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Four prime squares and powers of 2

by

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1. Introduction. Let

$$\mathcal{A}_3 = \{n : n \in \mathbb{N},\ n \equiv 3 \pmod{24},\ n \not\equiv 0 \pmod 5\},$$
$$\mathcal{A}_5 = \{n : n \in \mathbb{N},\ n \equiv 5 \pmod{24}\}.$$

In 1938 Hua [3] proved that almost all $n \in \mathcal{A}_3$ are representable as sums of three squares of primes, and all sufficiently large $n \in \mathcal{A}_5$ are representable as sums of five squares of primes. In view of these results and Lagrange's theorem of four squares, it is reasonable to conjecture that every large even integer $n \equiv 4 \pmod{24}$ is a sum of four squares of primes

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2. \tag{1.1}$$

In [6], Liu, Liu and Zhan proved that every large even integer N can be written as a sum of four squares of primes and powers of 2,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.2}$$

In [4], Liu and Liu showed that $k = 8330$ is acceptable in (1.2). Recently, Liu and Lü [7] improved this result and proved that $k = 165$ suffices.

In this note, we will prove the following:

Theorem. *Every sufficiently large even integer can be written as a sum of four squares of primes and* 151 *powers of* 2.

Throughout this paper, ε always denotes a sufficiently small positive number, though it may be different at each appearance.

2. Outline and preliminary results. Suppose N, our main parameter, is "sufficiently large". We write

$$P = N^{1/5-\varepsilon}, \qquad Q = NP^{-1}L^{14}, \qquad L = \log_2 N. \tag{2.1}$$

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The circle method, in the form we require here, begins with the observation that

$$R(N) := \sum_{\substack{N=p_1^2+\cdots+p_4^2+2^{\nu_1}+\cdots+2^{\nu_k} \\ p_1,\ldots,p_4 \le N^{1/2}}} (\log p_1)\cdots(\log p_4) = \int_0^1 f^4(\alpha) g^k(\alpha) e(-\alpha N)\, d\alpha, \tag{2.2}$$

where we write $e(x) = \exp(2\pi i x)$ and

$$f(\alpha) = \sum_{p^2 \le N} (\log p) e(\alpha p^2), \qquad g(\alpha) = \sum_{2^\nu \le N} e(\alpha 2^\nu) := \sum_{\nu \le L} e(\alpha 2^\nu). \tag{2.3}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ can be written as

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le q^{-1} Q^{-1}, \tag{2.4}$$

for some integers a, q with $1 \le a \le q \le Q$, $(a,q) = 1$. We denote by $I(a,q)$ the set of α satisfying (2.4), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{1 \le q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I(a,q), \qquad \mathfrak{m} = [1/Q, 1+1/Q] \setminus \mathfrak{M}. \tag{2.5}$$

It follows from $2P < Q$ that the major arcs $I(a,q)$ are mutually disjoint.

By (2.2) we have

$$\begin{aligned} R(N) &= \int_{\mathfrak{M}} f^4(\alpha) g^k(\alpha) e(-\alpha N)\, d\alpha + \int_{\mathfrak{m}} f^4(\alpha) g^k(\alpha) e(-\alpha N)\, d\alpha \\ &=: R_{\mathfrak{M}}(N) + R_{\mathfrak{m}}(N). \end{aligned} \tag{2.6}$$

We will prove that $R(N) > 0$ for sufficiently large N; this proves the Theorem.

For the integral on the major arcs, we need the following lemma.

LEMMA 1 ([7, Lemma 2.1]). *Let $\mathfrak{M}$ be as in* (2.5) *with P, Q determined by* (2.1). *Then for $2 \le n \le N$, we have*

$$\int_{\mathfrak{M}} f^4(\alpha) e(-\alpha n)\, d\alpha = \frac{\pi^2}{16} \mathfrak{S}(n) n + O\left(\frac{N}{\log N}\right), \tag{2.7}$$

where $\mathfrak{S}(n)$ is defined in (4.4), *and satisfies $\mathfrak{S}(n) \gg 1$ for $n \equiv 4 \pmod{24}$.*

On the minor arcs, we need estimates for the measure of the set

$$\mathcal{E}_\lambda := \{\alpha \in (0,1] : |g(\alpha)| \ge \lambda L\}. \tag{2.8}$$

The following lemma is due to Heath-Brown and Puchta [1] and calculated by Liu and Lü [7].

LEMMA 2. *We have* $\mathrm{meas}(\mathcal{E}_\lambda) \ll N^{-E(\lambda)}$, *with* $E(0.887167) > 3/4 + 10^{-10}$.

3. Estimation of an integral. In this section we shall estimate the integral $\int_0^1 |f(\alpha)g(\alpha)|^4\,d\alpha$. We have

LEMMA 3. *Let* $f(\alpha)$ *and* $g(\alpha)$ *be as in* (2.3). *Then*

$$\int_0^1 |f(\alpha)g(\alpha)|^4\,d\alpha \le c_1\,\frac{\pi^2}{16}\,NL^4,$$

where

$$c_1 \le \left(\frac{32^4\cdot 101\cdot 1.620767}{3} + \frac{8\cdot\log^2 2}{\pi^2}\right)(1+\varepsilon)^9.$$

To show this we need

LEMMA 4. *For odd* q, *let* $\varepsilon(q)$ *be the order of* 2 *in the multiplicative group of integers modulo* q. *Then the series* $\sum_{q=1,\,2\nmid q}^{\infty} \mu^2(q)/q\varepsilon(q)$ *is convergent, and its value* c_2 *satisfies* $c_2 < 1.620767$.

In Lemma 4.2 of [7], one has $c_2 < 43/25$.

Proof of Lemma 3. By Proposition 3 in [2], we know that the conclusion of Lemma 3.1 of [7] holds for $D = N^{1/16-2\varepsilon}$. By the argument in Section 3 of [7], in the proof of Lemma 2.2 of [7], we can fix $z = N^{1/32-\varepsilon}$, and then we can get $c_1 \le (1+\varepsilon)^6\cdot 101\cdot 32^4$ in Lemma 2.2 of [7]. Following the argument of the proof of Lemma 4.1 of [7], by Lemma 4 we get the assertion of Lemma 3.

Proof of Lemma 4. For the estimation of c_2, we follow [1]. We set

$$m = \prod_{e\le x}(2^e - 1), \tag{3.1}$$

$$s(x) = \sum_{\varepsilon(d)\le x} k(d), \qquad h(n) = \sum_{d\mid n} k(d), \tag{3.2}$$

where $k(d)$ is the multiplicative function defined by taking

$$k(p^\alpha) = \begin{cases} 0, & p = 2 \text{ or } \alpha \ge 2,\\ 1/p, & \text{otherwise.}\end{cases} \tag{3.3}$$

Hence

$$\begin{aligned} s(x) &\le \sum_{d\mid m} k(d) = h(m)\\ &= \prod_{\substack{p\mid m\\ p>2}}\left(1+\frac{1}{p}\right) \le \prod_{\substack{p\mid m\\ p>2}}\left(1-\frac{1}{p^2}\right)\prod_{p\mid m}\frac{p}{p-1} = \prod_{\substack{p\mid m\\ p>2}}\left(1-\frac{1}{p^2}\right)\frac{m}{\phi(m)}.\end{aligned}$$

Moreover, we have

$$\frac{m}{\phi(m)} \le e^{\gamma} \log x \qquad \text{for } x \ge 9,$$

as shown in (3.9) of [5]. When $x \ge 9$ we have

$$\prod_{\substack{p|m \\ p>2}} \left(1 - \frac{1}{p^2}\right) \le 0.831951343,$$

hence for $x \ge 9$, we have

$$s(x) \le 1.4817719 \log x.$$

It then follows that

$$\begin{aligned} c_2 = \int_1^{\infty} s(x) \frac{dx}{x^2} &= \int_1^{9} s(x) \frac{dx}{x^2} + \int_9^{\infty} s(x) \frac{dx}{x^2} \\ &\le \sum_{\varepsilon(d) \le 9} \int_{\varepsilon(d)}^{9} k(d) \frac{dx}{x^2} + 1.4817719 \int_9^{\infty} \log x \frac{dx}{x^2} \\ &\le \sum_{\varepsilon(d) < 9} k(d) \left(\frac{1}{\varepsilon(d)} - \frac{1}{9}\right) + 1.4817719 \frac{1 + \log 9}{9}. \end{aligned}$$

Let

$$\sum_{\varepsilon(d) = e} k(d) = \kappa(e).$$

Then

$$\sum_{e|d} \kappa(e) = \sum_{\varepsilon(e)|d} k(e).$$

However, $\varepsilon(e) \mid d$ if and only if $e \mid 2^d - 1$, hence

$$\sum_{e|d} \kappa(e) = \sum_{e|2^d-1} k(e) = h(2^d - 1).$$

Therefore

$$\kappa(e) = \sum_{d|e} \mu(e/d) h(2^d - 1),$$

and then

$$\sum_{\varepsilon(d) < 9} k(d) \left(\frac{1}{\varepsilon(d)} - \frac{1}{9}\right) = \sum_{m < 9} \kappa(m) \left(\frac{1}{m} - \frac{1}{9}\right).$$

By using the information on the prime factorization of $2^d - 1$ for $d < 9$, we find that

$$\sum_{m<9} \kappa(m)\left(\frac{1}{m} - \frac{1}{9}\right) = 1.094371632\ldots,$$

and hence we have

$$c_2 \le \sum_{m<9} \kappa(m)\left(\frac{1}{m} - \frac{1}{9}\right) + 1.4817719\,\frac{1+\log 9}{9} \le 1.6207669\ldots. \tag{3.4}$$

This completes the proof of the lemma.

4. Proof of Theorem. For the proof, we need the following lemmas.

LEMMA 5. *Let* $\mathcal{A}(N,k) = \{n \ge 2 : n = N - 2^{\nu_1} - \cdots - 2^{\nu_k}\}$ *with* $k \ge 100$. *Then for* $N \equiv 4 \pmod 8$,

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 4 \,(\mathrm{mod}\, 24)}} n \ge (1/3 - 2^{-90})NL^k.$$

In Lemma 6.1 of [4], one has $1/3$ replaced by $1/4$.

Proof. Following the argument of Lemma 6.1 in [4], we have

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 4 \,(\mathrm{mod}\, 24)}} n \ge \sum_{((\nu))} (N - 2^{\nu_1} - \cdots - 2^{\nu_k}) \ge (N - N/L)\sum_{((\nu))} 1, \tag{4.1}$$

where $((\nu))$ means $\nu_1, \ldots, \nu_k$ satisfy

$$3 \le \nu_1, \ldots, \nu_k \le \log_2(N/kL), \qquad 2^{\nu_1} + \cdots + 2^{\nu_k} \equiv N - 4 \pmod 3. \tag{4.2}$$

Let

$$H(d,N,K) = \sharp\Big\{(\nu_1, \ldots, \nu_K) : 1 \le \nu_i \le \varepsilon(d),\ d \mid N - \sum 2^{\nu_i}\Big\}.$$

When $d = 3$, $\varepsilon(3) = 2$, and it is an easy exercise to check that

$$H(3,N,K) = \begin{cases} \frac{1}{3}(2^K - (-1)^K), & 3 \nmid N, \\ \frac{1}{3}(2^K + (-1)^K), & 3 \mid N. \end{cases}$$

Thus if $K > 100$ we have

$$H(3,N,K)\varepsilon(3)^{-K} \ge \frac{1}{3}\,(1 - 2^{-98}).$$

And

$$\sum_{((\nu))} 1 \ge \frac{1}{3}\,(1 - 2^{-98})H(3,N,k)([\log_2(N/kL)/\varepsilon(3)] - 2)^k \ge \frac{1}{3}\,(1 - 2^{-96})L^k.$$

From this and (4.1) we get Lemma 5.

LEMMA 6. *Let*

$$C(q,a)=\sum_{\substack{m=1\\(m,q)=1}}^{q} e\left(\frac{am^2}{q}\right),\qquad B(n,q)=\sum_{\substack{a=1\\(a,q)=1}}^{q} C^4(q,a)e\left(-\frac{an}{q}\right),\tag{4.3}$$

$$A(n,q)=\frac{B(n,q)}{\varphi^4(q)},\qquad \mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q).\tag{4.4}$$

Then for $n\equiv 4 \pmod{24}$, *one has*

$$\mathfrak{S}(n)>c_3$$

with $c_3=4.99457$, *while for* $n\not\equiv 4 \pmod{24}$, *one has* $\mathfrak{S}(n)=0$.

In Lemma 5.2 of [7], one has $\mathfrak{S}(n)>4.952$.

Proof. This is Proposition 4.3 in [6] except for the value of c_3. It has been shown in [6] that

$$\mathfrak{S}(n)=(1+A(n,2)+A(n,4)+A(n,8))\prod_{p\geq 3}(1+A(n,p)),\tag{4.5}$$

where $A(n,p)$ is defined in (4.4). By the proof of Lemma 4.2 in [6], for $n\equiv 4 \pmod{24}$ we have

$$1+A(n,2)+A(n,4)+A(n,8)=8,\qquad 1+A(n,3)=3.\tag{4.6}$$

By the proof of Lemma 5.2 in [7] we have

$$B(n,p)\geq\begin{cases}-5p^2+2p-1 & \text{if } p\nmid n, p\equiv 3 \pmod 4,\\ -5p^2-10p-1 & \text{if } p\nmid n, p\equiv 1 \pmod 4,\\ (p-1)(p^2-6p+1) & \text{if } p\mid n.\end{cases}\tag{4.7}$$

Hence

$$\begin{aligned}\prod_{p\geq 5}(1+A(n,p))\geq &\prod_{\substack{p\equiv 1 \pmod 4\\ p\geq 5,\, p\nmid n}}\left(1-\frac{5p^2+10p+1}{(p-1)^4}\right)\\ &\times\prod_{\substack{p\equiv 3 \pmod 4\\ p\geq 5,\, p\nmid n}}\left(1-\frac{5p^2-2p+1}{(p-1)^4}\right)\prod_{\substack{p\geq 5\\ p\mid n}}\left(1+\frac{p^2-6p+1}{(p-1)^3}\right)\\ >&\prod_{\substack{p\equiv 1 \pmod 4\\ p\geq 5}}\left(1-\frac{5p^2+10p+1}{(p-1)^4}\right)\prod_{\substack{p\equiv 3 \pmod 4\\ p\geq 5}}\left(1-\frac{5p^2-2p+1}{(p-1)^4}\right).\end{aligned}$$

We apply the elementary inequality

$$(1+x)^a<1+ax-\frac{a(a-1)}{2}\,x^2\qquad \text{if } a>2,\ -1<x<0.$$

For $p > 82$ and $p \equiv 1 \pmod 4$, we have

$$1 - \frac{5p^2 + 10p + 1}{(p-1)^4} \ge \left(1 - \frac{1}{(p-1)^2}\right)^{5.25},$$

and for $p > 35$ and $p \equiv 3 \pmod 4$, we have

$$1 - \frac{5p^2 - 2p + 1}{(p-1)^4} \ge \left(1 - \frac{1}{(p-1)^2}\right)^{5.25}.$$

Thus

$$\begin{aligned}
&\prod_{p\ge 5}(1 + A(n,p)) \\
&\quad \ge \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}} \left(1 - \frac{5p^2 + 10p + 1}{(p-1)^4}\right) \prod_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ 5\le p<35}} \left(1 - \frac{5p^2 - 2p + 1}{(p-1)^4}\right) \\
&\qquad \times \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ p>82}} \left(1 - \frac{1}{(p-1)^2}\right)^{5.25} \prod_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ p>35}} \left(1 - \frac{1}{(p-1)^2}\right)^{5.25} \\
&\quad = \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}} \left(1 - \frac{5p^2 + 10p + 1}{(p-1)^4}\right) \prod_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ 5\le p<35}} \left(1 - \frac{5p^2 - 2p + 1}{(p-1)^4}\right) \\
&\qquad \times \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}} \left(1 - \frac{1}{(p-1)^2}\right)^{-5.25} \prod_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ 3\le p<35}} \left(1 - \frac{1}{(p-1)^2}\right)^{-5.25} \\
&\qquad \times \prod_{p\ge 3} \left(1 - \frac{1}{(p-1)^2}\right)^{5.25} \\
&\quad \ge 1.8422 \cdot (0.6601)^{5.25} > 0.208107568,
\end{aligned}$$

where we have used the well known result $\prod_{p\ge 3}(1 - 1/(p-1)^2) = 0.6601\ldots$. By (4.5) and (4.6) the lemma follows.

Now we prove the Theorem. Following the argument of [7], suppose first $N \equiv 4 \pmod 8$, let $\mathcal{E}_\lambda$ be defined in (2.8), and $\mathfrak{M}$ and $\mathfrak{m}$ as in (2.5) with P, Q determined in (2.1). Then (2.2) becomes

$$R(N) = \int_0^1 f^4(\alpha) g^k(\alpha) e(-\alpha N)\, d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}\cap\mathcal{E}_\lambda} + \int_{\mathfrak{m}\setminus\mathcal{E}_\lambda}. \tag{4.8}$$

For the major arcs, by Lemma 1 we have

$$
\begin{aligned}
(4.9)\qquad \int_{\mathfrak{M}} f^4(\alpha)g^k(\alpha)e(-\alpha N)\,d\alpha &= \sum_{n\in\mathcal{A}(N,k)} \int_{\mathfrak{M}} f^4(\alpha)e(-\alpha n)\,d\alpha \\
&= \frac{\pi^2}{16} \sum_{n\in\mathcal{A}(N,k)} \mathfrak{S}(n)n + O(NL^{k-1}) \\
&\geq c_3\,\frac{\pi^2}{16}\Big\{ \sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)}} n\Big\} + O(NL^{k-1}) \\
&\geq \frac{4.99457}{3}\,(1-2^{-90})\,\frac{\pi^2}{16}\,NL^k,
\end{aligned}
$$

where we have used Lemmas 5 and 6.

For the second integral in (4.8), by the estimation in [7], we have

$$\max_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll N^{1/2-1/16+\varepsilon}.$$

Therefore

$$(4.10)\qquad \int_{\mathfrak{m}\cap\mathcal{E}_\lambda} \ll N^{-E(0.887167)}N^{2-1/4+\varepsilon}L^k \ll N^{1-\varepsilon},$$

where we have used Lemma 2 for $\lambda = 0.887167$.

For the last integral in (4.8), by the definition of $\mathcal{E}_\lambda$ and Lemma 3, we have

$$(4.11)\qquad \int_{\mathfrak{m}\backslash\mathcal{E}_\lambda} \leq (\lambda L)^{k-4}\int_0^1 |f(\alpha)g(\alpha)|^4\,d\alpha \leq c_1\lambda^{k-4}\,\frac{\pi^2}{16}\,NL^k.$$

Combining this and (4.9) and (4.10), we get

$$(4.12)\qquad R(N) \geq \frac{\pi^2}{16}\,NL^k\bigg(\frac{4.99456}{3} - c_1\lambda^{k-4}\bigg).$$

When $k \geq 149$, for $\lambda = 0.887167$, by the above estimate we have

$$R(N) > 0.$$

This means that every large even integer N with $N \equiv 4 \pmod{24}$ can be written in the form of (1.2) for $k \geq 149$.

If N is a large even integer but $N \not\equiv 4 \pmod{24}$, then by the argument of [7], N can be written in the form (1.2) for $k \geq 151$. This completes the proof of the Theorem.

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