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The universality theorem for class group L-functions

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1. Introduction. Throughout this paper, for a set S we denote by $\sharp S$ the cardinality of S. Let D be the strip $\{s = \sigma + it \in \mathbb{C} \mid 1/2 < \sigma < 1\}$ and Λ be the set of all negative fundamental discriminants $-d$. Our purpose is to investigate the functional distribution of a family of class group L-functions over the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ as $-d$ varies in Λ. Before stating our results, we recall some related results on other L-functions. B. Bagchi [1] and S. M. Gonek [4] independently proved the following result.

THEOREM 1. *Let C be a simply connected compact subset of D and $f(s)$ be a non-vanishing and continuous function on C which is analytic in the interior of C. Then for any small positive number ε we have*

$$\liminf_{q\to\infty}{}' \frac{1}{q-1}\sharp\{\chi \pmod q \mid \max_{s\in C}|L(s,\chi) - f(s)| < \varepsilon\} > 0,$$

where $L(s,\chi)$ is the Dirichlet L-function associated with the Dirichlet character χ and $\liminf'$ denotes the limit inferior over prime numbers q.

Theorem 1 asserts that any analytic function can be uniformly approximated by Dirichlet L-functions associated with suitable Dirichlet characters, and that the set of such characters has positive lower density. K. M. Eminyan [3] obtained the same property for a set of Dirichlet L-functions $\{L(s,\chi) \mid \chi \pmod{p^n}\}$ where $p > 2$ is a fixed prime and $n \to \infty$. This type of property for a set of zeta functions is called "universality".

The author and Nagoshi [5] showed that the universality property also holds for a family of Dirichlet L-functions associated with quadratic Dirichlet characters.

THEOREM 2. *For a negative fundamental discriminant $-d$, denote by $\chi_{-d}(\cdot) = \left(\frac{\cdot}{-d}\right)$ the Kronecker symbol, which is a quadratic Dirichlet character modulo d. Let Ω be a simply connected bounded region in D which is*

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symmetric with respect to the real axis. Let $f(s)$ be an analytic and non-vanishing function on Ω which is positive on $\Omega \cap \mathbb{R}$, and C be a compact subset of Ω. For $X > 0$ put $\Lambda_X = \{-d \in \Lambda \mid d \leq X\}$. Then for any $\varepsilon > 0$ we have

$$\liminf_{X\to\infty} \frac{1}{\sharp\Lambda_X} \sharp\{-d \in \Lambda_X \mid \max_{s\in C} |L(s,\chi_{-d}) - f(s)| < \varepsilon\} > 0.$$

An analogous statement also holds for positive fundamental discriminants.

In the following, for $-d \in \Lambda$, let K be the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, $H(-d)$ be the ideal class group of K, and $h(-d)$ be the class number of K. For a class group character $\chi \in \widehat{H}(-d)$, the attached Hecke L-function is defined by

$$L_K(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \quad (\Re s > 1), \tag{1.1}$$

where $\mathfrak{a}$ runs over all integral ideals of K other than 0, and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It is well known that the class number $h(-d)$ satisfies

$$d^{1/2-\varepsilon} \ll_\varepsilon h(-d) \ll d^{1/2}\log d,$$

where the lower bound is due to C. L. Siegel [7]. Therefore the number of class characters $\chi \in \widehat{H}(-d)$ goes to infinity as $d \to \infty$.

Now we state our main result, which is the universality theorem for a family of class group L-functions.

Theorem 3. *Let Ω be a simply connected bounded region in D which is symmetric with respect to $\mathbb{R}$. Let $f(s)$ be an analytic and non-vanishing function on Ω which is positive on $\mathbb{R} \cap \Omega$, and C be a compact subset of Ω. Then for any small positive number ε there is a subset $\Lambda_0 \subset \Lambda$ satisfying the following conditions:*

(1) *Λ_0 has positive density in Λ, namely*

$$\lim_{X\to\infty} \frac{\sharp\{-d \in \Lambda_0 \mid d \leq X\}}{\sharp\{-d \in \Lambda \mid d \leq X\}} = \frac{1}{8} \prod_{3\leq p\leq \nu} \frac{p}{2(p-1)},$$

where $\nu = \nu(f, C, \varepsilon)$ is a positive constant.

(2) *We have*

$$\liminf_{\substack{d\to\infty \\ -d\in\Lambda_0}} \frac{1}{h(-d)} \sharp\{\chi \in \widehat{H}(-d) \mid \max_{s\in C} |L_K(s,\chi) - f(s)| < \varepsilon\} > 0.$$

Remark. As we compare Theorem 3 with Theorems 1 and 2, it seems that the natural form of the universality theorem for class group L-functions is

$$\liminf_{d\to\infty} \frac{1}{h(-d)} \sharp\{\chi \in \widehat{H}(-d) \mid \max_{s\in C} |L_K(s,\chi) - f(s)| < \varepsilon\} > 0.$$

However the above form probably does not hold. In general, the proof of the universality theorem is divided into two steps.

(i) For a given analytic function $g(s)$ there is a Dirichlet polynomial $\sum_{p\le\nu} a_p p^{-s}$ which uniformly approximates $g(s)$.

(ii) The set of class characters χ for which the logarithms of the L-functions $L_K(s,\chi)$ uniformly approximate $\sum_{p\le\nu} a_p p^{-s}$ has positive lower density.

We have a problem in step (ii). For $\Re s > 1$ the logarithm of $L_K(s,\chi)$ has the Dirichlet series over rational primes

$$\log L_K(s,\chi) = \sum_{\substack{p \\ (p)=\mathfrak{p}\bar{\mathfrak{p}}}} \log\left(1 - \frac{2\cos(\arg\chi(\mathfrak{p}))}{p^s} + \frac{1}{p^{2s}}\right)^{-1} + l(s,\chi),$$

where $l(s,\chi)$ is an analytic function given by a convergent series in $\Re s > 1/2$. In order to have $\log L_K(s,\chi)$ uniformly approximating $\sum_{p\le\nu} a_p p^{-s}$, all primes p with $p \le \nu$ have to decompose completely in $K = \mathbb{Q}(\sqrt{-d})$. Therefore we need to consider the set Λ_0 which is defined in Lemma 2 in the next section.

Finally we consider the relation between quadratic Dirichlet L-functions and class group L-functions. Let $-d \in \Lambda$ and $\chi \in \widehat{H}(-d)$. If χ is a real character, that is, a genus character, we have a decomposition $d = d_1 d_2$ with $-d_1, -d_2 \in \Lambda$ such that the Kronecker factorization

$$L_K(s,\chi) = L(s,\chi_{-d_1}) L(s,\chi_{-d_2}) \tag{1.2}$$

holds. If we assume the general Riemann hypothesis for quadratic Dirichlet L-functions, we could obtain the universality theorem for a family of L-functions with genus characters.

THEOREM 4. *Assume that there exists a quadratic Dirichlet L-function $L(s,\chi_{-d_1})$ which satisfies the general Riemann hypothesis. Let Ω, $f(s)$ and C be as in Theorem 2. For any $\varepsilon > 0$ there is a subset $\Lambda_1 \subset \Lambda$ with positive lower density which has the following property. If a discriminant $-d$ belongs to Λ_1, then there is at least one genus character $\chi \in \widehat{H}(-d)$ such that*

$$\max_{s\in C} |L_K(s,\chi) - f(s)| < \varepsilon.$$

Proof. Since $L(s,\chi_{-d_1})$ is positive on $(1/2,1)$, the product $f(s)L(s,\chi_{-d_1})^{-1}$ satisfies the same condition as $f(s)$ in Theorem 2. Let Λ_2 be the set of negative fundamental discriminants $-d_2$ satisfying

$$\max_{s\in C} |L(s,\chi_{-d_2}) - f(s)L(s,\chi_{-d_1})^{-1}| < \frac{\varepsilon}{\max_{s\in C} |L(s,\chi_{-d_1})|}. \tag{1.3}$$

By Theorem 2 the set Λ_2 has positive lower density in Λ. Now we put $\Lambda_1 = \{-d_1 d_2 \in \Lambda \mid -d_2 \in \Lambda_2\}$. For $-d = -d_1 d_2 \in \Lambda_1$, there is a genus character $\chi \in \widehat{H}(-d)$ for which (1.2) holds. From (1.2) and (1.3), we deduce the conclusion of Theorem 4. ■

2. Lemmas. The first lemma, called the denseness lemma, asserts that any analytic function can be uniformly approximated by a certain type of Dirichlet polynomial.

LEMMA 1. *Let Ω be as in Theorem 3 and $g(s)$ be an analytic function on Ω which is real-valued on $\Omega \cap \mathbb{R}$. Let C be a compact subset of Ω and $\nu_0 \geq 3$. For any $\varepsilon > 0$ there exist $\nu \geq \nu_0$ and $\theta_p \in [0,1)$ for each prime p with $p \leq \nu$ which satisfy*

$$\max_{s\in C}\left| g(s) - \sum_{p\leq\nu} \log\left(1 - \frac{2\cos 2\pi\theta_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1}\right| < \varepsilon.$$

Proof. We invoke Proposition 2.4 in [5]. Let $y > 0$ and $g_1(s)$ be a function which satisfies the condition in Lemma 1. There exist $\nu > y$ and $\theta_p \in [0,1)$ for $y \leq p \leq \nu$ such that

$$\max_{s\in C}\left| g_1(s) - \sum_{y<p\leq\nu} \frac{2\cos 2\pi\theta_p}{p^s}\right| < \varepsilon. \tag{2.1}$$

Now we take a sufficiently large $y > 0$ satisfying

$$\sum_{y<p\leq\nu}\left| \frac{2\cos 2\pi\theta_p}{p^s} - \log\left(1 - \frac{2\cos 2\pi\theta_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1}\right| \ll_C \sum_{k\geq 2}\sum_{p>y} \frac{1}{p^{2\sigma_1}} \ll y^{1-2\sigma_1} < \varepsilon, \tag{2.2}$$

where $\sigma_1 = \min_{s\in C}\sigma > 1/2$. Also we take

$$g_1(s) = g(s) - \sum_{p\leq y} \log\left(1 - \frac{2}{p^s} + \frac{1}{p^{2s}}\right)^{-1}. \tag{2.3}$$

Put $\theta_p = 0$ for each prime p with $p \leq y$. Then (2.1)–(2.3) imply

$$\max_{s\in C}\left| g(s) - \sum_{p\leq\nu} \log\left(1 - \frac{2\cos 2\pi\theta_p}{p^s} + \frac{1}{p^{2s}}\right)^{-1}\right| < 2\varepsilon. \blacksquare$$

The next lemma follows from the quadratic reciprocity law.

LEMMA 2 ([5, Lemma 4.2]). *For $\nu \geq 3$ define*

$$\Lambda_0 = \{-d \in \Lambda \mid \chi_{-d}(p) = 1\ (3 \leq p \leq \nu),\ -d \equiv 1 \pmod 8\}.$$

Then Λ_0 has positive density in Λ, namely

$$\lim_{X\to\infty} \frac{\sharp\{-d \in \Lambda_0 \mid d \leq X\}}{\sharp\{-d \in \Lambda \mid d \leq X\}} = \frac{1}{8}\prod_{3\leq p\leq\nu} \frac{p}{2(p-1)}.$$

From the orthogonality of class group characters

$$(2.4)\qquad \frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \chi(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} \text{ is a principal ideal,} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the large sieve inequality for class group characters. If an integral ideal $\mathfrak{a}$ has no rational integer factors other than ± 1, then we call $\mathfrak{a}$ *primitive*.

LEMMA 3 ([2, Theorem A1]). *Let $X > 0$ and $c_{\mathfrak{a}} \in \mathbb{C}$ for integral ideals $\mathfrak{a}$ of K. Then*

$$\frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \Big| \sum_{N\mathfrak{a} \le X} c_{\mathfrak{a}} \chi(\mathfrak{a}) \Big|^2 = \Big\{ 1 + O\Big(\frac{X}{d^{1/2}} \Big) \Big\} \sum_{\mathfrak{a}}{}' \Big| \sum_{(l)} c_{(l)\mathfrak{a}} \Big|^2,$$

where $\sum'$ denotes summation over the set of primitive ideals, and (l) denotes the principal ideal generated by a rational integer l.

As a consequence of the orthogonality (2.4), we also obtain the uniform distribution of class group characters.

LEMMA 4. *For $\nu \ge 3$ define Λ_0 as in Lemma 2. If $-d \in \Lambda_0$, then each prime p with $p \le \nu$ splits completely in K:*

$$(2.5)\qquad (p) = \mathfrak{p}\bar{\mathfrak{p}}, \qquad \mathfrak{p} \ne \bar{\mathfrak{p}},\ N\mathfrak{p} = p.$$

Let $0 < \delta < 1/2$ and $\theta_p \in [0,1)$ for each prime p with $p \le \nu$. For $-d \in \Lambda_0$ define

$$A(-d) = \Big\{ \chi \in \widehat{H}(-d) \ \Big|\ \Big\| \frac{\arg \chi(\mathfrak{p})}{2\pi} - \theta_p \Big\| < \delta \ (p \le \nu) \Big\},$$

where $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. Then

$$\lim_{\substack{d \to \infty \\ -d \in \Lambda_0}} \frac{\sharp A(-d)}{h(-d)} = (2\delta)^{\pi(\nu)},$$

where $\pi(\nu)$ is the number of rational primes p with $p \le \nu$.

Proof. By Weyl's criterion,

$$\lim_{\substack{d \to \infty \\ -d \in \Lambda_0}} \frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \chi(\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r}) = 0$$

for any prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which satisfy $N\mathfrak{p}_i = p_i$ and are not conjugate to one another and any r-tuple $(k_1, \ldots, k_r) \in \mathbb{Z}^r$ other than $(0, \ldots, 0)$. Moreover, by the orthogonality (2.4), it is enough to show that for any $-d \in \Lambda_0$ with d sufficiently large the ideal $\mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r}$ is not principal. Since $(p_i) = \mathfrak{p}_i \bar{\mathfrak{p}}_i$, it suffices to show that one of the ideals $\mathfrak{p}_1^{\pm k_1} \cdots \mathfrak{p}_r^{\pm k_r}$ is not principal. Consider the case where all k_i are non-negative, that is, $\mathfrak{a} = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r}$

is integral. If $\mathfrak{a}$ is a principal ideal, then

$$\mathfrak{a} = \left(a + b\frac{1+\sqrt{-d}}{2}\right) \qquad (a, b \in \mathbb{Z}).$$

Taking the norm of both sides, we have

$$N\mathfrak{a} = p_1^{k_1} \cdots p_r^{k_r} = \left(a + \frac{b}{2}\right)^2 + \frac{b^2}{4}d.$$

Since the norm $p_1^{k_1} \cdots p_r^{k_r}$ does not depend on d, if d is sufficiently large then $b = 0$ and $(a) = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_r^{k_r}$. However, since $\mathfrak{p}_i$'s are not conjugate to one another, such a rational integer a does not exist. ∎

The class group L-function $L_K(s,\chi)$ satisfies the functional equation

$$L_K(s,\chi) = H(s,\chi)L_K(1-s,\bar{\chi}), \tag{2.6}$$

where

$$H(s,\chi) = w(\chi)\left(\frac{2\pi}{\sqrt{d}}\right)^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}, \tag{2.7}$$

$w(\chi)$ is the root number with $|w(\chi)| = 1$ and $\Gamma(s)$ is the Euler Γ-function. From this we can obtain the approximate functional equation.

Lemma 5. *Let $x, y > 0$, $0 < \beta < \alpha < 2$ and $0 < \gamma < 2$. For $\beta < \sigma < \alpha$ we have*

$$L_K(s,\chi) = \sum_{i=1}^{6} S_i,$$

where

$$S_1 = \sum_{N\mathfrak{a}\le x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}, \quad S_2 = H(s,\chi) \sum_{N\mathfrak{a}\le y} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s}},$$

$$S_3 = \sum_{N\mathfrak{a}>x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \exp\left(-\left(\frac{N\mathfrak{a}}{x}\right)^2\right),$$

$$S_4 = \frac{1}{2\pi i}\int_{(-\gamma)} x^w \frac{\Gamma(1+w/2)}{w} \sum_{N\mathfrak{a}\le x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s+w}}\, dw,$$

$$S_5 = -\frac{1}{2\pi i}\int_{(\beta)} x^w \frac{\Gamma(1+w/2)}{w} H(s+w,\chi) \sum_{N\mathfrak{a}\le y} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s-w}}\, dw,$$

$$S_6 = -\frac{1}{2\pi i}\int_{(-\alpha)} x^w \frac{\Gamma(1+w/2)}{w} H(s+w,\chi) \sum_{N\mathfrak{a}>y} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s-w}}\, dw.$$

Proof. For $X > 0$ we have

$$e^{-X^2} = \frac{1}{2\pi i} \int_{(1)} X^{-w} \frac{\Gamma(1+w/2)}{w}\, dw. \tag{2.8}$$

Combining (1.1) and (2.8) yields

$$\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \exp\left(-\left(\frac{N\mathfrak{a}}{x}\right)^2\right) = \frac{1}{2\pi i} \int_{(1)} L_K(s+w,\chi) x^w \frac{\Gamma(1+w/2)}{w}\, dw.$$

We move the contour on the right hand side from $\Re w = 1$ to $\Re w = -\alpha$, to obtain

$$\begin{aligned} L_K(s,\chi) = &\sum_{N\mathfrak{a}\le x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \exp\left(-\left(\frac{N\mathfrak{a}}{x}\right)^2\right) + S_3 \\ &- \frac{1}{2\pi i} \int_{(-\alpha)} L_K(s+w,\chi) x^w \frac{\Gamma(1+w/2)}{w}\, dw. \end{aligned} \tag{2.9}$$

Moving the contour in (2.8) from $\Re w = 1$ to $\Re w = -\gamma$, we obtain

$$e^{-X^2} = 1 + \frac{1}{2\pi i} \int_{(-\gamma)} X^{-w} \frac{\Gamma(1+w/2)}{w}\, dw.$$

Therefore the first term on the right hand side of (2.9) is

$$\sum_{N\mathfrak{a}\le x} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \exp\left(-\left(\frac{N\mathfrak{a}}{x}\right)^2\right) = S_1 + S_4. \tag{2.10}$$

By the functional equation (2.6), the third term on the right hand side of (2.9) is

$$\begin{aligned} &-\frac{1}{2\pi i} \int_{(-\alpha)} L_K(s+w,\chi) x^w \frac{\Gamma(1+w/2)}{w}\, dw \\ &\quad = S_6 - \frac{1}{2\pi i} \int_{(-\alpha)} x^w \frac{\Gamma(1+w/2)}{w} H(s+w,\chi) \sum_{N\mathfrak{a}\le y} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s-w}}\, dw. \end{aligned}$$

Moving the contour from $\Re w = -\alpha$ to $\Re w = \beta$ yields

$$-\frac{1}{2\pi i} \int_{(-\alpha)} L_K(s+w,\chi) x^w \frac{\Gamma(1+w/2)}{w}\, dw = S_2 + S_5 + S_6. \tag{2.11}$$

From (2.9)–(2.11), the approximate functional equation follows. ∎

Lastly we quote the following lemma.

LEMMA 6 ([5, Lemma 2.5]). *Let C and C' be compact subsets in $\mathbb{C}$ such that C is contained in the interior of C'. There exists a positive constant*

$a(C, C')$ with the following property. If an analytic function $f(s)$ on C' satisfies the estimate

$$\iint_{C'} |f(s)|^2 \, d\sigma \, dt < A$$

with some $A > 0$, then

$$\max_{s \in C} |f(s)| < a(C, C')\sqrt{A}.$$

3. Approximation by truncated Euler product. For $\sigma > 1$ the class group L-function has the Euler product

$$L_K(s, \chi) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1},$$

where $\mathfrak{p}$ runs over all prime ideals of K. For $z > 0$ we consider the truncated Euler product

$$L_K(s, \chi, z) = \prod_{N\mathfrak{p} \le z} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}.$$

Now we prove that if z is sufficiently large, then for almost all characters $\chi \in \widehat{H}(-d)$ the attached L-functions $L_K(s, \chi)$ can be uniformly approximated by the truncated Euler products $L_K(s, \chi, z)$.

PROPOSITION 1. *Let $\varepsilon > 0$, $z > 0$ and C be a compact subset of D. Define a set of characters*

$$B(-d) = \{\chi \in \widehat{H}(-d) \mid \max_{s \in C} |\log L_K(s, \chi) - \log L_K(s, \chi, z)| < \varepsilon\}$$

where the logarithm of $L_K(s, \chi)$ is defined as the analytic continuation of the series

$$\sum_{k=1}^{\infty} \sum_{\mathfrak{p}} \frac{\chi^k(\mathfrak{p})}{kN\mathfrak{p}^{ks}} \qquad (\sigma > 1)$$

along the path $[\sigma + it, 2 + it]$. For any small positive numbers ε and ε_1 there is a positive constant $z_0 > 0$ such that for $z > z_0$ and any sufficiently large d we have

$$\frac{\sharp B(-d)}{h(-d)} > 1 - \varepsilon_1.$$

Proof. Let C' be a simply connected compact subset in D such that C is contained in the interior of C'. We estimate the second moment

$$I = \frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \iint_{C'} |L_K(s, \chi) \cdot L_K(s, \chi, z)^{-1} - 1|^2 \, d\sigma \, dt.$$

Let $\sigma_1 = \min_{s \in C'} \sigma > 1/2$ and δ be a positive number with $\delta < \frac{1}{2}(\sigma_1 - 1/2)$. In Lemma 5 we take $x = d^{1/2-\delta}$, $y = d^{1/2+\delta}$ and $\alpha > 1 + \sigma_1$. Applying

Lemma 5 and the Cauchy–Schwarz inequality, we have

$$I \le \sum_{i=1}^{6} I_i, \tag{3.1}$$

where

$$I_1 = \frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \iint_{C'} |S_1\cdot L_K(s,\chi,z)^{-1}-1|^2\,d\sigma\,dt,$$

$$I_i = \frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \iint_{C'} |S_i\cdot L_K(s,\chi,z)^{-1}|^2\,d\sigma\,dt \qquad (2\le i\le 6).$$

First we calculate I_1. By the definition of S_1 and $L_K(s,\chi,z)$ we have

$$S_1\cdot L_K(s,\chi,z) = 1 + \sum_{z<N\mathfrak{a}\le xz'} \frac{\chi(\mathfrak{a})c_{\mathfrak{a}}}{N\mathfrak{a}^s},$$

where $z' \ll z^z$ and $c_{\mathfrak{a}}$ are numbers satisfying $|c_{\mathfrak{a}}| \ll N\mathfrak{a}^{\varepsilon_2}$ for arbitrarily small $\varepsilon_2>0$. It follows from Lemma 3 that

$$\begin{aligned} I_1 &= \frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \iint_{C'} \biggl|\sum_{z<N\mathfrak{a}\le xz'} \frac{\chi(\mathfrak{a})c_{\mathfrak{a}}}{N\mathfrak{a}^s}\biggr|^2 d\sigma\,dt \\ &\ll_{C'} \left(1+\frac{xz'}{d^{1/2}}\right)\sum_{N\mathfrak{a}>z}\frac{1}{N\mathfrak{a}^{2\sigma_1-\varepsilon_2}} \ll_{C'} (1+d^{-\delta}z')z^{1-2\sigma_1+\varepsilon_2}. \end{aligned} \tag{3.2}$$

Next we calculate I_2. From the Stirling formula

$$|\Gamma(x+iy)| \ll (1+|y|)^{x-1/2}e^{-\pi|y|/2} \qquad (|y|\to\infty), \tag{3.3}$$

we obtain

$$|H(s,\chi)| \ll d^{1-2\sigma}|t|^{1-2\sigma}. \tag{3.4}$$

From (3.4) and Lemma 3, it follows that

$$\begin{aligned} I_2 &\ll_{z,C'} d^{1-2\sigma_1} \iint_{C'} \biggl\{\frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \biggl|\sum_{N\mathfrak{a}\le y}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s}}\biggr|^2\biggr\} d\sigma\,dt \\ &\ll_{z,C'} d^{1-2\sigma_1}\left(1+\frac{y}{d^{1/2}}\right)\sum_{N\mathfrak{a}\le y}\frac{1}{N\mathfrak{a}^{2-2\sigma_1}} \ll_{z,C'} d^{2\sigma_1\delta+1/2-\sigma_1+\varepsilon_2}. \end{aligned} \tag{3.5}$$

Here we remark that

$$2\sigma_1\delta+\frac12-\sigma_1 < \sigma_1\left(\sigma_1-\frac12\right)+\frac12-\sigma_1 = (\sigma_1-1)\left(\sigma_1-\frac12\right) < 0.$$

As above, applying Lemma 3 and the estimates (3.3) and (3.4), we obtain

$$I_3, I_4 \ll_{z,C'} d^{(1/2-\delta)(1-2\sigma_1)+\varepsilon_2}, \tag{3.6}$$

$$I_5, I_6 \ll_{z,C'} d^{1/2-\sigma_1+2\sigma_1\delta+\varepsilon_2}. \tag{3.7}$$

From (3.1), (3.2) and (3.5)–(3.7) it follows that

$$I \ll_{z,C'} z^{1-2\sigma_1+\varepsilon_2} + O_z(d^{-\delta} + d^{(1/2-\delta)(1-2\sigma_1)+\varepsilon_2} + d^{1/2-\sigma_1+2\sigma_1\delta+\varepsilon_2}).$$

Now we take $z_0 > 0$ such that

$$z_0^{1-2\sigma_1+\varepsilon_2} < \left(\frac{\varepsilon\varepsilon_1}{2a(C,C')}\right)^2,$$

where $a(C,C')$ is the constant given by Lemma 6. For $z > z_0$ and sufficiently large d we have

$$\frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \iint_{C'} |L_K(s,\chi)\cdot L_K(s,\chi,z)^{-1} - 1|^2\, d\sigma\, dt < \left(\frac{\varepsilon\varepsilon_1}{2a(C,C')}\right)^2.$$

By Lemma 6, we obtain

$$\frac{1}{h(-d)} \sum_{\chi\in\widehat{H}(-d)} \max_{s\in C} |L_K(s,\chi)\cdot L_K(s,\chi,z)^{-1} - 1| < \frac{1}{2}\varepsilon\varepsilon_1. \tag{3.8}$$

Put

$$B'(-d) = \{\chi \in \widehat{H}(-d) \mid \max_{s\in C} |L_K(s,\chi)\cdot L_K(s,\chi,z)^{-1} - 1| < \varepsilon\}.$$

Then (3.8) implies the estimate

$$\frac{\sharp B'(-d)}{h(-d)} > 1 - \varepsilon_1.$$

As $e^x - 1 \asymp x$ for sufficiently small positive x, this completes the proof of Proposition 1. ∎

PROPOSITION 2. *Let C be a compact subset of D and ε be a small positive number. There is a positive constant ν_0 which has the following property. Let $\nu > \nu_0$, $z > \nu$, $0 < \delta < 1/2$ and $\theta_p \in [0,1)$ for each prime p with $p \le \nu$. For these parameters take the subsets $\Lambda_0 \subset \Lambda$ of Lemma 2 and $A(-d) \subset \widehat{H}(-d)$ of Lemma 4. Define*

$$A'(-d) = \left\{\chi \in A(-d) \;\middle|\; \max_{s\in C}\left|\sum_{\nu<N\mathfrak{p}\le z} \log\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}\right| < \varepsilon\right\}.$$

Then

$$\liminf_{\substack{d\to\infty\\ -d\in\Lambda_0}} \frac{\sharp A'(-d)}{h(-d)} > \frac{1}{2}(2\delta)^{\pi(\nu)}.$$

Proof. Let C' be a simply connected compact subset of D such that C is contained in the interior of C'. We take a positive number ν_0 sufficiently large such that

$$\nu_0^{1-2\sigma_1+\varepsilon_2} < \frac{1}{4}\left(\frac{\varepsilon}{a(C,C')}\right)^2, \tag{3.9}$$

where $\sigma_1 = \min_{s\in C'} \sigma > 1/2$, $a(C,C')$ is the constant in Lemma 6, and ε_2 denotes an arbitrarily small positive number. Let $\nu > \nu_0$ and $z > \nu$. We calculate the second moment

$$I = \sum_{\chi\in A(-d)} \iint_{C'} \left| \sum_{\nu < N\mathfrak{p} \le z} \log\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \right|^2 d\sigma\, dt. \tag{3.10}$$

Applying the estimate $\log(1-x)^{-1} = x + O(|x|^2)$, we have

$$I \ll_{C'} \sum_{\chi\in A(-d)} \iint_{C'} \left| \sum_{\nu < N\mathfrak{p} \le z} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \right|^2 d\sigma\, dt + \sharp A(-d) \cdot \nu^{2(1-2\sigma_1)+\varepsilon_2}. \tag{3.11}$$

In order to remove the condition "$\chi \in A(-d)$" from the first term in (3.11), we construct a continuous characteristic function of $A(-d)$. Let $\delta_1 > 0$ be a small number satisfying $0 < \delta \pm \delta_1 < 1/2$. Let $\xi : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period 1 which satisfies

$$\xi(x) = \begin{cases} 1 & (|x| \le \delta), \\ -\dfrac{|x|}{\delta_1} + \dfrac{\delta+\delta_1}{\delta_1} & (\delta < |x| \le \delta+\delta_1), \\ 0 & (\delta+\delta_1 < |x| \le 1/2). \end{cases}$$

Define $\xi_\nu : \widehat{H}(-d) \to \mathbb{R}$ by

$$\xi_\nu(\chi) = \prod_{p\le\nu} \xi\left(\frac{\arg\chi(\mathfrak{p})}{2\pi} - \theta_p\right). \tag{3.12}$$

Then for all $\chi \in \widehat{H}(-d)$ we have

$$0 \le \xi_\nu(\chi)^2 \le \xi_\nu(\chi) \le 1, \tag{3.13}$$

in particular for $\chi \in A(-d)$,

$$\xi_\nu(\chi) = \xi_\nu(\chi)^2 = 1. \tag{3.14}$$

Let $\sum_{n\in\mathbb{Z}} c_n e(nx)$ be the Fourier expansion of $\xi(x)$. The constant term is

$$c_0 = \int_{-1/2}^{1/2} \xi(x)\, dx = 2\delta + \delta_1. \tag{3.15}$$

We index all prime ideals with $N\mathfrak{p} \le \nu$ as $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, where $r = \pi(\nu)$. By the definition (3.12), $\xi_\nu(\chi)$ has the series expansion

$$\xi_\nu(\chi) = \prod_{i=1}^{r} \Big(\sum_{n_i} c_{n_i} \chi^{n_i}(\mathfrak{p}_i) e(-n_i \theta_{p_i}) \Big)$$
$$= \sum_{(n_1,\dots,n_r)} c_{n_1} \cdots c_{n_r} e\Big(- \sum_{i=1}^{r} n_i \theta_{p_i} \Big) \chi(\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}) = \sum_{\mathfrak{a}}{}' c_{\mathfrak{a}} \chi(\mathfrak{a}).$$

Since the series $\sum_n c_n e(nx)$ is uniformly convergent on $\mathbb{R}$, there is a constant $M > 0$ such that for any $\chi \in \widehat{H}(-d)$,

$$\Big| \xi_\nu(\chi) - \sum_{N\mathfrak{a} \le M}{}' c_{\mathfrak{a}} \chi(\mathfrak{a}) \Big| < \Big\{ \frac{1}{2} (2\delta)^{\pi(\nu)} \Big\}^{1/2}. \tag{3.16}$$

From (3.12)–(3.14) and (3.16) it follows that for any $s \in C'$,

$$\sum_{\chi \in A(-d)} \Big| \sum_{\nu < N\mathfrak{p} \le z} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \Big|^2 \le \sum_{\chi \in \widehat{H}(-d)} \Big| \xi_\nu(\chi) \sum_{\nu < N\mathfrak{p} \le z} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \Big|^2$$
$$\ll \sum_{\chi \in \widehat{H}(-d)} \Big| \sum_{N\mathfrak{a} \le M}{}' \sum_{\nu < N\mathfrak{p} \le z} \frac{c_{\mathfrak{a}}}{N\mathfrak{p}^s} \chi(\mathfrak{a}\mathfrak{p}) \Big|^2 + \frac{1}{2} (2\delta)^{\pi(\nu)} \sum_{\chi \in \widehat{H}(-d)} \Big| \sum_{\nu < N\mathfrak{p} \le z} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \Big|^2.$$

By Lemma 3, for sufficiently large d we have

$$\sum_{\chi \in A(-d)} \Big| \sum_{\nu < N\mathfrak{p} \le z} \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \Big|^2 \ll \Big\{ \sum_{N\mathfrak{a} \le M}{}' |c_{\mathfrak{a}}|^s + \frac{1}{2} (2\delta)^{\pi(\nu)} \Big\} h(-d) \nu^{1 - 2\sigma_1 + \varepsilon_2}. \tag{3.17}$$

Taking into account (3.13) and (3.15), we obtain

$$\sum_{N\mathfrak{a} \le M}{}' |c_{\mathfrak{a}}|^2 \le \sum_{\mathfrak{a}} |c_{\mathfrak{a}}|^2 = \lim_{d \to \infty} \frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \xi_\nu(\chi)^2 \tag{3.18}$$
$$\le \lim_{d \to \infty} \frac{1}{h(-d)} \sum_{\chi \in \widehat{H}(-d)} \xi_\nu(\chi) = c_0^{\pi(\nu)} = (2\delta + \delta_1)^{\pi(\nu)}.$$

Now we choose δ_1 sufficiently small such that $(2\delta + \delta_1)^{\pi(\nu)} + \frac{1}{2}(2\delta)^{\pi(\nu)} < 2(2\delta)^{\pi(\nu)}$. Combining (3.9), (3.11), (3.17) and (3.18), we obtain

$$\sum_{\chi \in A(-d)} \iint_{C'} \Big| \sum_{\nu < N\mathfrak{p} \le z} \log\Big(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \Big)^{-1} \Big|^2 d\sigma\, dt \tag{3.19}$$
$$< \frac{1}{4} h(-d) (2\delta)^{\pi(\nu)} \Big(\frac{\varepsilon}{a(C, C')} \Big)^2.$$

Define

$$A''(-d)=\Big\{\chi\in A(-d)\ \Big|\ \iint_{C'}\Big|\sum_{\nu<N\mathfrak{p}\le z}\log\Big(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\Big)^{-1}\Big|^2 d\sigma\,dt<\Big(\frac{\varepsilon}{a(C,C')}\Big)^2\Big\}.$$

Then from Lemma 4 and (3.19), it follows that

$$\liminf_{\substack{d\to\infty\\ -d\in\Lambda_0}}\frac{\sharp A''(-d)}{h(-d)}>\frac{1}{2}(2\delta)^{\pi(\nu)}.$$

In view of Lemma 6, this completes the proof of Proposition 2. ∎

4. Proof of Theorem 3. Assume that Ω, $f(s)$ and C satisfy the conditions in Theorem 3. Let $\nu_0>0$ be the constant in Proposition 2. Since $f(s)$ is positive on $\Omega\cap\mathbb{R}$, the logarithm of $f(s)$ satisfies the condition in Lemma 1. Therefore there exist $\nu>\nu_0$ and $\theta_p\in[0,1)$ for each prime p with $p\le\nu$ such that

$$\max_{s\in C}\Big|\log f(s)-\sum_{p\le\nu}\log\Big(1-\frac{2\cos 2\pi\theta_p}{p^s}+\frac{1}{p^{2s}}\Big)^{-1}\Big|<\varepsilon. \tag{4.1}$$

Let Λ_0 be the subset given by Lemma 2. From the decomposition (2.5), for $z>\nu$ and $-d\in\Lambda_0$ it follows that

$$\begin{aligned}\log L_K(s,\chi,z)&=\sum_{N\mathfrak{p}\le z}\log\Big(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\Big)^{-1}\\ &=\sum_{p\le\nu}\log\Big(1-\frac{2\cos\arg\chi(\mathfrak{p})}{p^s}+\frac{1}{p^{2s}}\Big)^{-1}+\sum_{\nu<N\mathfrak{p}\le z}\log\Big(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\Big)^{-1}.\end{aligned} \tag{4.2}$$

By continuity, there is a small constant $\delta>0$ such that if

$$\Big\|\theta_p-\frac{\arg\chi(\mathfrak{p})}{2\pi}\Big\|<\delta$$

for each p with $p\le\nu$ then

$$\begin{aligned}\max_{s\in C}\Big|&\sum_{p\le\nu}\log\Big(1-\frac{2\cos 2\pi\theta_p}{p^s}+\frac{1}{p^{2s}}\Big)^{-1}\\ &-\sum_{p\le\nu}\log\Big(1-\frac{2\cos\arg\chi(\mathfrak{p})}{p^s}+\frac{1}{p^{2s}}\Big)^{-1}\Big|<\varepsilon.\end{aligned} \tag{4.3}$$

For ε and $\varepsilon_1=\frac{1}{4}(2\delta)^{\pi(\nu)}$ we take a positive constant z_0 as in Proposition 1. Let $z>\max\{z_0,\nu\}$. For the above parameters taking the subsets $A(-d)$ of

Lemma 4, $B(-d)$ of Proposition 1 and $A'(-d)$ of Proposition 2, we have

$$\liminf_{\substack{d\to\infty \\ -d\in\Lambda_0}} \frac{\sharp(A'(-d)\cap B(-d))}{h(-d)} > \frac{1}{2}(2\delta)^{\pi(\nu)} - \frac{1}{4}(2\delta)^{\pi(\nu)} > 0.$$

Furthermore for any $\chi \in A'(-d)\cap B(-d)$ the inequalities

$$\max_{s\in C} |\log L_K(s,\chi) - \log L_K(s,\chi,z)| < \varepsilon, \tag{4.4}$$

$$\max_{s\in C} \left| \sum_{\nu < N\mathfrak{p}\le z} \log\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1} \right| < \varepsilon, \tag{4.5}$$

and (4.3) hold. Combining (4.1)–(4.5) we obtain

$$\max_{s\in C} |\log L_K(s,\chi) - \log f(s)| < 4\varepsilon.$$

This completes the proof of Theorem 3.

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