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A parametric family of sextic Thue equations

by

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1. Introduction. The first complete family of sextic Thue equations

$$(1.1)\qquad F_a(x,y) = x^6 - 2ax^5y - (5a+15)x^4y^2 - 20x^3y^3 + 5ax^2y^4 + (2a+6)xy^5 + y^6 \in \{\pm 1, \pm 27\}$$

was studied by Lettl–Pethő–Voutier (see [8] and [9]). In [8], they used the hypergeometric method to prove that the equation (1.1) has only trivial solutions for $a \in \mathbb{Z}$. Moreover, they solved the inequality $|F_a(x,y)| \le 2a + 323$, for $a \ge 89$. In [9], they found all solutions to $F_a(x,y) \in \{\pm 1, \pm 27\}$ for $a \ge 89$. In 2002, we obtained a partial result by means of a computational method (see [12]). In fact, we used a computational method due to Bilu–Hanrot [1] to prove that for $|n| \le 2.03 \cdot 10^6$, the Thue equation

$$F_n(x,y) = x^6 - 2A_nx^5y - 5(A_n+3)x^4y^2 - 20x^3y^3 + 5A_nx^2y^4 + 2(A_n+3)xy^5 + y^6 = 1$$

with $A_n = 54n^3 + 81n^2 + 54n + 12$, $n \in \mathbb{Z}$, has no integral solution except for the trivial ones: $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$, $(1,-1)$, $(-1,1)$.

In this paper, we consider the following parameterized polynomial in two variables x and y:

$$(1.2)\qquad \Phi_a(x,y) = x^6 - (a-2)x^5y - (a^2+a+6)x^4y^2 + (a^3 - 2a^2 + 6a - 10)x^3y^3 + (a^3 + 5a + 3)x^2y^4 + (a^2 - a + 4)xy^5 - y^6.$$

In fact, we tried the hypergeometric method but without success. So we solve the Thue equation $\Phi_a(x,y) = \pm 1$ using Baker's method. The main result is the following:

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THEOREM 1.1. *For $a \geq 1.078 \cdot 10^{12}$, the family of parameterized Thue equations*

$$\begin{aligned}(1.3)\quad \Phi_a(x,y) = x^6 &- (a-2)x^5y - (a^2+a+6)x^4y^2 \\ &+ (a^3-2a^2+6a-10)x^3y^3 \\ &+ (a^3+5a+3)x^2y^4 + (a^2-a+4)xy^5 - y^6 = \pm 1\end{aligned}$$

has only the integral solutions

$$(1.4)\qquad \pm\{(0,1),\ (1,0),\ (-1,1)\}.$$

Lecacheux [7] studied the extension $\mathbb{K}_a = \mathbb{Q}(H,h)$, where h and H are respectively roots of

$$(1.5)\qquad x^3 - (a+1)x^2 + (a-2)x + 1,$$

$$(1.6)\qquad x^2 - ax - 1.$$

In Section 2, we will give some elementary properties of these polynomials and recall the result of Lecacheux about a system of fundamental units of $\mathbb{K}_a$ that we will extend at the end of the section. We prove that the index of the subgroup of units generated by five roots of $\phi_a(x) = \Phi_a(x,1)$ in the full group of units is less than $92.60\log^5(a)$. Then we work with a system of almost fundamental units. In Section 3, we study approximation properties of solutions to (1.3) and we exclude all solutions with small $|y|$. The key result in this section is a lower bound for $\log|y|$, using small linear combinations of the unknown exponents of the almost fundamental units. In Section 4, we determine upper and lower bounds for linear forms in logarithms to prove Theorem 1.1. The main step in this section is to transform the linear forms of six logarithms into linear forms of two logarithms. This enables us to obtain a lower bound for the parameter a. In fact, we use an approach close to that in [3, 4, 11, 13, 14].

2. Elementary properties of the relevant polynomials. We have the following properties.

- The couples in (1.4) are solutions to (1.3).
- $\Phi_a(-x,-y) = \Phi_a(x,y)$; hence if (x,y) is a solution to (1.3), so is $(-x,-y)$. Without loss of generality, we will consider only the solutions (x,y) to (1.3) with y positive.

Let us recall some properties obtained by Lecacheux.

THEOREM 2.1. *There exists a constant C such that if $|a| > C$, and if $d := a^2+4$ and $D := a^2-a+7$ are squarefree, then the unit group of $\mathbb{K}_a = \mathbb{Q}(H,h)$ is generated by $-1, u = H-h$ and its conjugates.*

REMARK 2.2.

(1) The discriminants of the rings $\mathbb{Z}[H]$ and $\mathbb{Z}[h]$ are respectively $d = a^2+4$ and $D^2 = (a^2-a+7)^2$. The identity $-13 = d(a+2)-D(a+3)$ implies that d and D are relatively prime except for $a \equiv 3 \bmod 13$. The discriminant of $\mathbb{K}_a$ is $D^4 \cdot d^3$.

(2) The extension $\mathbb{K}_a$ is a real cyclic field of degree 6. The Galois group of $\mathbb{K}_a$ is $G = \langle\sigma\rangle$ such that

$$H^\sigma = -\frac{1}{H}, \qquad h^\sigma = \frac{h-1}{h}, \qquad h^{\sigma^2} = -\frac{1}{h-1}. \tag{2.1}$$

Therefore, we use Maple and the expressions (1.5), (1.6), (2.1) to determine the irreducible polynomial of $H-h$. We obtain

$$\begin{aligned}\phi_a(x) = x^6 &- (a-2)x^5 - (a^2+a+6)x^4 + (a^3-2a^2+6a-10)x^3 \\ &+ (a^3+5a+3)x^2 + (a^2-a+4)x - 1.\end{aligned} \tag{2.2}$$

Most of the computations involve manipulations with asymptotic approximations done with Maple. We will use the following variant of the usual O-notation, introduced in [5]. For two functions $g(a)$ and $h(a)$ and a positive number a_0 we write $g(a) = L_{a_0}(h(|a|))$ if $|g(a)| \le h(|a|)$ for all a with absolute value at least a_0.

Let $\theta^{(1)}, \dots, \theta^{(6)}$ be the roots of $\phi_a(x)$. We let H be the root of (1.6) whose asymptotic expression is given by

$$H = a + \frac{1}{a} - \frac{1}{a^3} + L_2\left(\frac{0.1}{a^4}\right) \tag{2.3}$$

and h the root of (1.5) whose asymptotic expression is given by

$$h = -\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} + L_2\left(\frac{0.1}{a^4}\right). \tag{2.4}$$

Then using (2.1), (2.3), and (2.4), we have, for $a \ge 2$,

$$\begin{cases} \theta^{(1)} = H - h = a + \frac{2}{a} + \frac{1}{a^2} - \frac{2}{a^3} + L_2\left(\frac{0.2}{a^4}\right), \\ \theta^{(2)} = -\frac{1}{H} - \frac{h-1}{h} = -a - \frac{3}{a} + \frac{3}{a^2} - \frac{4}{a^3} + L_2\left(\frac{8}{a^4}\right), \\ \theta^{(3)} = H + \frac{1}{h-1} = a - 1 + \frac{2}{a} - \frac{3}{a^3} + L_2\left(\frac{3}{a^4}\right), \\ \theta^{(4)} = -\frac{1}{H} - h = \frac{1}{a^2} - \frac{2}{a^5} + L_2\left(\frac{3}{a^7}\right), \\ \theta^{(5)} = H - \frac{h-1}{h} = -\frac{1}{a} + \frac{3}{a^2} - \frac{6}{a^3} + L_2\left(\frac{8}{a^4}\right), \\ \theta^{(6)} = -\frac{1}{H} + \frac{1}{h-1} = -1 - \frac{1}{a^3} + \frac{3}{a^4} + L_2\left(\frac{3}{a^5}\right). \end{cases} \tag{2.5}$$

Using the asymptotic expressions of $\theta^{(i)}$, we determine those of $\log|\theta^{(i)}|$ and $\log|\theta^{(i)} - \theta^{(k)}|$. In fact, we know that for the function $f(x) = \log(x) = \log(1+u)$, we have

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}, \qquad n \ge 0.$$

The error associated with the approximation of $f(x)$ by the third Taylor polynomial is

$$|R_3(1+u)| = \left|\frac{f^{(4)}(z)}{4!}\,u^4\right| = \frac{u^4}{4z^4}$$

where z is between 1 and $1+u$. In this interval, $|R_3(1+u)|$ is the largest when $z=1$. Therefore we obtain:

$$(2.6)\qquad \begin{aligned}
\log|\theta^{(1)}| &= \log(a) + \frac{2}{a^2} + \frac{1}{a^3} + L_2\left(\frac{3}{a^4}\right),\\
\log|\theta^{(2)}| &= \log(a) + \frac{3}{a^2} - \frac{3}{a^3} + L_2\left(\frac{1.8}{a^4}\right),\\
\log|\theta^{(3)}| &= \log(a) - \frac{1}{a} + \frac{3}{2a^2} + \frac{5}{3a^3} + L_2\left(\frac{7}{a^4}\right),\\
\log|\theta^{(4)}| &= -2\log(a) - \frac{2}{a^3} + L_4\left(\frac{0.2}{a^4}\right),\\
\log|\theta^{(5)}| &= -\log(a) - \frac{3}{a} + \frac{3}{2a^2} + \frac{9}{a^3} + L_3\left(\frac{0.1}{a^4}\right),\\
\log|\theta^{(6)}| &= \frac{1}{a^3} - \frac{3}{a^4} + L_2\left(\frac{0.1}{a^5}\right)
\end{aligned}$$

and

$$(2.7)\qquad \begin{aligned}
\log|\theta^{(1)}-\theta^{(2)}| &= \log(a) + \log(2) + \frac{5}{2a^2} - \frac{1}{a^3} + L_3\left(\frac{1}{a^4}\right),\\
\log|\theta^{(1)}-\theta^{(3)}| &= \frac{1}{a^2} + \frac{1}{a^3} + L_2\left(\frac{0.1}{a^4}\right),\\
\log|\theta^{(1)}-\theta^{(4)}| &= \log(a) + \frac{2}{a^2} + L_3\left(\frac{3}{a^4}\right),\\
\log|\theta^{(1)}-\theta^{(5)}| &= \log(a) + \frac{3}{a^2} - \frac{2}{a^3} + L_2\left(\frac{0.8}{a^4}\right),\\
\log|\theta^{(1)}-\theta^{(6)}| &= \log(a) + \frac{1}{a} + \frac{3}{2a^2} - \frac{2}{3a^3} + L_3\left(\frac{2}{a^4}\right),\\
\log|\theta^{(2)}-\theta^{(3)}| &= \log(a) + \log(2) - \frac{1}{2a} + \frac{19}{8a^2} - \frac{7}{24a^3} + L_2\left(\frac{1}{a^4}\right),\\
\log|\theta^{(2)}-\theta^{(4)}| &= \log(a) + \frac{3}{a^2} - \frac{2}{a^3} + L_2\left(\frac{1.2}{a^4}\right),\\
\log|\theta^{(2)}-\theta^{(5)}| &= \log(a) + \frac{2}{a^2} + L_2\left(\frac{2.4}{a^4}\right),\\
\log|\theta^{(2)}-\theta^{(6)}| &= \log(a) - \frac{1}{a} + \frac{5}{2a^2} - \frac{1}{3a^3} + L_5\left(\frac{0.2}{a^4}\right),\\
\log|\theta^{(3)}-\theta^{(4)}| &= \log(a) - \frac{1}{a} + \frac{3}{2a^2} + \frac{2}{3a^3} + L_2\left(\frac{21}{4a^4}\right),
\end{aligned}$$

$$\begin{aligned}
\log|\theta^{(3)}-\theta^{(5)}| &= \log(a)-\frac{1}{a}+\frac{5}{2a^2}-\frac{1}{3a^3}+L_5\Big(\frac{1}{a^4}\Big),\\
\log|\theta^{(3)}-\theta^{(6)}| &= \log(a)+\frac{2}{a^2}+L_2\Big(\frac{4.1}{a^4}\Big),\\
\log|\theta^{(4)}-\theta^{(5)}| &= -\log(a)-\frac{2}{a}+\frac{4}{a^2}+\frac{28}{3a^3}+L_4\Big(\frac{10}{a^4}\Big),\\
\log|\theta^{(4)}-\theta^{(6)}| &= \frac{1}{a^2}+\frac{1}{a^3}+L_2\Big(\frac{4.5}{a^4}\Big),\\
\log|\theta^{(5)}-\theta^{(6)}| &= -\frac{1}{a}+\frac{5}{2a^2}-\frac{7}{3a^3}+L_3\Big(\frac{10}{a^4}\Big).
\end{aligned} \tag{2.7 [cont.]}$$

Now let us extend Theorem 2.1:

Lemma 2.3. *Let $\mathcal{O}=\mathbb{Z}[\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}]$ and consider the subgroup $\langle -1,\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}\rangle$ of the unit group of $\mathcal{O}$. Then*

$$I := [\mathcal{O}^\times : \langle -1,\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}\rangle] < 92.60\log^5(a) \tag{2.8}$$

for $a\ge 35$.

Proof. We determine an upper bound for the index of $\langle -1,\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}\rangle$ in $\mathcal{O}^\times$ by estimating the regulators of the two groups.

Let R_θ be the regulator of $\langle -1,\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}\rangle$:

$$R_\theta = \left\|\begin{matrix}
\log|\theta^{(1)}| & \log|\theta^{(2)}| & \log|\theta^{(3)}| & \log|\theta^{(4)}| & \log|\theta^{(5)}|\\
\log|\theta^{(2)}| & \log|\theta^{(3)}| & \log|\theta^{(4)}| & \log|\theta^{(5)}| & \log|\theta^{(6)}|\\
\log|\theta^{(3)}| & \log|\theta^{(4)}| & \log|\theta^{(5)}| & \log|\theta^{(6)}| & \log|\theta^{(1)}|\\
\log|\theta^{(4)}| & \log|\theta^{(5)}| & \log|\theta^{(6)}| & \log|\theta^{(1)}| & \log|\theta^{(2)}|\\
\log|\theta^{(5)}| & \log|\theta^{(6)}| & \log|\theta^{(1)}| & \log|\theta^{(2)}| & \log|\theta^{(3)}|
\end{matrix}\right\|.$$

Applying (2.6), we obtain

$$19\log^5(a)-\frac{40\log^4(a)}{a} < R_\theta < 19\log^5(a)-\frac{38\log^4(a)}{a} \tag{2.9}$$

for $a\ge 35$. In fact, using asymptotic expressions, we obtain

$$R_\theta = (19+L_{35}(0.001))\log^5(a). \tag{2.10}$$

So $R_\theta>0$ and $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$, $\theta^{(4)}$, $\theta^{(5)}$ are independent units. By Theorem B in [2], the regulator of $\mathbb{K}_a$ can be bounded by $\operatorname{Reg}\mathbb{Z}_{\mathbb{K}_a}\ge 0.2052$. From [10, p. 361], we find a bound for the index I:

$$I = \frac{R_\theta}{\operatorname{Reg}\mathbb{Z}_{\mathbb{K}_a}} < 92.60\log^5(a) \tag{2.11}$$

for $a\ge 35$. ∎

Therefore, we can work without the conditions in Theorem 2.1. Instead, we will use Lemma 2.3.

3. Approximation properties of solutions. Let $(x,y)\in\mathbb{Z}^2$ be a solution to (1.3). We define $\beta^{(i)} := x-\theta^{(i)}y$. Since ϕ_a is irreducible by Theorem 2.1 (having six distinct real roots), we define the *type* of a solution (x,y) of (1.3) to be the j such that

$$|\beta^{(j)}| = \min_{1\le i\le 6} |\beta^{(i)}|. \tag{3.1}$$

The crucial step of the proof of Theorem 1.1 will be the following lemma, which excludes solutions with small y:

Lemma 3.1. *Let $a\ge 50$ and (x,y) be a solution to* (1.3) *of type j which is not contained in* (1.4). *Then*

$$|\beta^{(j)}| \le c_j \frac{1}{y^5}, \quad \textit{where} \quad c_j = \begin{cases} 160/11a^4 & \textit{if } j=1,\\ 320/39a^5 & \textit{if } j=2,\\ 320/19a^4 & \textit{if } j=3,\\ 320/9a^2 & \textit{if } j=4,5,\\ 960/29a^3 & \textit{if } j=6; \end{cases} \tag{3.2}$$

$$\log|\beta^{(i)}| = \log(y) + \log|\theta^{(i)}-\theta^{(j)}| + L_{50}\left(\frac{1}{2a}\right), \quad i\ne j. \tag{3.3}$$

Proof. For $i\ne j$, we have

$$y|\theta^{(i)}-\theta^{(j)}| \le 2|\beta^{(i)}|,$$

then

$$|\beta^{(j)}| = \frac{1}{\prod_{i\ne j}|\beta^{(i)}|} \le \frac{2^5}{y^5}\cdot\frac{1}{\prod_{i\ne j}|\theta^{(i)}-\theta^{(j)}|} = \frac{2^5}{y^5|\phi_a'(\theta^{(j)})|}. \tag{3.4}$$

For each $j=1,\dots,6$, we compute the asymptotic expressions of $\phi_a'(\theta^{(j)})$ and we deduce that for $a\ge 50$:

$$|\phi_a'(\theta^{(1)})| = \prod_{i\ne 1}|\theta^{(i)}-\theta^{(1)}| \ge \frac{11}{5}a^4, \tag{3.5}$$

$$|\phi_a'(\theta^{(2)})| = \prod_{i\ne 2}|\theta^{(i)}-\theta^{(2)}| \ge \frac{39}{10}a^5, \tag{3.6}$$

$$|\phi_a'(\theta^{(3)})| = \prod_{i\ne 3}|\theta^{(i)}-\theta^{(3)}| \ge \frac{19}{10}a^4, \tag{3.7}$$

$$|\phi_a'(\theta^{(4)})| = \prod_{i\ne 4}|\theta^{(i)}-\theta^{(4)}| \ge \frac{9}{10}a^2, \tag{3.8}$$

$$|\phi_a'(\theta^{(5)})| = \prod_{i\ne 5}|\theta^{(i)}-\theta^{(5)}| \ge \frac{9}{10}a^2, \tag{3.9}$$

$$|\phi_a'(\theta^{(6)})| = \prod_{i\ne 6}|\theta^{(i)}-\theta^{(6)}| \ge \frac{29}{30}a^3. \tag{3.10}$$

So we obtain (3.2). Additionally, we note that this implies together with

$$\frac{|\beta^{(i)}|}{y|\theta^{(i)}-\theta^{(j)}|}=\left|1-\frac{\beta^{(j)}}{y(\theta^{(i)}-\theta^{(j)})}\right|$$

that

$$\log|\beta^{(i)}|=\log(y)+\log|\theta^{(i)}-\theta^{(j)}|+L_{50}\left(\frac{1}{2a}\right). \tag{3.11}$$

In fact, we look at all the asymptotic expressions of $\log\left|1-\frac{\beta^{(j)}}{y(\theta^{(i)}-\theta^{(j)})}\right|$ to draw this conclusion. This completes the proof. ∎

The next result gives us a lower bound for $\log(y)$.

Lemma 3.2. *Let (x,y) be a solution to* (1.3) *with $y\geq 2$ and $a\geq 475$. Then*

$$\log(y)\geq 0.032a^2\log^2(a). \tag{3.12}$$

Proof. If (x,y) is a solution to (1.3), then $\beta^{(i)}$ is a unit in $\mathbb{Z}[\theta^{(1)},\theta^{(2)},\theta^{(3)},\theta^{(4)},\theta^{(5)}]$ for $1\leq i\leq 6$. By Lemma 2.3, there are integers u_1,u_2,u_3,u_4,u_5 such that

$$(\beta^{(1)})^I=\pm(\theta^{(1)})^{u_1}(\theta^{(2)})^{u_2}(\theta^{(3)})^{u_3}(\theta^{(4)})^{u_4}(\theta^{(5)})^{u_5}, \tag{3.13}$$

where I is the index obtained in Lemma 2.3. So for $i=1,\dots,6$, the conjugates of $\beta^{(1)}$ are given by

$$|\beta^{(i)}|^I=|\theta^{(i)}|^{u_1}|\theta^{(i+1)}|^{u_2}|\theta^{(i+2)}|^{u_3}|\theta^{(i+3)}|^{u_4}|\theta^{(i+4)}|^{u_5}, \tag{3.14}$$

with indices taken mod 6; therefore applying log, we obtain

$$\begin{aligned}\log|\beta^{(i)}|=\frac{u_1}{I}\log|\theta^{(i)}|+\frac{u_2}{I}\log|\theta^{(i+1)}|+\frac{u_3}{I}\log|\theta^{(i+2)}|\\ +\frac{u_4}{I}\log|\theta^{(i+3)}|+\frac{u_5}{I}\log|\theta^{(i+4)}|\quad\text{(indices mod 6)}.\end{aligned} \tag{3.15}$$

For each j, from (3.15), we consider the subsystem not containing $\beta^{(j)}$ that we solve for u_1/I and u_2/I using Cramer's method. Then we use the asymptotic expressions (2.5)–(2.7) and (3.3) to obtain

$$\begin{aligned}\frac{Ru_i}{I}\\ =\left(c_{i,1}\log^4(a)+\frac{c_{i,2}\log^3(a)}{a}+\frac{c_{i,3}\log^3(a)+c_{i,4}\log^2(a)}{a^2}+L_{50}\left(\frac{c_{i,5}}{a^3}\right)\right)\log(y)\\ +c_{i,6}\log^5(a)+c_{i,7}\log(2)\log^4(a)+\frac{c_{i,8}\log^4(a)+c_{i,9}\log(2)\log^3(a)}{a}\\ +\frac{c_{i,10}\log^4(a)+c_{i,11}\log^3(a)+c_{i,12}\log(2)\log^3(a)+c_{i,13}\log(2)\log^2(a)}{a^2}\\ +L_{50}\left(\frac{c_{i,14}}{a^3}\right),\end{aligned} \tag{3.16}$$

where, for $i = 1, \ldots, 5$ and $k = 1, \ldots, 14$, the coefficients $c_{i,k}$ are rational numbers. We use R instead of R_θ to simplify notation. Tables 1–5 give us the coefficients $c_{i,j}$:

Table 1. Values of $c_{1,k}$ for u_1

Case	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$	$c_{1,6}$	$c_{1,7}$	$c_{1,8}$	$c_{1,9}$	$c_{1,10}$	$c_{1,11}$	$c_{1,12}$	$c_{1,13}$	$c_{1,14}$
$j=1$	-11	34	-55	111	30.5	-4	-2	24	18	-51	90	-21	10	1
$j=2$	-1	78	-79	21	1	-1	3	$\frac{149}{2}$	41	$-\frac{725}{8}$	$-\frac{9}{2}$	$-\frac{79}{2}$	-13	1
$j=3$	31	44	0	-59	137	24	5	$\frac{67}{2}$	23	$\frac{175}{8}$	$-\frac{129}{2}$	$-\frac{37}{2}$	-23	80
$j=4$	65	-64	112	-15	255	19	0	-135	0	$\frac{655}{2}$	76	0	0	447
$j=5$	37	-66	109	33	10	-2	0	-126	0	282	134	0	0	25
$j=6$	-7	-2	21	79	5	0	0	-8	0	16	88	0	0	40

Table 2. Values of $c_{2,k}$ for u_2

Case	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$	$c_{2,5}$	$c_{2,6}$	$c_{2,7}$	$c_{2,8}$	$c_{2,9}$	$c_{2,10}$	$c_{2,11}$	$c_{2,12}$	$c_{2,13}$	$c_{2,14}$
$j=1$	-10	-44	24	90	1	-14	-7	-9	7	-56	51	$-\frac{53}{2}$	29	25
$j=2$	-32	34	-79	80	100	-32	-18	$\frac{91}{2}$	26	$-\frac{1309}{8}$	$\frac{115}{2}$	-49	9	10
$j=3$	-34	108	-112	-44	125	-30	-11	$\frac{183}{2}$	19	$-\frac{1137}{8}$	$-\frac{135}{2}$	$-\frac{45}{2}$	-20	595
$j=4$	28	2	3	-48	117	0	0	12	0	34	-21	0	0	12
$j=5$	44	-64	88	-46	180	12	0	-53	0	$\frac{249}{2}$	49	0	0	120
$j=6$	4	-36	76	-32	3	0	0	-28	0	56	-4	0	0	40

Table 3. Values of $c_{3,k}$ for u_3

Case	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$c_{3,4}$	$c_{3,5}$	$c_{3,6}$	$c_{3,7}$	$c_{3,8}$	$c_{3,9}$	$c_{3,10}$	$c_{3,11}$	$c_{3,12}$	$c_{3,13}$	$c_{3,14}$
$j=1$	21	0	24	31	10	18	9	9	-13	63	-25	$\frac{51}{2}$	2	36
$j=2$	33	-30	33	65	1	33	15	-38	-9	$\frac{441}{4}$	77	$\frac{53}{2}$	75	100
$j=3$	3	42	-3	-11	32	6	6	35	4	$\frac{17}{4}$	29	1	73	15
$j=4$	21	0	24	31	9	0	0	9	0	$\frac{51}{2}$	-13	0	0	10
$j=5$	33	-30	33	65	100	9	0	-37	0	$\frac{145}{2}$	24	0	0	1
$j=6$	3	42	-3	-11	40	0	0	36	0	4	-36	0	0	1

Table 4. Values of $c_{4,k}$ for u_4

Case	$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	$c_{4,4}$	$c_{4,5}$	$c_{4,6}$	$c_{4,7}$	$c_{4,8}$	$c_{4,9}$	$c_{4,10}$	$c_{4,11}$	$c_{4,12}$	$c_{4,13}$	$c_{4,14}$
$j=1$	-44	64	-88	46	100	-35	-8	48	14	-166	111	-22	41	100
$j=2$	-4	36	-76	32	8	-4	-7	$\frac{63}{2}$	28	$-\frac{733}{8}$	16	-40	22	0.1
$j=3$	10	44	-24	-90	75	1	1	$\frac{89}{2}$	14	$-\frac{241}{8}$	-170	-18	-19	4
$j=4$	32	-34	79	-80	107	0	0	11	0	$\frac{75}{2}$	-19	0	0	1
$j=5$	34	-108	112	44	130	11	0	-42	0	81	113	0	0	1
$j=6$	-28	-2	-3	48	120	-19	0	8	0	-61	57	0	0	300

Table 5. Values of $c_{5,k}$ for u_5

Case	$c_{5,1}$	$c_{5,2}$	$c_{5,3}$	$c_{5,4}$	$c_{5,5}$	$c_{5,6}$	$c_{5,7}$	$c_{5,8}$	$c_{5,9}$	$c_{5,10}$	$c_{5,11}$	$c_{5,12}$	$c_{5,13}$	$c_{5,14}$
$j=1$	-7	-2	21	79	45	-6	-3	17	8	-26	38	-11	29	2
$j=2$	-11	34	-55	111	18	-11	-5	43	26	$-\frac{361}{4}$	88	-32	39	1
$j=3$	-1	78	-79	21	54	-2	-2	71	18	$-\frac{291}{4}$	-11	-21	10	105
$j=4$	31	44	0	-59	37	0	0	16	0	39	-32	0	0	3
$j=5$	65	-64	112	-15	255	16	0	-81	0	$\frac{393}{2}$	60	0	0	1
$j=6$	37	-66	109	33	1	19	0	-52	0	149	31	0	0	4

We choose small and positive linear combinations of the u_k depending on j:

$$\frac{Rv_j}{I} := b_{j0}R + b_{j1}\,\frac{Ru_1}{I} + b_{j2}\,\frac{Ru_2}{I} + b_{j3}\,\frac{Ru_3}{I} + b_{j4}\,\frac{Ru_4}{I} + b_{j5}\,\frac{Ru_5}{I}, \tag{3.17}$$

where $b_{ji} \in \mathbb{Z}$. In fact, we use coefficients b_{ji} such that the coefficients of the main terms $\log(y)\log^4(a)$, $\log(y)\log^3(a)/a$, $\log^5(a)$, and $\log(2)\log^4(a)$ vanish:

$$\frac{Rv_j}{I} = \begin{cases} -3\frac{Ru_1}{I} - 3\frac{Ru_2}{I} + 2\frac{Ru_3}{I} + 0\frac{Ru_4}{I} + 15\frac{Ru_5}{I} & \text{if } j=1, \\ R - 75\frac{Ru_1}{I} - 444\frac{Ru_2}{I} - 403\frac{Ru_3}{I} + 246\frac{Ru_4}{I} + 0\frac{Ru_5}{I} & \text{if } j=2, \\ R + 138\frac{Ru_1}{I} + 138\frac{Ru_2}{I} + 41\frac{Ru_3}{I} + 0\frac{Ru_4}{I} - 291\frac{Ru_5}{I} & \text{if } j=3, \\ 0\frac{Ru_1}{I} - 203\frac{Ru_2}{I} + 313\frac{Ru_3}{I} - 21\frac{Ru_4}{I} - 7\frac{Ru_5}{I} & \text{if } j=4, \\ -53\frac{Ru_1}{I} - \frac{Ru_2}{I} - 667\frac{Ru_3}{I} - \frac{Ru_4}{I} + 370\frac{Ru_5}{I} & \text{if } j=5, \\ 139R + 69\frac{Ru_1}{I} + 366\frac{Ru_2}{I} + 547\frac{Ru_3}{I} + 276\frac{Ru_4}{I} + 138\frac{Ru_5}{I} & \text{if } j=6. \end{cases} \tag{3.18}$$

Then we get

$$\frac{Rv_j}{I} \tag{3.19}$$

$$= \begin{cases} \Big(\frac{456\log^3(a)+644\log^2(a)}{a^2} + L_{50}\big(\frac{1}{a^3}\big)\Big)\log(y) + \frac{228\log^4(a)+19\log(2)\log^3(a)}{a} \\ \quad +\frac{57\log^4(a)+97\log^3(a)+\frac{57}{2}\log(2)\log^3(a)+322\log(2)\log^2(a)}{a^2} + L_{50}\big(\frac{90}{a^3}\big) & \text{if } j=1, \\ \Big(\frac{9006\log^3(a)-55418\log^2(a)}{a^2} + L_{50}\big(\frac{1000}{a^3}\big)\Big)\log(y) \\ \quad +19\log^5(a) - \frac{5533\log^4(a)+8208\log(2)\log^3(a)}{2a} \\ \quad +\frac{\frac{100199}{8}\log^4(a)-\frac{104449}{2}\log^3(a)+4199\log(2)\log^3(a)-27834\log(2)\log^2(a)}{a^2} \\ \quad + L_{50}\big(\frac{100}{a^3}\big) & \text{if } j=2, \\ \Big(\frac{7410\log^3(a)-20776\log^2(a)}{a^2} + L_{50}\big(\frac{2}{a^3}\big)\Big)\log(y) \\ \quad +19\log^5(a) + \frac{-2016\log^4(a)+722\log(2)\log^3(a)}{a} \\ \quad +\frac{4799\log^4(a)-13763\log^3(a)+494\log(2)\log^3(a)-5851\log(2)\log^2(a)}{a^2} \\ \quad + L_{50}\big(\frac{10}{a^3}\big) & \text{if } j=3, \\ \Big(\frac{5244\log^3(a)+21540\log^2(a)}{a^2} + L_{50}\big(\frac{245}{a^3}\big)\Big)\log(y) \\ \quad +\frac{38\log^4(a)}{a} + \frac{19\log^4(a)+817\log^3(a)}{a^2} + L_{50}\big(\frac{300}{a^3}\big) & \text{if } j=4, \\ \Big(\frac{13452\log^3(a)-50652\log^2(a)}{a^2} + L_{50}\big(\frac{2}{a^3}\big)\Big)\log(y) \\ \quad +\frac{1482\log^4(a)}{a} + \frac{9196\log^4(a)-1072\log^3(a)}{a^2} + L_{50}\big(\frac{85}{a^3}\big) & \text{if } j=5, \\ \Big(\frac{41838\log^3(a)+5524\log^2(a)}{a^2} + L_{50}\big(\frac{50}{a^3}\big)\Big)\log(y) \\ \quad +19\log^5(a) - \frac{1636\log^4(a)}{a} + \frac{34325\log^4(a)+13683\log^3(a)}{a^2} + L_{50}\big(\frac{3055}{a^3}\big) & \text{if } j=6. \end{cases}$$

From the above expressions, one can check that each $v_j > 0$ for $a \geq 475$. Therefore if we use (3.19) and (2.10), the inequality $Rv_j/I \geq R$ leads to the following lower bounds for $\log(y)$ according to j:

$$\log(y) \geq \begin{cases} 0.032a^2\log^2(a) & \text{if } j=1 \text{ and } a \geq 180, \\ 0.31a\log(a) & \text{if } j=2 \text{ and } a \geq 475, \\ 0.27a\log(a) & \text{if } j=3 \text{ and } a \geq 50, \\ 0.0018a^2\log^2(a) & \text{if } j=4 \text{ and } a \geq 64, \\ 0.0014a^2\log^2(a) & \text{if } j=5 \text{ and } a \geq 44, \\ 0.03a\log(a) & \text{if } j=6 \text{ and } a \geq 50. \end{cases} \tag{3.20}$$

This implies (3.12) and completes the proof. ∎

4. Proof of Theorem 1.1. Suppose that $(x,y)\in\mathbb{Z}^2$ is a nontrivial solution of type j (a solution not in (1.4)). We define indices (i,k) depending on j:

$$(i,k):=\begin{cases}(2,3) & \text{if } j=1,\\ (3,4) & \text{if } j=2,\\ (4,5) & \text{if } j=3,\\ (5,6) & \text{if } j=4,\\ (6,1) & \text{if } j=5,\\ (1,2) & \text{if } j=6.\end{cases}$$

We consider the following Siegel identity:

$$\frac{\beta^{(k)}(\theta^{(j)}-\theta^{(i)})}{\beta^{(i)}(\theta^{(j)}-\theta^{(k)})}-1=\frac{\beta^{(j)}(\theta^{(k)}-\theta^{(i)})}{\beta^{(i)}(\theta^{(j)}-\theta^{(k)})}.$$

We put

$$\lambda_j=\frac{\theta^{(j)}-\theta^{(i)}}{\theta^{(j)}-\theta^{(k)}},\qquad \tau_j=\frac{\beta^{(j)}}{\beta^{(i)}}\left(\frac{\theta^{(k)}-\theta^{(i)}}{\theta^{(j)}-\theta^{(k)}}\right)$$

and by (3.13) we obtain the following linear form in logarithms:

$$\begin{aligned}(4.1)\qquad \varLambda_j&=\frac{u_1}{I}\log\left|\frac{\theta^{(k)}}{\theta^{(i)}}\right|+\frac{u_2}{I}\log\left|\frac{\theta^{(k+1)}}{\theta^{(i+1)}}\right|+\frac{u_3}{I}\log\left|\frac{\theta^{(k+2)}}{\theta^{(i+2)}}\right|\\ &\quad+\frac{u_4}{I}\log\left|\frac{\theta^{(k+3)}}{\theta^{(i+3)}}\right|+\frac{u_5}{I}\log\left|\frac{\theta^{(k+4)}}{\theta^{(i+4)}}\right|+\log|\lambda_j|\\ &=\log|1+\tau_j|\end{aligned}$$

(indices mod 6).

Lemma 4.1. *We have $\varLambda_j\neq 0$.*

Proof. Suppose that $\varLambda_j=0$, then from (4.1) we have $\tau_j=0$ or $\tau_j=-2$. The first is impossible because the polynomial $\phi_a(x)$ has six distinct nonzero roots. On the other hand, if $\tau_j=-2$, then by the Siegel identity the conjugate τ_{j+1} (with index reduced mod 6) of τ_j would be 1. This is also impossible in the normal closure of $\mathbb{K}_a$. ∎

We know that for any real $z\geq 0.2032$ the inequality $|\log(z)|<2|z-1|$ holds. By (4.1) we have

$$(4.2)\qquad \log|\varLambda_j|=\log|\log|1+\tau_j||\leq\log|2\tau_j|=\log\left|2\,\frac{\beta^{(j)}}{\beta^{(i)}}\left(\frac{\theta^{(k)}-\theta^{(i)}}{\theta^{(j)}-\theta^{(k)}}\right)\right|.$$

Therefore, by (3.4) and (3.11), we obtain

$$\log|\Lambda_j| \le -6\log(y) + \begin{cases} -4\log(a) + \log\left(\frac{320}{11}\right) & \text{if } j = 1, \\ -6\log(a) + \log\left(\frac{320}{39}\right) & \text{if } j = 2, \\ -7\log(a) + \log\left(\frac{640}{19}\right) & \text{if } j = 3, \\ -\log(a) + \log\left(\frac{640}{9}\right) + \frac{1}{a} & \text{if } j = 4, \\ -2\log(a) + \log\left(\frac{640}{9}\right) + \frac{2}{a} & \text{if } j = 5, \\ -4\log(a) + \log\left(\frac{3840}{29}\right) & \text{if } j = 6. \end{cases} \tag{4.3}$$

Observing the absolute values of the coefficients $c_{i,1}$, one can conclude that

$$U_j = \max\{|u_1/I|, \ldots, |u_6/I|\} = \begin{cases} -u_4/I & \text{if } j = 1, \\ u_3/I & \text{if } j = 2, \\ -u_2/I & \text{if } j = 3, \\ u_1/I & \text{if } j = 4, \\ u_5/I & \text{if } j = 5, 6. \end{cases}$$

In order to apply the result due to Laurent, Mignotte, and Nesterenko (see [6, Corollaire 2, p. 288]) on linear forms in two logarithms to determine lower bounds of Λ_j, we rewrite (4.1) to obtain the following linear form in two logarithms:

$$k_j\Lambda_j = U_j \log|\delta_j| + \log|\nu_1^{W_{1j}}\nu_2^{W_{2j}}\nu_3^{W_{3j}}\nu_4^{W_{4j}}\nu_5^{W_{5j}}\nu_6^{W_{6j}}\lambda_j^{k_j}| \tag{4.4}$$

with

$$k_1 = 44, \quad k_2 = 33, \quad k_3 = 34, \quad k_4 = 65, \quad k_5 = 65, \quad k_6 = 37; \tag{4.5}$$

$$\begin{aligned} |\nu_1| &= -\frac{\theta^{(3)}}{\theta^{(2)}}, \quad |\nu_2| = \frac{\theta^{(4)}}{\theta^{(3)}}, \quad |\nu_3| = -\frac{\theta^{(5)}}{\theta^{(4)}}, \\ |\nu_4| &= \frac{\theta^{(6)}}{\theta^{(5)}}, \quad |\nu_5| = \frac{\theta^{(1)}}{\theta^{(6)}}, \quad |\nu_6| = -\frac{\theta^{(2)}}{\theta^{(1)}}; \end{aligned} \tag{4.6}$$

$$\delta_j = \begin{cases} \nu_1^{-11}\nu_2^{-10}\nu_3^{21}\nu_4^{-44}\nu_5^{-7} & \text{if } j = 1, \\ \nu_2^{-1}\nu_3^{-32}\nu_4^{33}\nu_5^{-4}\nu_6^{-11} & \text{if } j = 2, \\ \nu_1^{-1}\nu_3^{31}\nu_4^{-34}\nu_5^{3}\nu_6^{10} & \text{if } j = 3, \\ \nu_1^{32}\nu_2^{31}\nu_4^{65}\nu_5^{28}\nu_6^{21} & \text{if } j = 4, \\ \nu_1^{33}\nu_2^{34}\nu_3^{65}\nu_5^{37}\nu_6^{44} & \text{if } j = 5, \\ \nu_1^{4}\nu_2^{3}\nu_3^{-28}\nu_4^{37}\nu_6^{-7} & \text{if } j = 6; \end{cases} \tag{4.7}$$

$$\begin{aligned} \lambda_1 &= \frac{\theta^{(1)} - \theta^{(2)}}{\theta^{(1)} - \theta^{(3)}}, \quad \lambda_2 = \frac{\theta^{(2)} - \theta^{(3)}}{\theta^{(2)} - \theta^{(4)}}, \quad \lambda_3 = \frac{\theta^{(3)} - \theta^{(4)}}{\theta^{(3)} - \theta^{(5)}}, \\ \lambda_4 &= \frac{\theta^{(4)} - \theta^{(5)}}{\theta^{(4)} - \theta^{(6)}}, \quad \lambda_5 = \frac{\theta^{(5)} - \theta^{(6)}}{\theta^{(5)} - \theta^{(1)}}, \quad \lambda_6 = \frac{\theta^{(6)} - \theta^{(1)}}{\theta^{(6)} - \theta^{(2)}}. \end{aligned} \tag{4.8}$$

Table 6. Choice of W_{ij}

Case	W_{1j}	W_{2j}	W_{3j}	W_{4j}	W_{5j}	W_{6j}
$j=1$	$\frac{44u_1-11u_4}{I}$	$\frac{44u_2-10u_4}{I}$	$\frac{44u_3+21u_4}{I}$	0	$\frac{44u_5-7u_4}{I}$	0
$j=2$	0	$\frac{33u_1+u_3}{I}$	$\frac{33u_2+32u_3}{I}$	0	$\frac{33u_4+4u_3}{I}$	$\frac{33u_5+11u_3}{I}$
$j=3$	$\frac{34u_5-u_2}{I}$	0	$\frac{34u_1+31u_2}{I}$	0	$\frac{34u_3+3u_2}{I}$	$\frac{34u_4+10u_2}{I}$
$j=4$	$\frac{65u_4-32u_1}{I}$	$\frac{65u_5-31u_1}{I}$	0	0	$\frac{65u_2-28u_1}{I}$	$\frac{65u_3-21u_1}{I}$
$j=5$	$\frac{65u_3-33u_5}{I}$	$\frac{65u_4-34u_5}{I}$	0	0	$\frac{65u_1-37u_5}{I}$	$\frac{65u_2-44u_5}{I}$
$j=6$	$\frac{37u_2-4u_5}{I}$	$\frac{37u_3-3u_5}{I}$	$\frac{37u_4+28u_5}{I}$	0	0	$\frac{37u_1+7u_5}{I}$

In fact, to determine the exponents W_{ij}, first we choose U_j as coefficients of one logarithm and then we try to eliminate the terms containing $\frac{c_{i1}}{19\log(a)}\log(y)$ in the asymptotic expressions of u_i/I. Using asymptotic expressions, one can see that

$$\begin{aligned} &\log|\nu_1| \sim -\frac{1}{a}, && \log|\nu_4| \sim \log(a)+\frac{3}{a},\\ &\log|\nu_2| \sim -3\log(a)+\frac{1}{a}, && \log|\nu_5| \sim \log(a)+\frac{2}{a^2},\\ &\log|\nu_3| \sim \log(a)-\frac{3}{a}, && \log|\nu_6| \sim \frac{1}{a^2}; \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\log|\lambda_1| \sim \log(a)+\log(2), && \log|\lambda_4| \sim -\log(a),\\ &\log|\lambda_2| \sim \log(2), && \log|\lambda_5| \sim -\log(a),\\ &\log|\lambda_3| \sim -\frac{1}{a^2}, && \log|\lambda_6| \sim \frac{2}{a}. \end{aligned} \tag{4.10}$$

We take $D=6$ and we obtain

$$h(\nu_i) \le \frac{1}{6}\log|\nu_2^{-1}\nu_3\nu_4\nu_5| \le \log(a)-\frac{1}{6a}+\frac{2}{3a^2}; \tag{4.11}$$

and

$$\begin{aligned} h(\lambda_j) &= \frac{1}{6}\log\left|(4a^3+2a^2+10a+13)\left(\frac{\lambda_1\lambda_2}{\lambda_4\lambda_5}\right)\right|\\ &\le \log(a)+\frac{2}{3}\log(2)+\frac{1}{2a}+\frac{1}{8a^2}, \end{aligned} \tag{4.12}$$

where $4a^3+2a^2+10a+13$ is the leading coefficient of the irreducible polynomial of λ_j. Now set

$$\gamma_1=\delta_j, \qquad \gamma_2=\nu_1^{W_{1j}}\nu_2^{W_{2j}}\nu_3^{W_{3j}}\nu_4^{W_{4j}}\nu_5^{W_{5j}}\nu_6^{W_{6j}}\lambda_j^{k_j}. \tag{4.13}$$

Then using the properties of the absolute logarithmic height of algebraic numbers, we have

$$h(\gamma_1) = h(\delta_j) \leq s_j h(\nu_1), \tag{4.14}$$

where s_j is the sum of the absolute values of the exponents of ν_i in the expression of δ_j, i.e.

$$s_1 = 93, \quad s_2 = 81, \quad s_3 = 79, \quad s_4 = 177, \quad s_5 = 213, \quad s_6 = 79, \tag{4.15}$$

and

$$h(\gamma_2) \leq \Big(\sum_{i=1}^{6} |W_{ij}|\Big) h(\nu_1) + k_j h(\lambda_1) \tag{4.16}$$

for $j \in \{1, \ldots, 6\}$. The choice of h_1, h_2, and b' depending on j is given in Table 7 below.

Table 7. Choice of h_1, h_2, and b' depending on j

Case	h_1	h_2	b'
$j=1$	$93\log(a) - \frac{31}{2a} + \frac{62}{a^2}$	$(\frac{5248}{19\log(a)a} - \frac{0.34}{a^2})\log(y) + \frac{172}{3}\log(2) + 73\log(a) + \frac{72}{a}$	$\frac{88a^2}{520125}$
$j=2$	$81\log(a) - \frac{27}{2a} + \frac{189}{4a^2}$	$(\frac{4566}{19\log(a)a} + \frac{21}{a^2})\log(y) + 13\log(2) + 33\log(a) + \frac{429}{a}$	$\frac{2a^2}{7125}$
$j=3$	$79\log(a) - \frac{79}{6a} + \frac{553}{12a^2}$	$(\frac{11716}{19\log(a)a} - \frac{50}{a^2})\log(y) + \frac{47}{3}\log(2) + 18\log(a) + \frac{658}{a}$	$\frac{34a^2}{200070}$
$j=4$	$177\log(a) - \frac{59}{2a} + \frac{413}{4a^2}$	$(\frac{8272}{19\log(a)a} - \frac{51}{a^2})\log(y) + \frac{130}{3}\log(2) + 17\log(a) + \frac{14039}{38a}$	$\frac{65a^2}{24225}$
$j=5$	$213\log(a) - \frac{71}{2a} + \frac{497}{4a^2}$	$(\frac{8272}{19\log(a)a} - \frac{51}{a^2})\log(y) + \frac{130}{3}\log(2) + 93\log(a) + \frac{41801}{114a}$	$\frac{26a^2}{132525}$
$j=6$	$79\log(a) - \frac{79}{6a} + \frac{553}{12a^2}$	$(\frac{5278}{19\log(a)a} - \frac{55}{a^2})\log(y) + \frac{130}{3}\log(2) + 68\log(a) + \frac{256}{a}$	$\frac{37a^2}{242250}$

Thus we get

$$\log|\Lambda_j| \geq -31544.64(\log(b') + .14)^2 h_1 h_2 - \log(k_j), \tag{4.17}$$

where b', h_1, h_2 are given in Table 7, and k_j is given in (4.5). By combining (4.3), (4.17) and Lemma 3.2, we obtain inequalities that are not true for

$$a \geq N_j = \begin{cases} 261910993786 & \text{if } j=1, \\ 197991298372 & \text{if } j=2, \\ 558417390554 & \text{if } j=3, \\ 951648836636 & \text{if } j=4, \\ 1077675177955 & \text{if } j=5, \\ 219113871094 & \text{if } j=6. \end{cases} \tag{4.18}$$

This completes the proof of Theorem 1.1.

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