ACTA ARITHMETICA
169.2 (2015)

On a sum involving the Möbius function

by

I. Kiuchi (Yamaguchi), M. Minamide (Yamaguchi)
and Y. Tanigawa (Nagoya)

1. Introduction. Let $c_q(n)$ be the Ramanujan sum [1, p. 160] defined by

$$c_q(n) = \sum_{\substack{h=1\\(h,q)=1}}^{q} e^{2\pi ihn/q} = \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right),$$

where $\mu(n)$ is the Möbius function. We recall a well-known identity [9, p. 10]

$$\sum_{q=1}^{\infty} \frac{c_q(n)}{q^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \quad (\operatorname{Re} s > 1)$$

with $\sigma_{1-s}(n) = \sum_{d|n} d^{1-s}$ and the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Recently, T. H. Chan and A. V. Kumchev [2] studied a new type of sums,

$$C_k(x,y) = \sum_{n\le y}\Big(\sum_{q\le x} c_q(n)\Big)^k \quad (k=1,2) \tag{1.1}$$

for any sufficiently large positive numbers x and y. They showed

$$C_1(x,y) = y - \frac{x^2}{4\zeta(2)} + O(xy^{1/3}\log x + x^3/y) \tag{1.2}$$

for $y \ge x$,

$$C_2(x,y) = \frac{yx^2}{2\zeta(2)} + O(x^4 + xy\log x) \tag{1.3}$$

for $y \ge x^2(\log x)^B$ $(B>0)$, and

$$C_2(x,y) = \frac{yx^2}{2\zeta(2)}(1+2\kappa(u)) + O\left(yx^2(\log x)^{10}\left(\frac{1}{\sqrt{x}} + \left(\frac{x}{y}\right)^{1/2}\right)\right) \tag{1.4}$$

2010 *Mathematics Subject Classification*: 11N37, 11L03.
Key words and phrases: Ramanujan's sum, Möbius function, asymptotic formulas.

DOI: 10.4064/aa169-2-3

for $x \le y \le x^2(\log x)^B$ $(B > 0)$ and $u = \log(yx^{-2})$. Here $\kappa(u)$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\zeta(1-it)}{\zeta(1+it)}\,\frac{1}{(1+it)^2(1-it)}e^{-itu}\,dt.$$

Their work stems from their unpublished paper concerned with Diophantine approximation of reals by sums of rational numbers. In the present paper, as a problem on arithmetical functions, we shall consider a certain sum which is a modification of (1.1).

Let $\widehat{c}_q(n)$ be the arithmetical function defined by

$$\widehat{c}_q(n) = \sum_{d|(n,q)} d\left|\mu\left(\frac{q}{d}\right)\right|. \tag{1.5}$$

This can be regarded as a modification of the Ramanujan sum and also as a restricted divisor function (a sum over modified square-free divisors). Note that the Dirichlet series with the coefficients $\widehat{c}_q(n)$ is given by

$$\sum_{q=1}^{\infty} \frac{\widehat{c}_q(n)}{q^s} = \sigma_{1-s}(n)\frac{\zeta(s)}{\zeta(2s)} \tag{1.6}$$

for $\operatorname{Re} s > 1$. Following [2], we let

$$D_k(x,y) = \sum_{n\le y}\Big(\sum_{q\le x}\widehat{c}_q(n)\Big)^k \qquad (k = 1, 2). \tag{1.7}$$

The purpose of this paper is to obtain formulas for $D_k(x,y)$ analogous to (1.2)–(1.4).

In the case $k = 1$, we have the following theorem:

Theorem 1.1. *Let x and y be large real numbers such that $y \ge x$, and let $\varepsilon(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$. Then*

$$D_1(x,y) = \frac{1}{\zeta(2)}xy\log x + \frac{1}{\zeta(2)}\left(2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right)xy - \frac{\zeta(2)}{4\zeta(4)}x^2 + O(x^{1/2}y\exp(-C\varepsilon(x)) + xy^{1/3}\log x + x^3/y),$$

where γ is the Euler constant and C is a certain positive constant.

In the case $k = 2$, we have two types of formulas. To state the first formula, define a polynomial $P(u)$ by

$$P(u) = \frac{1}{3\zeta^3(2)}u^3 + C_1u^2 + C_2u, \tag{1.8}$$

where

$$C_1 = \frac{1}{\zeta^3(2)}\left(3\gamma - 1 - \frac{3\zeta'(2)}{\zeta(2)}\right), \tag{1.9}$$

$$C_2 = \frac{1}{\zeta^3(2)}\left\{2\gamma_1 + 8\gamma^2 - 6\gamma\left(1 + \frac{3\zeta'(2)}{\zeta(2)}\right) + 1 + \frac{6\zeta'(2)}{\zeta(2)} + \frac{10(\zeta'(2))^2}{\zeta^2(2)} - \frac{\zeta''(2)}{\zeta(2)}\right\}, \tag{1.10}$$

where γ_1 is the coefficient of $s-1$ in the Laurent expansion of $\zeta(s)$ at $s=1$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots .$$

In fact, these values are determined by

$$C_1 = \frac{A_1 + A_2}{\zeta^2(2)}, \quad C_2 = \frac{A_1^2 + 2A_1A_2}{\zeta(2)} - \frac{2A_3}{\zeta^2(2)}, \tag{1.11}$$

where A_1, A_2 and A_3 are constants defined by (2.1), (2.7) and (2.8) below, respectively.

THEOREM 1.2. *Let the notation be as above. Then for large real numbers x and y, we have*

$$D_2(x,y) = x^2 y P(\log x) + O(x^2 y + x^4). \tag{1.12}$$

This (1.12) gives an asymptotic formula for $D_2(x,y)$ when $y \gg x^2/\log^3 x$.

For the second formula, we introduce another polynomial $Q(u)$ by

$$Q(u) = -\frac{1}{6\zeta^3(2)}u^3 + C_3u^2 + C_4u + C_5, \tag{1.13}$$

where

$$C_3 = \frac{1}{2\zeta^3(2)}\left(-2\gamma + 1 + \frac{4\zeta'(2)}{\zeta(2)}\right) \tag{1.14}$$

$$C_4 = -\frac{2}{\zeta^3(2)}\left\{2\gamma_1 - \gamma\left(1 + \frac{4\zeta'(2)}{\zeta(2)}\right) + 1 + \frac{2\zeta'(2)}{\zeta(2)} + \frac{6(\zeta'(2))^2}{\zeta^2(2)} - \frac{2\zeta''(2)}{\zeta(2)}\right\} \tag{1.15}$$

and C_5 is a certain constant.

Under this notation we have

THEOREM 1.3. *Let x and y be large real numbers such that $y \ll x^M$ for some constant M. Then*

$$D_2(x,y) = x^2yP(\log x) + x^2yQ\bigg(\log\frac{x^2}{y}\bigg) + O\bigg(x^2y\bigg((x^{-3/8}+y^{-1/2})\log^{10}x + \bigg(\frac{x}{y}\bigg)^{1/2}\log^4 x + \bigg(\frac{y}{x^2}\bigg)^{1/2}\log^2 x\bigg)\bigg), \tag{1.16}$$

where the implied constant depends on M.

This gives an asymptotic formula for $D_2(x,y)$ when $x\log^2 x \ll y \ll x^2\log^2 x$.

These theorems are proved in the same way as in [2]. The change in the definition of the Ramanujan sum $c_q(n)$ causes a little complication in the behaviour of $D_k(x,y)$. However this may be of arithmetical interest, especially in connection with modified square-free numbers.

Remarks. (i) In Theorems 1.2 and 1.3, the asymptotic behaviour is obtained only for $y \gg x\log^2 x$. It is an interesting problem to investigate the asymptotic behaviour e.g. for $y \ll x\log^2 x$.

(ii) In the proof of Theorem 1.3 (see Section 5), we will observe by direct calculation that the first three terms containing $x^2y\log^j x$ $(j=3,2,1)$ are the same as those of Theorem 1.2. If we ignore the error term $O(x^4)$ of Theorem 1.2, this is easily derived by considering the asymptotic behaviour of these two theorems with the special choice $y = x^2/\log^4 x$. Unfortunately we cannot deduce it from the present error terms, but this observation may suggest that the error term $O(x^4)$ in Theorem 1.2 could be smaller.

The identity (1.6) leads to problems similar to those above. Let $\overline{c}_q(n;l)$ be the qth coefficient of the Dirichlet series

$$\sigma_{1-s}(n)\frac{\zeta(s)}{\zeta(ls)} = \sum_{q=1}^{\infty}\frac{\overline{c}_q(n;l)}{q^s} \quad (\operatorname{Re} s > 1).$$

The function $\overline{c}_q(n;l)$ can be regarded as a sum over modified l-free numbers. We shall write

$$U_k(x,y) = \sum_{n\le y}\Big(\sum_{q\le x}\overline{c}_q(n;l)\Big)^k.$$

Moreover, let $\widetilde{c_q}(n)$ be the qth coefficient of the series

$$\sigma_{1-s}(n)\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} = \sum_{q=1}^{\infty}\frac{\widetilde{c_q}(n)}{q^s} \quad (\operatorname{Re} s > 1/2),$$

which can be regarded as a sum over modified square-full numbers. Similarly we write

$$V_k(x,y) = \sum_{n\le y}\Big(\sum_{q\le x}\widetilde{c_q}(n)\Big)^k$$

for any positive integer k. The method of the proofs of Theorems 1.1–1.3 may be applied to studying $U_k(x,y)$ and $V_k(x,y)$ $(k=1,2)$, which will be done elsewhere.

2. Some lemmas. In order to prove our theorems, we prepare several lemmas.

LEMMA 2.1. *Let $\omega(m)$ be the number of distinct prime divisors of a positive integer m, and $\varepsilon(x)=(\log x)^{3/5}(\log\log x)^{-1/5}$ as in Theorem 1.1. For $x\ge 1$, we have*

$$\sum_{m\le x} 2^{\omega(m)} = \frac{1}{\zeta(2)} x\log x + A_1 x + O(x^{1/2}\exp(-C\varepsilon(x))),$$

where $C>0$ is a positive constant and

$$A_1 = \frac{1}{\zeta(2)}\left(2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right). \tag{2.1}$$

See A. Ivić [7, p. 394]. It is easy to see that A_1 is indeed given explicitly by (2.1), though this form is not given in [7].

In the proof of Theorem 1.1, we need an upper bound on the sum $\sum_{n\in I}\psi(y/n)$, where $\psi(x)=x-[x]-1/2$ denotes the first periodic Bernoulli function. This kind of sum is estimated effectively by exponent pairs. Roughly speaking, an *exponent pair* (κ,λ) is a pair of numbers $0\le\kappa\le 1/2\le\lambda\le 1$ such that

$$\sum_{n\in I} e^{2\pi i f(n)} \ll A^\kappa N^\lambda,$$

where $I\subset(N,2N]$ and $A\ll|f'(u)|\ll A$ for $u\in I$. For the precise definition and properties, the reader should consult S. W. Graham and G. Kolesnik [5] and [7]. Now applying [5, Lemma 4.3] with $f(n)=y/n$, we have

LEMMA 2.2. *Let (κ,λ) be an exponent pair. If I is a subinterval of $(N,2N]$, then*

$$\sum_{n\in I}\psi\left(\frac{y}{n}\right) \ll y^{\frac{\kappa}{\kappa+1}} N^{\frac{\lambda-\kappa}{\kappa+1}} + N^2 y^{-1}.$$

In particular, if we take $(\kappa,\lambda)=(1/2,1/2)$, we get

$$\sum_{n\in I}\psi\left(\frac{y}{n}\right) \ll y^{1/3} + N^2 y^{-1}. \tag{2.2}$$

LEMMA 2.3. *Let q be a non-negative integer. For $y\ge 1$, we have*

$$\sum_{n\le y}\frac{\log^q n}{n} = \frac{1}{q+1}\log^{q+1} y - \frac{\log^q y}{y}\psi(y) + C(q) + O\left(\frac{\log^q(y+1)}{y^2}\right), \tag{2.3}$$

where $C(q)$ is the constant given by

$$C(q) = \frac{\delta_q}{2} + \int_1^\infty \frac{q\log^{q-1} t - \log^q t}{t^2}\psi(t)\,dt$$

with $\delta_0 = 1$ *and* $\delta_q = 0$ *for* $q \geq 1$*, and in particular* $C(0) = \gamma$ *and* $C(1) = -\gamma_1$*.*

This lemma is derived immediately by applying the Euler–Maclaurin summation formula (see also [3]).

Lemma 2.4. *Let* $\phi(n)$ *be the Euler totient function and define*

$$F(x) = \sum_{n\leq x} \frac{\mu(n)}{n}\psi\left(\frac{x}{n}\right).$$

For $x \geq 2$*, we have*

$$\sum_{n\leq x} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)}\log x + A_2 - \frac{1}{x}F(x) + O\left(\frac{1}{x}\right), \tag{2.4}$$

$$\sum_{n\leq x} \frac{\phi(n)\log n}{n^2} = \frac{1}{2\zeta(2)}\log^2 x + A_3 - \frac{\log x}{x}F(x) + O\left(\frac{\log x}{x}\right), \tag{2.5}$$

$$\sum_{n\leq x} \frac{\phi(n)\log^2 n}{n^2} = \frac{1}{3\zeta(2)}\log^3 x + A_4 - \frac{\log^2 x}{x}F(x) + O\left(\frac{\log^2 x}{x}\right), \tag{2.6}$$

where the constants A_j $(j = 2, 3, 4)$ *are given by*

$$A_2 = \frac{\gamma}{\zeta(2)} - \sum_{n=1}^\infty \frac{\mu(n)\log n}{n^2} = \frac{\gamma}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}, \tag{2.7}$$

$$\begin{aligned} A_3 &= \frac{C(1)}{\zeta(2)} + \gamma\sum_{n=1}^\infty \frac{\mu(n)\log n}{n^2} - \frac{1}{2}\sum_{n=1}^\infty \frac{\mu(n)\log^2 n}{n^2} \\ &= -\frac{\gamma_1}{\zeta(2)} + \gamma\frac{\zeta'(2)}{\zeta^2(2)} + \frac{1}{2}\frac{\zeta''(2)}{\zeta^2(2)} - \frac{(\zeta'(2))^2}{\zeta^3(2)}, \\ A_4 &= \frac{C(2)}{\zeta(2)} + 2C(1)\sum_{n=1}^\infty \frac{\mu(n)\log n}{n^2} + \gamma\sum_{n=1}^\infty \frac{\mu(n)\log^2 n}{n^2} \\ &\quad - \frac{1}{3}\sum_{n=1}^\infty \frac{\mu(n)\log^3 n}{n^2}. \end{aligned} \tag{2.8}$$

Proof. We shall give a proof of (2.5) only, since (2.4) and (2.6) are similar. Using the well-known formula $\phi(n) = n\sum_{d\mid n}\mu(d)/d$ and changing the order of summation, we obtain

$$S := \sum_{n\leq x} \frac{\phi(n)\log n}{n^2} = \sum_{d\leq x} \frac{\mu(d)}{d^2} \sum_{n\leq x/d} \frac{\log dn}{n}.$$

For the sum over n we apply (2.3) with $q=0,1$ to get

$$S=\sum_{d\le x}\frac{\mu(d)}{d^2}\bigg\{\log d\bigg(\log\frac{x}{d}+\gamma-\frac{d}{x}\psi\Big(\frac{x}{d}\Big)+O\Big(\frac{d^2}{x^2}\Big)\bigg)$$
$$+\frac{1}{2}\log^2\frac{x}{d}+C(1)-\frac{d}{x}\psi\Big(\frac{x}{d}\Big)\log\frac{x}{d}+O\Big(\frac{d^2}{x^2}\log\Big(\frac{x}{d}+1\Big)\Big)\bigg\}$$
$$=\frac{1}{2}\Big(\sum_{d\le x}\frac{\mu(d)}{d^2}\Big)\log^2 x-\frac{1}{2}\sum_{d\le x}\frac{\mu(d)\log^2 d}{d^2}+\gamma\sum_{d\le x}\frac{\mu(d)\log d}{d^2}$$
$$+C(1)\sum_{d\le x}\frac{\mu(d)}{d^2}-\frac{\log x}{x}F(x)$$
$$+O\Big(\frac{1}{x^2}\sum_{d\le x}\log d\Big)+O\Big(\frac{1}{x^2}\sum_{d\le x}\log\Big(\frac{x}{d}+1\Big)\Big). \tag{2.9}$$

From the prime number theorem we observe that

$$\sum_{d\le x}\frac{\mu(d)\log^j d}{d^2}=\sum_{d=1}^{\infty}\frac{\mu(d)\log^j d}{d^2}+O\big(x^{-1}\exp(-c\sqrt{\log x})\big)$$

for $j=0,1,2$. Substituting this into (2.9) we get (2.5). ∎

LEMMA 2.5. *If $\sigma_0>\max(0,\sigma_a)$ and $x,T>0$, then*

$$\sideset{}{'}\sum_{n\le x}a_n=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}\,ds+R,$$

where

$$R\ll\sum_{\substack{x/2<n<2x\\ n\ne x}}|a_n|\min\Big(1,\frac{x}{T|x-n|}\Big)+\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma_0}},$$

and $\sum'$ indicates that the last term is to be halved if x is an integer.

This is the famous Perron formula (see H. L. Montgomery and R. C. Vaughan [8, Theorem 5.2 and Corollary 5.3]).

LEMMA 2.6 ([2, (4.12)]). *Let*

$$G(s_1,s_2;y)=\sum_{n\le y}\sigma_{1-s_1}(n)\sigma_{1-s_2}(n) \tag{2.10}$$

and $L=\log y$. Then

$$G(s_1,s_2,y)=\sum_{j=1}^{4}R_j(s_1,s_2,y)+O(L^6(y^{1/2}+y/T)), \tag{2.11}$$

where

$$R_1(s_1,s_2,y) = y\frac{\zeta(s_1)\zeta(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)},$$
$$R_2(s_1,s_2,y) = y^{2-s_1}\frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_2)}{(2-s_1)\zeta(2-s_1+s_2)},$$
$$R_3(s_1,s_2,y) = y^{2-s_2}\frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta(s_1)}{(2-s_2)\zeta(2+s_1-s_2)},$$
$$R_4(s_1,s_2,y) = y^{3-s_1-s_2}\frac{\zeta(3-s_1-s_2)\zeta(2-s_2)\zeta(2-s_1)}{(3-s_1-s_2)\zeta(4-s_1-s_2)}.$$

3. Proof of Theorem 1.1. From (1.5) and (1.7), we have

$$D_1(x,y) = \sum_{n\le y}\sum_{q\le x}\widehat{c}_q(n) = \sum_{n\le y}\sum_{q\le x}\sum_{\substack{d|q\\ d|n}} d\left|\mu\left(\frac{q}{d}\right)\right| = \sum_{n\le y}\sum_{\substack{dk\le x\\ d|n}} d|\mu(k)|.$$

Changing the order of summation, we find that

$$\begin{aligned}(3.1)\qquad D_1(x,y) &= \sum_{dk\le x} d|\mu(k)|\sum_{\substack{n\le y\\ d|n}} 1 = \sum_{dk\le x} d|\mu(k)|\left[\frac{y}{d}\right]\\ &= y\sum_{dk\le x}|\mu(k)| - \frac{1}{2}\sum_{dk\le x} d|\mu(k)| - \sum_{dk\le x} d|\mu(k)|\psi\left(\frac{y}{d}\right)\\ &=: D_{1,1}(x,y) - D_{1,2}(x,y) - D_{1,3}(x,y).\end{aligned}$$

For the first term, we apply Lemma 2.1 to get

$$\begin{aligned}(3.2)\qquad D_{1,1}(x,y) &= y\sum_{dk\le x}|\mu(k)| = y\sum_{m\le x}\sum_{k|m}|\mu(k)| = y\sum_{m\le x}2^{\omega(m)}\\ &= y\left(\frac{1}{\zeta(2)}x\log x + A_1x + O(x^{1/2}\exp(-C\varepsilon(x)))\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}(3.3)\qquad D_{1,2}(x,y) &= \frac{1}{2}\sum_{dk\le x} d|\mu(k)| = \frac{1}{2}\sum_{k\le x}|\mu(k)|\sum_{d\le x/k} d\\ &= \frac{1}{2}\sum_{k\le x}|\mu(k)|\left(\frac{x^2}{2k^2} + O\left(\frac{x}{k}\right)\right)\\ &= \frac{1}{4}x^2\sum_{k\le x}\frac{|\mu(k)|}{k^2} + O(x\log x) = \frac{\zeta(2)}{4\zeta(4)}x^2 + O(x\log x).\end{aligned}$$

To estimate $D_{1,3}(x,y)$ we use the theory of exponent pairs. Let $N_j = N_{j,k} = (x/k)2^{-j}$. Then

$$\begin{aligned} D_{1,3}(x,y) &= \sum_{k\le x} |\mu(k)| \sum_{d\le x/k} d\psi\Big(\frac{y}{d}\Big) \\ &\ll \sum_{k\le x} |\mu(k)| \sum_{j=0}^{\infty} N_j \sup_I \Big|\sum_{d\in I} \psi\Big(\frac{y}{d}\Big)\Big|, \end{aligned}$$

where the sup is over all subintervals I of $(N_j, 2N_j]$. From (2.2) we have

$$\begin{aligned} D_{1,3}(x,y) &\ll \sum_{k\le x} |\mu(k)| \sum_{j=0}^{\infty} \{N_j y^{1/3} + N_j^3 y^{-1}\} \qquad (3.4) \\ &\ll \sum_{k\le x} |\mu(k)| \Big\{\Big(\frac{x}{k}\Big) y^{1/3} + \Big(\frac{x}{k}\Big)^3 y^{-1}\Big\} \\ &\ll \sum_{k\le x} \frac{|\mu(k)|}{k} \cdot xy^{1/3} + \sum_{k\le x} \frac{|\mu(k)|}{k^3} \cdot x^3 y^{-1} \\ &\ll xy^{1/3}\log x + x^3 y^{-1}. \end{aligned}$$

Substituting (3.2)–(3.4) in (3.1), we get the assertion of Theorem 1.1. ■

4. Proof of Theorem 1.2. We follow the method of Chan and Kumchev [2]. From (1.5), we have

$$\begin{aligned} D_2(x,y) &= \sum_{n\le y} \Big(\sum_{q\le x} \widehat{c}_q(n)\Big)^2 = \sum_{n\le y} \Big(\sum_{\substack{dk\le x\\ d\mid n}} d|\mu(k)|\Big)^2 \\ &= \sum_{d_1k_1\le x} d_1|\mu(k_1)| \sum_{d_2k_2\le x} d_2|\mu(k_2)| \sum_{\substack{n\le y\\ d_1\mid n,\, d_2\mid n}} 1. \end{aligned}$$

The sum over n can be written as

$$\sum_{\substack{n\le y\\ d_1\mid n,\, d_2\mid n}} 1 = \sum_{[d_1,d_2]m\le y} 1 = \sum_{m\le y/[d_1,d_2]} 1 = \Big[\frac{y}{[d_1,d_2]}\Big],$$

where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2. Hence

$$\begin{aligned} D_2(x,y) &= \sum_{d_1k_1\le x} \sum_{d_2k_2\le x} d_1d_2|\mu(k_1)|\,|\mu(k_2)| \Big[\frac{y}{[d_1,d_2]}\Big] \qquad (4.1) \\ &= y \sum_{d_1k_1\le x} \sum_{d_2k_2\le x} (d_1,d_2)|\mu(k_1)|\,|\mu(k_2)| + O(E), \end{aligned}$$

where

$$E = \sum_{d_1k_1\le x}\sum_{d_2k_2\le x} d_1d_2|\mu(k_1)|\,|\mu(k_2)| \ll \sum_{d_1\le x} d_1\left[\frac{x}{d_1}\right]\sum_{d_2\le x} d_2\left[\frac{x}{d_2}\right] \ll x^2\cdot x^2 = x^4.$$

Now we shall evaluate the main term of (4.1):

$$\sum_{d_1k_1\le x}\sum_{d_2k_2\le x}(d_1,d_2)|\mu(k_1)|\,|\mu(k_2)|$$
$$= \sum_{d\le x} d \sum\sum_{\substack{dl_1k_1\le x,\, dl_2k_2\le x\\ (l_1,l_2)=1}} |\mu(k_1)|\,|\mu(k_2)|$$
$$= \sum_{d\le x} d \sum_{dl_1k_1\le x}\sum_{dl_2k_2\le x} |\mu(k_1)|\,|\mu(k_2)| \sum_{l|(l_1,l_2)}\mu(l)$$
$$= \sum_{dl\le x} d\mu(l)\Big(\sum_{mk\le x/(dl)}|\mu(k)|\Big)^2$$
$$= \sum_{dl\le x} d\mu(l)\Big(\sum_{n\le x/(dl)}\sum_{k|n}|\mu(k)|\Big)^2.$$

By Lemma 2.1, for large x,

$$(4.2)\qquad \Big(\sum_{n\le x/(dl)}\sum_{k|n}|\mu(k)|\Big)^2 = \Big(\sum_{n\le x/(dl)} 2^{\omega(n)}\Big)^2$$
$$= \frac{1}{\zeta^2(2)}\frac{x^2}{d^2l^2}\log^2\frac{x}{dl} + \frac{2A_1}{\zeta(2)}\frac{x^2}{d^2l^2}\log\frac{x}{dl} + A_1^2\frac{x^2}{d^2l^2} + O\Big(\Big(\frac{x}{dl}\Big)^{3/2}\Big)$$
$$= \frac{x^2\log^2 x}{\zeta^2(2)}\frac{1}{d^2l^2} - \frac{2x^2\log x}{\zeta^2(2)}\frac{\log dl}{d^2l^2} + \frac{x^2}{\zeta^2(2)}\frac{\log^2 dl}{d^2l^2} + \frac{2A_1x^2\log x}{\zeta(2)}\frac{1}{d^2l^2}$$
$$- \frac{2A_1x^2}{\zeta(2)}\frac{\log dl}{d^2l^2} + A_1^2x^2\frac{1}{d^2l^2} + O\Big(\Big(\frac{x}{dl}\Big)^{3/2}\Big)$$
$$= \Big\{\frac{x^2\log^2 x}{\zeta^2(2)} + \frac{2A_1x^2\log x}{\zeta(2)} + A_1^2x^2\Big\}\frac{1}{d^2l^2} - \Big\{\frac{2x^2\log x}{\zeta^2(2)} + \frac{2A_1x^2}{\zeta(2)}\Big\}\frac{\log dl}{d^2l^2}$$
$$+ \frac{x^2}{\zeta^2(2)}\frac{\log^2 dl}{d^2l^2} + O\Big(\Big(\frac{x}{dl}\Big)^{3/2}\Big).$$

Write $G(x,dl)$ for the first three terms of the right hand side of (4.2). Since

$$\sum_{dl\le x} d\mu(l)\cdot\frac{\log^j dl}{d^2l^2} = \sum_{n\le x}\frac{\phi(n)\log^j n}{n^2}$$

we can apply Lemma 2.4 to get

$$\begin{aligned}(4.3)\qquad &\sum_{dl\le x} d\mu(l)G(x,dl)\\ &= \frac{x^2\log^3 x}{3\zeta^3(2)} + \frac{A_1+A_2}{\zeta^2(2)}x^2\log^2 x + \left(\frac{A_1^2+2A_1A_2}{\zeta(2)} - \frac{2A_3}{\zeta^2(2)}\right)x^2\log x\\ &\quad + \left(A_1^2A_2 - \frac{2A_1A_3}{\zeta(2)} + \frac{A_4}{\zeta^2(2)}\right)x^2 - A_1^2xF(x) + O(x\log^2 x).\end{aligned}$$

Since $F(x) \ll \log x$ trivially, $xF(x)$ is included in the last error term.

On the other hand, the contribution from the error term of (4.2) is bounded above by

$$\sum_{dl\le x} d\left(\frac{x}{dl}\right)^{3/2} \ll x^{3/2}\sum_{n\le x}\frac{\sigma(n)}{n^{3/2}} \ll x^2.$$

Hence the terms lower than x^2 in (4.3) are absorbed in the error. Thus using (1.8)–(1.11), we finally obtain

$$D_2(x,y) = x^2yP(\log x) + O(x^2y + x^4).$$

This completes the proof of Theorem 1.2. ∎

5. Proof of Theorem 1.3. In this section we assume $1 \le y \le x^M$ for some constant M. Without loss of generality we can assume $x, y \in \mathbb{Z} + 1/2$. We apply Lemma 2.5 with

$$\alpha(s) = \sum_{q=1}^{\infty}\frac{\widehat{c}_q(n)}{q^s} = \sigma_{1-s}(n)\frac{\zeta(s)}{\zeta(2s)}.$$

Then we have, for $x^\varepsilon \ll T \ll x$,

$$(5.1)\qquad \sum_{q\le x}\widehat{c}_q(n) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\sigma_{1-s}(n)\frac{\zeta(s)}{\zeta(2s)}\frac{x^s}{s}\,ds + E_1(x,n)$$

with $\alpha \ge 1 + 1/\log x$, where $E_1(x,n)$ is the error term given by

$$E_1(x,n) \ll \sum_{x/2<q<2x}|\widehat{c}_q(n)|\min\left(1, \frac{x}{T|x-q|}\right) + \frac{x^\alpha}{T}\sum_{q=1}^{\infty}\frac{|\widehat{c}_q(n)|}{q^\alpha}.$$

It is easy to see that

$$E_1(x,n) \ll \frac{x}{T}\sigma_0(n)\log x.$$

Let $\alpha_j = 1 + j/\log x$ $(j = 1, 2)$. Applying (5.1) with $\alpha = \alpha_j$ we get

$$(5.2)\qquad \Big(\sum_{q\le x} \widehat{c}_q(n)\Big)^2 = \frac{1}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} F(s_1, s_2, n)\, ds_2\, ds_1 + E_2(x, n),$$

where

$$F(s_1, s_2, n) = \sigma_{1-s_1}(n)\sigma_{1-s_2}(n) \frac{\zeta(s_1)\zeta(s_2)}{\zeta(2s_1)\zeta(2s_2)} \frac{x^{s_1+s_2}}{s_1 s_2}$$

and

$$E_2(x, n) = E_1(x, n)\bigg(\frac{1}{2\pi i} \int_{\alpha_1-iT}^{\alpha_1+iT} \sigma_{1-s_1}(n) \frac{\zeta(s_1)}{\zeta(2s_1)} \frac{x^{s_1}}{s_1}\, ds_1 + \frac{1}{2\pi i} \int_{\alpha_2-iT}^{\alpha_2+iT} \sigma_{1-s_2}(n) \frac{\zeta(s_2)}{\zeta(2s_2)} \frac{x^{s_2}}{s_2}\, ds_2 + E_1(x, n) \bigg).$$

We can see easily that

$$E_2(x, n) \ll \frac{x^2}{T} \sigma_0(n)^2 \log^3 x.$$

Summing (5.2) over n and using the estimate

$$\sum_{n\le y} \sigma_0(n)^2 \ll y \log^3 y,$$

we get

$$(5.3)\quad D_2(x, y) = \frac{1}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} G(s_1, s_2, y) \frac{\zeta(s_1)\zeta(s_2)}{\zeta(2s_1)\zeta(2s_2)} \frac{x^{s_1+s_2}}{s_1 s_2}\, ds_2\, ds_1 + O(x^2 y L^6/T).$$

where $G(s_1, s_2; y)$ is defined by (2.10) and $L = \log x$. Here we note that $\log y \le M \log x$ by the assumption.

Now we shall evaluate the integral of (5.3). Substituting (2.11) in (5.3), we obtain

$$(5.4)\qquad D_2(x, y) = \sum_{j=1}^{4} D_{2,j}(x, y) + O(yx^2 L^{10}(y^{-1/2} + 1/T)),$$

where

$$D_{2,j}(x, y) = \frac{1}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} R_j(s_1, s_2, y) \frac{\zeta(s_1)\zeta(s_2)}{\zeta(2s_1)\zeta(2s_2)} \frac{x^{s_1+s_2}}{s_1 s_2}\, ds_2\, ds_1$$

with $\alpha_1 = 1 + 1/\log x$ and $\alpha_2 = 1 + 2/\log x$.

First we deal with $D_{2,1}(x,y)$. From the definition of $R_1(s_1,s_2,y)$, we get

(5.5)
$$D_{2,1}(x,y)=\frac{y}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta^2(s_1)\zeta^2(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)\zeta(2s_1)\zeta(2s_2)}\,\frac{x^{s_1+s_2}}{s_1s_2}\,ds_2\,ds_1.$$

As in [2], let $\Gamma(\alpha,\beta,T)$ denote the contour consisting of the line segments $[\alpha-iT,\beta-iT]$, $[\beta-iT,\beta+iT]$ and $[\beta+iT,\alpha+iT]$. In (5.5), we move the line of integration with respect to s_2 to $\Gamma(\alpha_2,1/2,T)$. We denote the integrals over the horizontal line segments by $J_{1,1}$ and $J_{1,3}$, and the integral over the vertical line segment by $J_{1,2}$. Then

$$\begin{aligned}
&J_{1,1},J_{1,3}\\
&\ll\frac{xy}{T}\int_{-T}^{T}\frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|}\,dt_1\int_{1/2}^{\alpha_2}\frac{|\zeta^2(\sigma_2+iT)\zeta(\alpha_1+\sigma_2-1+i(t_1+T))|x^{\sigma_2}}{|\zeta(2\sigma_2+2iT)|}\,d\sigma_2\\
&\ll\frac{xyL^4}{T}\int_{-T}^{T}\frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|}\,dt_1\int_{1/2}^{\alpha_2}T^{\frac{2}{3}(1-\sigma_2)}T^{\frac{1}{3}(1-\sigma_2-1/\log x)}x^{\sigma_2}\,d\sigma_2\\
&\ll\frac{xyL^5}{T}(x+x^{1/2}T^{1/2})\ll yx^2\frac{L^5}{T},
\end{aligned}$$

where we have used the estimate $\int_1^T|\zeta(\alpha_1+it)|^2\,dt\ll T$.

For the integral along the vertical line we have

$$\begin{aligned}
J_{1,2}&\ll yx^{3/2}\int_{-T}^{T}\int_{-T}^{T}\frac{|\zeta^2(\alpha_1+it_1)\zeta^2(1/2+it_2)\zeta(\alpha_1-1/2+i(t_1+t_2))|}{|\zeta(1+2it_2)|(1+|t_1|)(1+|t_2|)}\,dt_1\,dt_2\\
&\ll yx^{3/2}L^3\int_{-T}^{T}\int_{-T}^{T}\frac{|\zeta^2(1/2+it_2)\zeta(\alpha_1-1/2+i(t_1+t_2))|}{(1+|t_1|)(1+|t_2|)}\,dt_1\,dt_2\\
&\ll yx^{3/2}L^3\int_{-2T}^{2T}\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|\int_{-T}^{T}\frac{|\zeta^2(1/2+it)|}{(1+|t|)(1+|t-u|)}\,dt\,du\\
&\ll yx^{3/2}L^3\int_{2}^{2T}\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|\int_{-T}^{T}\frac{|\zeta^2(1/2+it)|}{(1+|t|)(1+|t-u|)}\,dt\,du.
\end{aligned}$$

Here we note that

$$\int_{-T}^{T}\frac{|\zeta^2(1/2+it)|}{(1+|t|)(1+|t-u|)}\,dt=\int_{|t-u|>\frac{1}{2}|u|}+\int_{|t-u|\le\frac{1}{2}|u|}\ll\frac{L}{1+|u|}+\frac{|u|^\delta}{1+|u|},$$

where δ is a positive number such that

$$\int_0^X |\zeta^2(1/2+it)|\,dt = cX\log X + c'X + O(X^\delta).$$

Hence,

$$J_{1,2} \ll yx^{3/2}L^4 \int_{-2T}^{2T} \left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right| \frac{|u|^\delta}{1+|u|}\,du \ll yx^{3/2}T^\delta L^5.$$

For simplicity, we take $\delta = 1/3$ in what follows.

It remains to evaluate the residues of the poles of the integrand when we move the line of integration to $\Gamma(\alpha_2, 1/2, T)$. There is a simple pole at $s_2 = 2 - s_1$ with residue

$$\frac{\zeta^2(s_1)\zeta^2(2-s_1)x^2}{\zeta(2)\zeta(2s_1)\zeta(4-2s_1)s_1(2-s_1)} =: H_1(s_1)x^2,$$

and a double pole at $s_2 = 1$ with residue

$$\begin{aligned}\frac{\zeta^2(s_1)}{\zeta(2s_1)}\frac{x^{s_1+1}}{s_1}\bigg\{\frac{\zeta(s_1)}{\zeta(s_1+1)}\left(\frac{\log x}{\zeta(2)}+A_1\right)& \\ +\frac{1}{\zeta(2)}\left(\frac{\zeta'(s_1)}{\zeta(s_1+1)}-\frac{\zeta(s_1)\zeta'(s_1+1)}{\zeta^2(s_1+1)}\right)\bigg\}& \\ =: x^{s_1+1}\{H_2(s_1)\log x + H_3(s_1)\},&\end{aligned}$$

where A_1 is defined by (2.1). The contributions to $D_{2,1}(x,y)$ from these residues are

$$\begin{aligned}\frac{x^2y}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_1(s_1)\,ds_1 + \frac{xy\log x}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_2(s_1)x^{s_1}\,ds_1& \\ + \frac{xy}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_3(s_1)x^{s_1}\,ds_1 =: I_1+I_2+I_3,&\end{aligned}$$

say.

For I_1, moving the line of integration to $\Gamma(\alpha_1, 5/4, T)$, we get

$$\begin{aligned}I_1 &= \frac{x^2y}{2\pi i}\int_{5/4-i\infty}^{5/4+i\infty} H_1(s_1)\,ds_1 + O\left(x^2y\int_T^\infty \left|H_1\left(\frac{5}{4}+it_1\right)\right|dt_1\right) \\ &\quad + O(x^2yL^4T^{-11/6}) \\ &= cx^2y + O(x^2y/T),\end{aligned}$$

where we have set

$$c=\frac{1}{2\pi i}\int_{5/4-i\infty}^{5/4+i\infty}H_1(s_1)\,ds_1.$$

For I_2, we move the line of integration to $\Gamma(\alpha_1,1/2,T)$. The integrals over the horizontal lines are

$$\ll xyL^5\int_{1/2}^{\alpha_1}T^{1-\sigma_1}T^{-1}x^{\sigma_1}\,d\sigma_1\ll xyL^5\biggl(\frac{x}{T}+\Bigl(\frac{x}{T}\Bigr)^{1/2}\biggr),$$

and the integral over the vertical line is

$$\ll xyL^2\int_{-T}^{T}\frac{|\zeta(1/2+it_1)|^3}{1+|t_1|}x^{1/2}\,dt_1\ll x^{3/2}yL^6,$$

where we have used the well-known estimate $\int_0^T|\zeta(1/2+it)|^3\,dt\ll T\log^3T$. Furthermore, when moving the line of integration we encounter a triple pole at $s_1=1$. Hence by Cauchy's theorem we get

$$I_2=x^2y\log xP_1(\log x)+O\biggl(xyL^5\biggl(\frac{x}{T}+\Bigl(\frac{x}{T}\Bigr)^{1/2}\biggr)\biggr)+O(x^{3/2}yL^6),$$

where $P_1(u)$ is a polynomial in u of degree 2. By direct computation we find that

$$P_1(u)=a_1u^2+a_2u+a_3 \tag{5.6}$$

with

$$a_1=\frac{1}{2\zeta^3(2)},\quad a_2=\frac{1}{\zeta^3(2)}\biggl(3\gamma-1-\frac{3\zeta'(2)}{\zeta(2)}\biggr), \tag{5.7}$$

$$\begin{aligned}a_3=\frac{1}{\zeta^3(2)}\biggl\{&3(\gamma_1+\gamma^2)-3\gamma\biggl(1+\frac{3\zeta'(2)}{\zeta(2)}\biggr)\\&+1+\frac{3\zeta'(2)}{\zeta(2)}-\frac{5\zeta''(2)}{2\zeta(2)}+\frac{7(\zeta'(2))^2}{\zeta^2(2)}\biggr\}.\end{aligned} \tag{5.8}$$

In the same way as for I_2, we find that there exists a polynomial $P_2(t)$ in t of degree 3 such that

$$I_3=x^2yP_2(\log x)+O\biggl(xyL^6\biggl(\frac{x}{T}+\Bigl(\frac{x}{T}\Bigr)^{1/2}\biggr)\biggr)+O(x^{3/2}yL^6).$$

Here we have used the mean square estimate $\int_0^T|\zeta'(1/2+it)|^2\,dt\ll T\log^3T$ due to A. E. Ingham [6], and the bound $\zeta'(\sigma+it)\ll|t|^{\frac{1}{3}(1-\sigma)}\log^3|t|$ for $1/2\le\sigma\le1$ (see S. M. Gonek [4]). In this case we find that

$$P_2(u)=b_1u^3+b_2u^2+b_3u+b_4 \tag{5.9}$$

with

$$b_1 = -\frac{1}{6\zeta^3(2)}, \quad b_2 = 0, \tag{5.10}$$

$$\begin{aligned} b_3 = &-\frac{\gamma_1}{\zeta^3(2)} + \frac{5\gamma^2}{\zeta^3(2)} - \frac{3\gamma}{\zeta^3(2)}\left(1 + \frac{3\zeta'(2)}{\zeta(2)}\right) \\ &+ \frac{1}{\zeta^4(2)}\left(3\zeta'(2) + \frac{3(\zeta'(2))^2}{\zeta(2)} + \frac{3\zeta''(2)}{2}\right). \end{aligned} \tag{5.11}$$

From (5.6)–(5.11), (1.9) and (1.10) we find that

$$a_2 + b_2 = a_2 = C_1 \quad \text{and} \quad a_3 + b_3 = C_2,$$

hence

$$uP_1(u) + P_2(u) = P(u) + b_4,$$

where $P(u)$ is the polynomial defined by (1.8). The constant term b_4 can also be computed explicitly. Combining these results we get

$$\begin{aligned} D_{2,1}(x,y) = &\, x^2 y\big(P(\log x) + b_4 + c\big) \\ &+ O(x^2 y L^6/T) + O(x^{3/2} y T^{1/3} L^5). \end{aligned} \tag{5.12}$$

Next we consider the term $D_{2,4}(x,y)$. It is given explicitly by

$$\begin{aligned} D_{2,4}(x,y) = \frac{y^3}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} &\frac{\zeta(3 - s_1 - s_2)\zeta(2 - s_1)\zeta(2 - s_2)\zeta(s_1)\zeta(s_2)}{\zeta(4 - s_1 - s_2)\zeta(2s_1)\zeta(2s_2)(3 - s_1 - s_2)} \\ &\times \frac{(x/y)^{s_1 + s_2}}{s_1 s_2}\, ds_2\, ds_1. \end{aligned}$$

We move the line of integration with respect to s_2 to $\Gamma(\alpha_2, \beta, T)$, where $\beta = 5/2 - \alpha_1 = 3/2 - 1/\log x$. There are no poles when we deform the path of integration over s_2. The contributions from the horizontal lines are

$$\begin{aligned} J_{4,1}, J_{4,3} \ll &\, xy^2 \left(\frac{x}{y}\right)^{\frac{1}{\log x}} \int_{-T}^{T} \frac{\left|\zeta\left(1 - \frac{1}{\log x} - it_1\right)\zeta\left(1 + \frac{1}{\log x} + it_1\right)\right|}{1 + |t_1|}\, dt_1 \\ &\times \int_{\alpha_2}^{\beta} \frac{\left|\zeta\left(2 - \frac{1}{\log x} - \sigma_2 - i(t_1 + T)\right)\zeta(2 - \sigma_2 - iT)\zeta(\sigma_2 + iT)\right|}{(1 + |t_1 + T|)T} \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2. \end{aligned}$$

The inner integral is estimated as

$$\begin{aligned} &\ll \frac{1}{T(1 + |t_1 + T|)} \left(L^3 \left(\frac{x}{y}\right)^{1 + \frac{2}{\log x}} + T^{1/3} \left(\frac{x}{y}\right)^{\frac{3}{2} - \frac{1}{\log x}}\right) \\ &\ll \frac{L^3}{T(1 + |t_1 + T|)} \left(\frac{x}{y}\right) \left(1 + T^{1/2} \left(\frac{x}{y}\right)^{1/2}\right), \end{aligned}$$

where we have used the assumption $y \ll x^M$. Hence,

$$J_{4,1}, J_{4,3} \ll x^2 y \frac{L^3}{T}\left(1+T^{1/3}\left(\frac{x}{y}\right)^{1/2}\right)\int_{-T}^{T}\frac{\left|\zeta\left(1-\frac{1}{\log x}-it_1\right)\zeta\left(1+\frac{1}{\log x}+it_1\right)\right|}{(1+|t_1|)(1+|t_1|+T)}\,dt_1 \ll x^2 y \frac{L^4}{T^2}\left(1+T^{1/3}\left(\frac{x}{y}\right)^{1/2}\right).$$

For the integral on the vertical line we find that

$$\begin{aligned}J_{4,2} &\ll y^3 \int_{-T}^{T}\int_{-T}^{T}\frac{\left|\zeta\left(\frac12-i(t_1+t_2)\right)\zeta\left(1-\frac{1}{\log x}-it_1\right)\zeta\left(\frac12+\frac{1}{\log x}-it_2\right)\zeta\left(1+\frac{1}{\log x}+it_1\right)\right|}{(1+|t_1+t_2|)(1+|t_1|)(1+|t_2|)} \\ &\qquad\times\left(\frac{x}{y}\right)^{5/2} dt_1\,dt_2 \\ &\ll y^3\left(\frac{x}{y}\right)^{5/2} L^2 \int_{-T}^{T}\int_{-T}^{T}\frac{\left|\zeta\left(\frac12-i(t_1+t_2)\right)\zeta\left(\frac12+\frac{1}{\log x}-it_2\right)\right|}{(1+|t_1|)(1+|t_2|)(1+|t_1+t_2|)}\,dt_1\,dt_2 \\ &\ll y^3\left(\frac{x}{y}\right)^{5/2} L^2 \int_{-2T}^{2T}\frac{\left|\zeta\left(\frac12-iu\right)\right|}{1+|u|}\int_{-T}^{T}\frac{\left|\zeta\left(\frac12+\frac{1}{\log x}-it_2\right)\right|}{(1+|t_2|)(1+|u-t_2|)}\,dt_2\,du \\ &\ll x^2 y\left(\frac{x}{y}\right)^{1/2} L^4.\end{aligned}$$

Hence we get

$$D_{2,4}(x,y) \ll x^2 y L^4\left\{\frac{1}{T^2}+\left(\frac{x}{y}\right)^{1/2}\right\}. \tag{5.13}$$

Now we shall evaluate the integral $D_{2,3}(x,y)$. It is given explicitly by

$$D_{2,3}(x,y) = \frac{y^2}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta^2(s_1)\zeta(s_2)}{(2-s_2)\zeta(2+s_1-s_2)\zeta(2s_1)\zeta(2s_2)}\,\frac{x^{s_1+s_2}y^{-s_2}}{s_1 s_2}\,ds_2\,ds_1.$$

We move the line of integration with respect to s_2 to $\Gamma(\alpha_2, 3/2, T)$. Note that there exist no poles under this deformation. The contributions from the horizontal lines are

$$\begin{aligned}J_{3,1}, J_{3,3} &\ll \frac{y^2 x}{T^2}\int_{-T}^{T}\frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|}\int_{\alpha_2}^{3/2}\left|\zeta(2-\sigma_2-iT)\zeta\left(1+\alpha_1-\sigma_2+i(t_1-T)\right)\right. \\ &\qquad\left.\times\,\zeta(\sigma_2+iT)\right|\left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2\,dt_1\end{aligned}$$

$$\ll \frac{y^2 x L^3}{T^2} \int_{-T}^{T} \frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|} \int_{\alpha_2}^{3/2} T^{(-1+\sigma_2)/3}(1+|t_1-T|)^{(-1+\sigma_2)/3} \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2\, dt_1$$
$$\ll \frac{y^2 x L^3}{T^2} \int_{-T}^{T} \frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|}\, dt_1 \int_{\alpha_2}^{3/2} T^{2(-1+\sigma_2)/3} \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2$$
$$\ll \frac{y^2 x L^4}{T^2} \left(T^{1/3}\left(\frac{x}{y}\right)^{3/2} + \frac{x}{y}\right) \ll y x^2 L^4 \left(T^{-2} + T^{-5/3}\left(\frac{x}{y}\right)^{1/2}\right).$$

On the other hand, the contribution from the vertical line is

$$J_{3,2} \ll y^2 x \int_{-T}^{T} \frac{|\zeta^2(\alpha_1+it_1)|}{1+|t_1|} \int_{-T}^{T} \frac{\left|\zeta\left(\frac{1}{2}-it_2\right)\zeta\left(\frac{1}{2}+\frac{1}{\log x}+i(t_1-t_2)\right)\right|}{(1+|t_2|)^2} \left(\frac{x}{y}\right)^{\frac{3}{2}} dt_2\, dt_1$$
$$\ll y^2 x \left(\frac{x}{y}\right)^{3/2} L.$$

Hence

$$D_{2,3}(x,y) \ll y x^2 L \left\{\frac{L^3}{T^2} + \left(\frac{x}{y}\right)^{1/2}\right\}. \tag{5.14}$$

Finally we consider the integral $D_{2,2}(x,y)$. Its explicit form is

$$D_{2,2}(x,y) = \frac{y^2}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} \frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_1)\zeta^2(s_2)}{(2-s_1)\zeta(2-s_1+s_2)\zeta(2s_1)\zeta(2s_2)} \frac{x^{s_1+s_2}y^{-s_1}}{s_1 s_2}\, ds_2\, ds_1. \tag{5.15}$$

This time we first move the line of integration over s_1 to $\Gamma(\alpha_1, 3/2, T)$ [1]. The estimates over the horizontal lines and the vertical line are the same as those of $D_{2,3}(x,y)$, but there is a simple pole at $s_1 = s_2$ inside this contour. The residue of the integrand of (5.15) at this pole is

$$-\frac{\zeta(2-s_2)\zeta(s_2)^3 x^{2s_2} y^{-s_2}}{\zeta(2)\zeta(2s_2)^2(2-s_2)s_2^2},$$

hence

$$D_{2,2}(x,y) = \frac{x^2 y}{2\pi i} \int_{\alpha_2-iT}^{\alpha_2+iT} \frac{\zeta(2-s_2)\zeta(s_2)^3 (y/x^2)^{1-s_2}}{\zeta(2)\zeta(2s_2)^2(2-s_2)s_2^2}\, ds_2 + y x^2 L \left\{\frac{L^3}{T^2} + \left(\frac{x}{y}\right)^{1/2}\right\}.$$

---

[1] In [2, p. 8, line 4], Chan and Kumchev wrote that “Similarly, by moving the line of integration to $\Gamma(\alpha_2, 3/2, T)$, we find” formulas (4.16) and (4.17). But it seems that to derive (4.17) of [2], we need to move the line of integration over s_1 to $\Gamma(\alpha_1, 3/2, T)$.

The treatment of the integral on the right hand side is easier than the corresponding integral of [2]. In fact, we move the line of integration to $\Gamma(\alpha_2, 1/2, T)$. By the same method as before, the integrals over the horizontal lines are estimated as

$$\ll \frac{x^2y}{T^3}\left(L^4\left(\frac{y}{x^2}\right)^{-2/\log x} + L^2T^{1/2}\left(\frac{y}{x^2}\right)^{1/2}\right) \ll \frac{x^2yL^4}{T^3}\left(1 + T^{1/2}\left(\frac{y}{x^2}\right)^{1/2}\right),$$

and those over the vertical line are estimated as $\ll x^2y(y/x^2)^{1/2}L^2$. Furthermore, there is a contribution from the pole $s_2 = 1$ of order 4, hence

$$\begin{aligned}(5.16)\qquad D_{2,2}(x,y) &= x^2yQ_0\left(\log\frac{x^2}{y}\right) + yx^2L\left(\frac{L^3}{T^2} + \left(\frac{x}{y}\right)^{1/2}\right)\\ &\quad + O\left(\frac{x^2yL^4}{T^3}\left(1 + T^{1/2}\left(\frac{y}{x^2}\right)^{1/2}\right)\right) + O\left(x^2y\left(\frac{y}{x^2}\right)^{1/2}L^2\right),\end{aligned}$$

where $Q_0(u)$ is a polynomial in u of degree 3. By Cauchy's residue theorem, we have

$$Q_0(u) = -\frac{1}{6\zeta^3(2)}u^3 + C_3u^2 + C_4u + C_5',$$

where C_3 and C_4 are the constants defined by (1.14) and (1.15), respectively, and C_5' is another constant.

Now we substitute (5.12)–(5.14) and (5.16) into (5.4), and take $T = x^{3/8}$. Then we obtain

$$\begin{aligned}D_2(x,y) &= x^2y\big(P(\log x) + b_4 + c\big) + x^2yQ_0\left(\log\frac{x^2}{y}\right)\\ &\quad + O\left(x^2y\left(L^{10}x^{-3/8} + L^{10}y^{-1/2} + L^4\left(\frac{x}{y}\right)^{1/2} + L^2\left(\frac{y}{x^2}\right)^{1/2}\right)\right).\end{aligned}$$

Taking $C_5 = b_4 + c + C_5'$ and defing $Q(u)$ by (1.13), we get the assertion of Theorem 1.3. ∎

Acknowledgements. The authors would like to thank the referee for careful reading of the earlier version of this paper, giving many valuable suggestions and pointing out misprints.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] T. H. Chan and A. V. Kumchev, *On sums of Ramanujan sums*, Acta Arith. 152 (2012), 1–10.

[3] J. Furuya, M. Minamide and Y. Tanigawa, *Representations and evaluations of the error term in a certain divisor problem*, Math. Slovaca, to appear.

[4] S. M. Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Invent. Math. 75 (1984), 123–141.

[5] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, London Math. Soc. Lecture Note Ser. 126, Cambridge Univ. Press, 1991.
[6] A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. 27 (1926), 273–300.
[7] A. Ivić, *The Riemann Zeta-Function*, Dover Publ., 2003.
[8] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, 2007.
[9] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* 2nd ed., Oxford Univ. Press, 1986.

I. Kiuchi, M. Minamide
Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
Yoshida 1677-1
Yamaguchi 753-8512, Japan
E-mail: kiuchi@yamaguchi-u.ac.jp
minamide@yamaguchi-u.ac.jp

Y. Tanigawa
Graduate School of Mathematics
Nagoya University
Nagoya 464-8602, Japan
E-mail: tanigawa@math.nagoya-u.ac.jp

Received on 21.4.2014
and in revised form on 13.12.2014 (7780)