta^{c_1}}{\zeta^{18} - 1} + \frac{\zeta^{c_2}}{\zeta^{30} - 1}
\]
is not relatively prime to 2 with integers \( c_1, c_2 \) in \( \{0, \ldots, 48\} \), that is,
\[
\sum_{a \in Q_0} \zeta^a + \sum_{a \in Q_1} \zeta^a + \sum_{a \in Q_2} \zeta^a
\]
is not relatively prime to 2, where
\[
Q_0 = \{0, 18, 30, 48\}, \quad Q_1 = \{c_1, c_1 + 1, c_1 + 30, c_1 + 31\}, \\
Q_2 = \{c_2, c_2 + 1, c_2 + 18, c_2 + 19\}.
\]
It is useful to treat the elements of \( Q_0 \cup Q_1 \cup Q_2 \) modulo 7; for each pair \((u, w)\) in \( \{0, \ldots, 6\} \times \{0, 1, 2\} \), we put
\[
Q_w(u) = \{a \in Q_w \mid a \equiv u \pmod{7}\}.
\]
Since the cardinality of each \( Q_w(u) \) is 0 or 1, the above assumption implies by Lemma 6 that
\[
\sum_{w=0}^{2} |Q_w(u)| \not= 1, \quad \text{i.e.,} \quad \sum_{w=0}^{2} |Q_w(u)| \in \{0, 2, 3\}
\]
for every integer \( u \) in \( \{0, \ldots, 6\} \). This condition is satisfied only when \( c_1 \equiv 5 \pmod{7} \) and \( c_2 \equiv 4 \pmod{7} \), and then
\[
Q_0(0) = \{0\}, \quad Q_1(0) = \{c_1 + 30\}, \quad Q_1(1) = \{c_1 + 31\}, \quad Q_2(1) = \{c_2 + 18\}, \\
Q_0(2) = \{30\}, \quad Q_2(2) = \{c_2 + 19\}, \quad Q_0(4) = \{18\}, \quad Q_2(4) = \{c_2\}, \\
Q_1(5) = \{c_1\}, \quad Q_2(5) = \{c_2 + 1\}, \quad Q_0(6) = \{48\}, \quad Q_1(6) = \{c_1 + 1\}.
\]
In particular, we see from Lemma 6 that neither \( 1 + \zeta^{c_1+30} \) nor \( \zeta^{48} + \zeta^{c_1+1} \) is a unit, which means that
\[
1 = \zeta^{c_1+30}, \quad \zeta^{48} = \zeta^{c_1+1}.
\]
but these equalities obviously contradict each other. We therefore find that
\[
\sum_{a \in R_1} \frac{\zeta^{f(a)}}{\zeta^a - 1}
\]
is relatively prime to 2 for any map \( f : R_1 \to \{0, \ldots, 48\} \). Hence, by Lemma 5, \( h_n/h_{n-1} \) is odd whenever \( n \geq 2m - 1 = 3 \).

We next let \( m = 1 \), still with \( R = \{1, 2, 4\} \). Let \( \psi \) be a primitive Dirichlet character of order 49 with conductor 73, and put \( \omega = \psi(50) \). Then \( \omega \) is a primitive 7th root of unity. In the case \( n = 2 \), putting \( c = 172 = (1 + 7^3)/2 \), we have
\[
\psi(c)^{-1}\psi(c_*) = \psi(2)\psi(25) = \omega, \quad \psi(c)^{-1}\psi((18c)_*) = \psi(18) = 1, \\
\psi(c)^{-1}\psi((30c)_*) = \psi(30) = \omega^4.
\]
so that
\[ \sum_{b \in R_1} \frac{\psi(c)^{-1}\psi((bc)_*)}{\omega^{(bc)_*} - 1} = -\frac{1}{\omega^2 + 1}. \]

Therefore Lemma 4 shows that \( h_2^- / h_1^- \) is odd. In the case \( n = 1 \), noting that \( \omega = \psi^7(8) \), we have
\[ \sum_{b \in R} \psi^7(b_*) = -\frac{\omega^3}{\omega + 1} \]
and hence Lemma 4 shows as well that \( h_1^- / h_0^- \) is odd. The conclusion for \( p = 7 \) is thus proved.

Assertion IV of [AF] implies that, if \( h_n \) is odd and the order of 2 modulo \( p \) is even, then \( h_n \) is also the class number of \( \mathbb{B}_n \) in the narrow sense. Hence, as already remarked in the introduction, the conclusion of Theorem 2 follows from Theorem 1 when \( p = 11 \) or \( p = 13 \); nonetheless, for this case, we shall give another proof of Theorem 2 without using Theorem 1 but along the same lines as in the case \( p \leq 7 \).

We now suppose that \( p = 11 \). Let \( R = \{1, 2, 4, 5, 8\} \) and let \( \xi \) be any primitive 11th root of unity. Let us consider the congruence
\[(2) \quad \frac{1}{\xi - 1} + \sum_{w=1}^{4} \frac{\xi^{c_w}}{\xi^{2w} - 1} \equiv 0 \pmod{2} \]
with integers \( c_1, c_2, c_3, c_4 \) in \( \{0, \ldots, 10\} \). Since
\[ \prod_{w=0}^{3} (\xi^{2w} + 1) \equiv \sum_{a=5}^{10} \xi^a \pmod{2}, \]
\[ \xi^{c_1} \prod_{w=1}^{3} (\xi^{2w} + 1) \equiv \xi^{c_1+5} + \xi^{c_1+7} + \xi^{c_1+9} \pmod{2}, \]
we find that (2) is equivalent to the congruence
\[ \sum_{a=5}^{10} \xi^a + \xi^{c_1+5} + \xi^{c_1+7} + \xi^{c_1+9} + \xi^{c_2+1} + \xi^{c_2+4} + \xi^{c_2+8} + \xi^{c_3} + \xi^{c_3+8} + \xi^{c_4} \equiv 0 \pmod{2}. \]
Hence, setting \( c_2 = 0, \ldots, c_2 = 10 \) successively in the above, we see without difficulty that (2) holds if and only if
\[(3) \quad (c_1, c_2, c_3, c_4) = (5, 8, 6, 7) \quad \text{or} \quad (c_1, c_2, c_3, c_4) = (7, 6, 1, 8). \]

Now, let \( m = 2 \). Let \( \zeta \) be a primitive 112th root of unity such that \( \zeta^{11} = \xi \), and note that \( R_1 = \{1, 27, 81, 112, 118\} \). Then, for any map \( f : \)}
\[ R_1 \to \{0, \ldots, 120\}, \]
\[ \sum_{a \in R_1} \frac{\zeta^{f(a)}}{\zeta^a - 1} = \zeta^{f(1)} \left( \frac{1}{\zeta - 1} + \sum_{w=1}^{4} \frac{\zeta^{f(\hat{w}) - f(1)}}{\zeta^{\hat{w}} - 1} \right), \]
where \((\hat{1}, \hat{2}, \hat{3}, \hat{4}) = (112, 81, 118, 27)\). We assume that there exist integers \(d_1, d_2, d_3, d_4\) in \(\{0, \ldots, 120\}\) satisfying
\[ \sum_{w=0}^{4} \zeta^{d_w} \equiv 0 \pmod{2}. \]
For each \(w \in \{0, 1, 2, 3, 4\}\), define a set \(Q_w\) of non-negative integers as follows. Let \(Q_0\) denote the set of
\[ \sum_{b \in R_1 \setminus \{1\}} \varepsilon(b) b \]
for all maps \(\varepsilon : R_1 \setminus \{1\} \to \{0, 1\}\). If \(w \geq 1\), let \(Q_w\) denote the set of
\[ d_w + \sum_{b \in R_1 \setminus \{\hat{w}\}} \varepsilon(b) b, \]
for all maps \(\varepsilon : R_1 \setminus \{\hat{w}\} \to \{0, 1\}\). Given any \(u \in \{0, \ldots, 120\}\), we then put
\[ Q_w^1(u) = \{a \in Q_w \mid a \equiv u \pmod{11}\}, \]
\[ Q_w^2(u) = \{a \in Q_w \mid a \equiv u \pmod{11^2}\}. \]
Direct computations show that the cardinality of each \(Q_w^2(u)\) is 0 or 1, and that the cardinality of each \(Q_w^1(u)\) does not exceed 2, whence
\[ \sum_{w=0}^{4} |Q_w^1(u)| \leq 10. \]
Furthermore, 2 remains prime in \(\mathbb{K}_{11^2}\), and (4) is equivalent to
\[ \sum_{w=0}^{4} \sum_{a \in Q_w} \zeta^a \equiv 0 \pmod{2}, \]
which, together with Lemma 6, gives
\[ \sum_{w=0}^{4} \sum_{a \in Q_w(u)} \zeta^a \equiv 0 \pmod{2}. \]
Therefore, in view of the form of the \(11^2\)th cyclotomic polynomial, we obtain
\[ \sum_{w=0}^{4} |Q_w^2(u)| \equiv 0 \pmod{2}. \]
Consequently,
\[ \sum_{w=0}^{4} |Q^1_w(u')| \equiv 0 \pmod{2} \quad \text{for every } u' \in \{0, \ldots, 10\}. \]

This implies that
\[ \sum_{w=0}^{4} \sum_{a \in Q_w} \xi^a \equiv 0 \pmod{2}, \]
that is,
\[ \frac{1}{\xi - 1} + \sum_{w=1}^{4} \frac{\xi^{d_w}}{\xi^w - 1} = \frac{1}{\xi - 1} + \sum_{w=1}^{4} \frac{\xi^{d_w}}{\xi^w - 1} \equiv 0 \pmod{2}. \]

Hence it follows from (2) that there exist integers \(c'_1, c'_2, c'_3, c'_4\) in \(\{0, \ldots, 10\}\) satisfying
\[ d_w = 11c'_w + c_w \quad \text{for every } w \in \{1, 2, 3, 4\}. \]

Therefore, by (4),
\[ 1 \xi - 1 + \sum_{w=1}^{4} \xi^{c'_w} T \left( \frac{\xi^{c_w}}{\xi^w - 1} \right) \equiv 0 \pmod{2}; \]
here \(T\) denotes the trace map from \(\mathcal{K}_{11^2}\) to \(\mathcal{K}_{11}\), so that
\[ T \left( \frac{1}{\zeta - 1} \right) = \frac{11}{\xi - 1} \]
and, for each positive integer \(a \leq 10\),
\[ T \left( \frac{\zeta^a}{\zeta - 1} \right) = T \left( \sum_{b=0}^{a-1} \xi^b + \frac{1}{\zeta - 1} \right) = \frac{11\xi}{\xi - 1}. \]

Since (3) is equivalent to (2), let us first consider the case \((c_1, c_2, c_3, c_4) = (5, 8, 6, 7)\). In this case, (5) is written as
\[ \frac{1}{\xi - 1} + \frac{\xi^{c'_1 - 2}}{\xi^2 - 1} + \frac{\xi^{c'_2 + 1}}{\xi^4 - 1} + \frac{\xi^{c'_3 - 1}}{\xi^8 - 1} + \frac{\xi^{c'_4 - 3}}{\xi^5 - 1} \equiv 0 \pmod{2}. \]

Hence, again by the equivalence of (2) and (3),
\[(c'_1, c'_2, c'_3, c'_4) = (7, 7, 6, 10) \quad \text{or} \quad (c'_1, c'_2, c'_3, c'_4) = (9, 5, 1, 0).\]

We thus deduce that
\[(d_1, d_2, d_3, d_4) = (82, 85, 72, 117) \quad \text{or} \quad (d_1, d_2, d_3, d_4) = (104, 63, 17, 7).\]
However,
\[
\frac{1}{\zeta - 1} + \frac{\zeta^{82}}{\zeta^{112} - 1} + \frac{\zeta^{85}}{\zeta^{81} - 1} + \frac{\zeta^{72}}{\zeta^{118} - 1} + \frac{\zeta^{117}}{\zeta^{27} - 1}
\equiv \left( \prod_{a \in R_1} \frac{1}{\zeta^a - 1} \right) \sum_b \zeta^b \pmod{2},
\]
\[
\frac{1}{\zeta - 1} + \frac{\zeta^{104}}{\zeta^{112} - 1} + \frac{\zeta^{63}}{\zeta^{81} - 1} + \frac{\zeta^{17}}{\zeta^{118} - 1} + \frac{\zeta^{7}}{\zeta^{27} - 1}
\equiv \left( \prod_{a \in R_1} \frac{1}{\zeta^a - 1} \right) \sum_{b'} \zeta'^{b'} \pmod{2},
\]
where \( b \) ranges over the integers
\[
0, 3, 4, 5, 7, 14, 16, 23, 25, 26, 29, 32, 33, 36, 37, 38, 39, 42, 43, 47, 48, 49, 50, 58, 62, 63,
64, 65, 67, 68, 71, 73, 75, 76, 79, 82, 84, 85, 86, 90, 92, 93, 95, 96, 101, 102, 106, 107,
\]
and \( b' \) ranges over the integers
\[
0, 2, 11, 13, 15, 17, 19, 20, 21, 27, 30, 31, 32, 36, 41, 42, 43, 44, 45, 46, 51, 53, 55,
57, 60, 62, 64, 68, 69, 72, 74, 75, 77, 79, 80, 82, 88, 89, 90, 92, 97, 102, 104, 107.
\]
We are therefore led to a contradiction, whence the case \((c_1, c_2, c_3, c_4) = (5, 8, 6, 7)\) does not occur. In the case \((c_1, c_2, c_3, c_4) = (7, 6, 1, 8)\), as \((5)\) means
\[
\frac{1}{\xi - 1} + \frac{\xi_{c_1} - 1}{\xi^2 - 1} + \frac{\xi_{c_2} - 3}{\xi^4 - 1} + \frac{\xi_{c_3} - 1}{\xi^8 - 1} + \frac{\xi_{c_4} + 2}{\xi^5 - 1}
\equiv 0 \pmod{2}
\]
and as \((2)\) is equivalent to \((3)\), it follows that
\[
(c_1', c_2', c_3', c_4') = (6, 0, 7, 5) \quad \text{or} \quad (c_1', c_2', c_3', c_4') = (8, 9, 2, 6),
\]
so that
\[
(d_1, d_2, d_3, d_4) = (73, 6, 78, 63) \quad \text{or} \quad (d_1, d_2, d_3, d_4) = (95, 105, 23, 74);
\]
but we have
\[
\frac{1}{\zeta - 1} + \frac{\zeta^{73}}{\zeta^{112} - 1} + \frac{\zeta^{6}}{\zeta^{81} - 1} + \frac{\zeta^{78}}{\zeta^{118} - 1} + \frac{\zeta^{63}}{\zeta^{27} - 1}
\equiv \left( \prod_{a \in R_1} \frac{1}{\zeta^a - 1} \right) \sum_b \zeta^b \pmod{2},
\]
\[
\frac{1}{\zeta - 1} + \frac{\zeta^{95}}{\zeta^{112} - 1} + \frac{\zeta^{105}}{\zeta^{81} - 1} + \frac{\zeta^{23}}{\zeta^{118} - 1} + \frac{\zeta^{74}}{\zeta^{27} - 1}
\equiv \left( \prod_{a \in R_1} \frac{1}{\zeta^a - 1} \right) \sum_{b'} \zeta'^{b'} \pmod{2},
\]
Lemma 5, \( h \) 5-tuples of integers: We thus conclude from the above contradiction that (4) is not satisfied by \( b \) and \( \psi \).

Furthermore, whether \( b \) be a primitive Dirichlet character of order 121 with conductor 11 \( \psi \)

Lemma 4 shows that \( h \psi^{1} \), \( 3^{1} \), \( 3^{0} \), \( 68 \), \( 53 \), \( 11 \) ranges over the integers

\[
\sum_{b \in R_{1}} \frac{\psi(b)}{\xi^{b^{*}} - 1} = \frac{1}{\xi - 1} + \frac{\psi(94)}{\xi^{2} - 1} + \frac{\psi(3)}{\xi^{4} - 1} + \frac{\psi(40)}{\xi^{8} - 1} + \frac{\psi(9)}{\xi^{5} - 1}.
\]

Furthermore, whether \( n = 1 \) or not,

\[
\sum_{b \in R} \frac{\psi^{1}(b)}{\xi^{b^{*}} - 1} = \frac{1}{\xi - 1} + \frac{\psi^{1}(6)}{\xi^{2} - 1} + \frac{\psi^{1}(3)}{\xi^{4} - 1} + \frac{\psi^{1}(7)}{\xi^{8} - 1} + \frac{\psi^{1}(9)}{\xi^{5} - 1}.
\]

We know, however, that \( \psi(3) = 1 \). Hence, by the equivalence of (2) and (3), Lemma 4 shows that \( h_{n} / h_{n-1} \) is odd even if \( n = 2 \) or \( n = 1 \).

We finally deal with the case \( p = 13 \). Let \( R = \{1, 2, 3, 4, 6, 8\} \), let \( \xi_{1} \) be a primitive 13th root of unity, and let \( U \) denote the set of the following 5-tuples of integers:

\[
(5, 6, 5, 1, 4), (3, 7, 1, 2, 4), (4, 7, 1, 3, 4), (5, 7, 2, 3, 4),
(5, 7, 5, 1, 5), (3, 6, 1, 2, 5), (4, 6, 1, 3, 5), (5, 6, 2, 3, 5),
(10, 0, 5, 6, 5), (9, 10, 7, 6, 5), (9, 0, 5, 7, 5), (10, 10, 7, 7, 5),
(12, 8, 0, 2, 6), (2, 6, 1, 2, 6), (12, 9, 0, 3, 6), (11, 9, 0, 4, 6),
(9, 10, 8, 6, 6), (10, 10, 8, 7, 6), (12, 10, 0, 3, 7), (11, 10, 0, 4, 7),
(9, 9, 8, 6, 7), (10, 9, 8, 7, 7), (3, 4, 0, 10, 7), (2, 4, 0, 11, 7),
(8, 9, 8, 6, 8), (5, 11, 9, 6, 8), (5, 12, 9, 7, 8), (4, 12, 9, 8, 8),
(3, 5, 0, 10, 8), (2, 5, 0, 11, 8), (2, 6, 0, 12, 8), (12, 8, 1, 12, 8),
(9, 7, 9, 1, 9), (8, 7, 9, 2, 9), (5, 0, 9, 7, 9), (4, 0, 9, 8, 9),
(10, 8, 1, 11, 9), (11, 8, 1, 12, 9), (9, 8, 9, 1, 10), (8, 8, 9, 2, 10),
(5, 11, 8, 3, 10), (8, 9, 9, 3, 10), (10, 7, 1, 11, 10), (11, 7, 1, 12, 10),
(2, 11, 7, 2, 11), (3, 11, 8, 2, 11), (4, 11, 8, 3, 11), (2, 10, 4, 4, 11),
(10, 7, 2, 11, 11), (11, 4, 4, 11, 11), (11, 7, 2, 12, 11), (10, 4, 4, 12, 11),
(2, 10, 7, 2, 12), (3, 10, 8, 2, 12), (4, 10, 8, 3, 12), (2, 11, 4, 4, 12).
\]
Using a computer, we can check that integers \(c_1, c_2, c_3, c_4, c_5\) in \(\{0, \ldots, 12\}\) satisfy

\[
\frac{1}{\xi_1 - 1} + \frac{\xi_1^{2w} c_w}{\xi_1^{2w} - 1} \equiv 0 \pmod{2}
\]

if and only if \((c_1, c_2, c_3, c_4, c_5)\) belongs to \(U\).

Now, let \(m = 3\). Let \(\zeta\) be a primitive \(13^3\)th root of unity such that \(\zeta^{13} = \xi_2\), with \(\xi_2\) a primitive \(169\)th root of unity such that \(\xi_2^{13} = \xi_1\). We note that \(2\) remains prime in \(\mathbb{K}_{13^3}, R_2 = \{1, 418, 1160, 1161, 1540, 1958\}\), and for any map \(f : R_2 \to \{0, \ldots, 13^3 - 1\},\)

\[
\sum_{a \in R_2} \frac{\zeta^{f(a)}}{\zeta^a - 1} = \zeta^{f(1)} \left( \frac{1}{\zeta - 1} + \sum_{w=1}^{5} \frac{\zeta^{f(w) - f(1)}}{\zeta^w - 1} \right),
\]

where \((\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}) = (418, 1161, 1958, 1160, 1540)\). Assume the congruence

(6) \[
\frac{1}{\zeta - 1} + \sum_{w=1}^{5} \frac{\zeta^{\hat{w}d_w}}{\zeta^w - 1} \equiv 0 \pmod{2}
\]

to be satisfied by non-negative integers \(d_1, d_2, d_3, d_4, d_5\) smaller than \(13^3\). Putting \(d_0 = 0\), let \(Q_w\) denote for each \(w \in \{0, \ldots, 5\}\) the set of the integers

\[
d_w + \sum_{b \in R_2 \setminus \{\hat{w}\}} \varepsilon(b)b
\]

for all maps \(\varepsilon : R_2 \setminus \{\hat{w}\} \to \{0, 1\}\). Let \(u\) range over the non-negative integers smaller than \(13^3\). We then put

\[
Q_w^2(u) = \{a \in Q_w \mid a \equiv u \pmod{13^2}\},
\]

\[
Q_w^3(u) = \{a \in Q_w \mid a \equiv u \pmod{13^3}\}.
\]

As in the case \(p = 11\), we see that each \(|Q_w^3(u)|\) is 0 or 1 and that no \(|Q_w^2(u)|\) exceeds 2. Hence not only is (6) equivalent to

\[
\sum_{w=0}^{5} \sum_{a \in Q_w} \zeta^a \equiv 0 \pmod{2}
\]

but also we have

\[
\sum_{w=0}^{5} |Q_w^2(u)| \leq 12.
\]

Lemma 6 therefore shows that

\[
\sum_{w=0}^{5} |Q_w^3(u)| \equiv 0 \pmod{2},
\]
Hence, by the first congruence of (7),

\[ \sum_{w=0}^{5} \sum_{a \in Q_w} \xi_2^a \equiv 0 \pmod{2}, \quad \sum_{w=0}^{5} \sum_{a \in Q_w} \xi_1^a \equiv 0 \pmod{2}. \]

Since the second congruence above gives

\[ \frac{1}{\xi_1 - 1} + \sum_{w=1}^{5} \frac{\xi_1^{2w} d_w}{\xi_1^{2w} - 1} = \frac{1}{\xi_1 - 1} + \sum_{w=1}^{5} \frac{\xi_1^{2w} d_w}{\xi_1^{2w} - 1} \equiv 0 \pmod{2}, \]

there exist integers \( d'_1, d'_2, d'_3, d'_4, d'_5, d''_1, d''_2, d''_3, d''_4, d''_5 \) in \( \{0, \ldots, 12\} \) such that \((d'_1, d'_2, d'_3, d'_4, d'_5)\) belongs to \( U \) and

\[ d_w \equiv 13d''_w + d'_w \pmod{13^2} \quad \text{for every } w \in \{1, \ldots, 5\}. \]

Hence, by the first congruence of (7),

\[ \frac{1}{\xi_2 - 1} + \sum_{w=1}^{5} \frac{\xi_1^{2w} d''_w}{\xi_2^{2w} - 1} \equiv 0 \pmod{2}. \]

The trace map from \( \mathbb{K}_{13^2} \) to \( \mathbb{K}_{13} \) transforms the above into

\[ \frac{1}{\xi_1 - 1} + \sum_{w=1}^{5} \frac{\xi_1^{2w (d''_w + \kappa(d'_w))}}{\xi_1^{2w} - 1} \equiv 0 \pmod{2}, \]

where, for each integer \( c, \kappa(c) = 0 \) or \( \kappa(c) = 1 \) according to whether \( c \) is divisible by 13 or not. Thus there exists a 5-tuple \((c_1, c_2, c_3, c_4, c_5)\) in \( U \) satisfying

\[ d''_w + \kappa(d'_w) \equiv c_w \pmod{13} \quad \text{for every } w \in \{1, \ldots, 5\}. \]

In particular,

\[ \frac{1}{\xi_2 - 1} + \sum_{w=1}^{5} \frac{\xi_2^{w (13(c_w - \kappa(d'_w)) + d'_w)}}{\xi_2^{w} - 1} \equiv 0 \pmod{2}. \]

However, for any given \((c'_1, c'_2, c'_3, c'_4, c'_5), (c''_1, c''_2, c''_3, c''_4, c''_5) \in U\), we can check by computer that

\[ \frac{1}{\xi_2 - 1} + \sum_{w=1}^{5} \frac{\xi_2^{w (13(c'_w - \kappa(c'_w)) + c'_w)}}{\xi_2^{w} - 1} \not\equiv 0 \pmod{2}. \]

This contradiction shows that no 5-tuple \((d_1, d_2, d_3, d_4, d_5)\) of integers in \( \{0, \ldots, 13^3 - 1\} \) satisfies the congruence (6). Hence, by Lemma 5, \( h_{-1}^{-}/h_{-1}^{-} \) is odd if \( n \geq 5 \).

We now let \( m = 1, R = \{1, 5, 7, 9, 10, 11\} \), and so

\[ R_3 = \{1, 239, 5051, 7627, 7628, 23749\}. \]

Let \( \xi_1 \) be any primitive 13th root of unity as before. Let \( \psi \) be a primitive Dirichlet character of order \( 13^4 \) with conductor \( 13^5 \) such that \( \xi_1 = \psi(1+13^4) \),
whence
\[\xi_1 = \psi^{13}(1 + 13^3) = \psi^{13^2}(1 + 13^2) = \psi^{13^3}(14).\]

We then see that, in the case \( n = 4 \),
\[
\sum_{b \in R_3} \frac{\psi(b_*)}{\xi_1^{b_*} - 1} = \frac{1}{\xi_1 - 1} + \psi(5051) + \psi(7628) + \psi(239) + \psi(7627) + \psi(23749),
\]
in the case \( n = 3 \),
\[
\sum_{b \in R_2} \frac{\psi^{13}(b_*)}{\xi_1^{b_*} - 1} = \frac{1}{\xi_1 - 1} + \psi^{13}(657) + \psi^{13}(1037) + \psi^{13}(239) + \psi^{13}(1036) + \psi^{13}(1779),
\]
in the case \( n = 2 \),
\[
\sum_{b \in R_1} \frac{\psi^{13^2}(b_*)}{\xi_1^{b_*} - 1} = \frac{1}{\xi_1 - 1} + \psi^{13^2}(150) + \psi^{13^2}(23) + \psi^{13^2}(70) + \psi^{13^2}(22) + \psi^{13^2}(89),
\]
and in any case,
\[
\sum_{b \in R} \frac{\psi^{13^3}(b)}{\xi_1^{b_*} - 1} = \frac{1}{\xi_1 - 1} + \psi^{13^3}(7) + \psi^{13^3}(10) + \psi^{13^3}(5) + \psi^{13^3}(9) + \psi^{13^3}(11).
\]

Furthermore,
\[
\psi^{13}(1779) = \psi^{13^2}(89) = (\xi_1^{32})^3, \quad \psi(7627) = (\xi_1^{16})^8, \quad \psi(23749) = (\xi_1^{32})^5,
\]
\[
\psi^{13^3}(9) = (\xi_1^{16})^{12}, \quad \psi^{13^3}(11) = (\xi_1^{32})^7.
\]

Therefore, viewing the elements of \( U \), we know from Lemma 4 that \( h_n^+/h_{n-1}^- \) is odd if \( n \leq 4 \). Consequently, the theorem is completely proved.

**5. Final remark.** Let \( H \) denote the class number of \( \mathbb{B}_1 \) in the narrow sense. By class field theory, \( H \) must be odd when the narrow 2-class group of \( \mathbb{B}_\infty \) is trivial. For each integer \( a \) relatively prime to \( p \), we define an integer \( v(a) \) by
\[1 - a^{p-1} = pv(a)\]
The following assertion does not need the assumption \( 2^{p-1} \neq 1 \pmod{p^2} \).

**Proposition.** Assume that \( m = 1 \), and take any primitive \( p \)th root \( \zeta \) of unity. Then \( H \) is odd if and only if
\[
\sum_{b \in R} \frac{\zeta^{v(b)}}{\zeta^{b_*} - 1}
\]
is relatively prime to 2.
\textit{Proof.} Let \( \psi \) be a primitive Dirichlet character of order \( p \) with conductor \( p^2 \) such that \( \psi(1+p) = \zeta \). Let \( \chi \) be any primitive Dirichlet character of order \( 2^t \) with conductor \( p \), that is, \( \chi \in \mathcal{X} \). As in the proof of Lemma 4, we have

\[
-\frac{1}{2p^2} \sum_{a=1}^{p^2} \chi \psi(a) a \equiv \sum_{b \in R} \frac{\psi(b)}{\zeta^{b^*} - 1} \pmod{i},
\]

where \( i \) denotes the integral ideal of \( \mathbb{K}_{2^t,p} \) generated by \( e^{\pi i/2^t-1} - 1 \). Since \( i \) is the product of all prime ideals of \( \mathbb{K}_{2^t,p} \) dividing 2, it then follows from (1) that \( h_{l^-}^{-1}/h_{l^-}^{-0} \) is odd if and only if

\[
\sum_{b \in R} \frac{\psi(b)}{\zeta^{b^*} - 1}
\]

is relatively prime to 2. Furthermore, for any \( b \in R \), an integer \( b' \) with \( \zeta^{b'} = \psi(b) \) satisfies \((1+p)^{b'(p-1)} = b p^{(p-1)} \) (mod \( p^2 \)), i.e., \( b' \equiv v(b) \pmod{p} \). Hence Lemma 3, together with the fact \( 2 \nmid h_{l^-}^{-1} \), proves the proposition.

By means of the above proposition, we have checked by computer that, if \( p \leq 487 \), then \( H \) is odd.

Now, take any prime number \( q \) different from \( p \). Let \( F \) be the decomposition field of \( q \) for the abelian extension \( \mathbb{K}_p \mathbb{B}_\infty / \mathbb{Q} \). Note that \( F \) is of finite degree and that the case \( q = l \) is none other than the case \( F = \mathbb{Q} \). It is shown in [H3] that, if \( q \) is sufficiently large with the degree of \( F \) bounded, then the \( q \)-class group of \( \mathbb{B}_\infty \) is trivial, whence \( q \) does not divide \( h_n \), the class number of \( \mathbb{B}_n \). This result implies that the primes \( q' \) for which the \( q' \)-class group of \( \mathbb{B}_\infty \) is trivial distribute with natural density 1 in the set of all prime numbers. On the other hand, we have not found any example of \((p,n)\) such that \( h_n > 1 \). Hence the question arises whether the ideal class group of \( \mathbb{B}_\infty \) is trivial (cf. also J. Buhler, C. Pomerance and L. Robertson [BPR], J. P. Cerri [Ce], H. Cohn [Co], T. Fukuda and K. Komatsu [FK], and [H1]). Moreover, in connection with our results on the narrow class group of \( \mathbb{B}_\infty \), it might be an interesting problem to find whether \( H \) is always odd.

\textbf{References}


[H1] K. Horie, Ideal class groups of Iwasawa-theoretical abelian extensions over the rational field, J. London Math. Soc. (2) 66 (2002), 257–275 (“\psi_2^d(b) = 1” in line 11 on page 260 should be “\psi_2(b^d = 1”).


[H3] —, The ideal class group of the basic \Z_p\-extension over an imaginary quadratic field, Tohoku Math. J. 57 (2005), 375–394 (“\(p - 1\)” in line 7 on page 391 should be “\(\varphi(q)\)”; so “(\(p - 1\))\(f\)” in lines 16, 20 on page 389 and in lines 9, 11 on page 391, along with “\(f(p - 1)\)” in line 19 on page 391, should be “\(\varphi(q)f\)” (cf. also [H4, §4]).


