ACTA ARITHMETICA
147.2 (2011)

A note on simultaneous Diophantine approximation in positive characteristic

by

MICHAEL FUCHS (Hsinchu)

1. Introduction. Let $\mathbb{F}_q$ be a finite field with q elements and denote by $\mathbb{F}_q((T^{-1}))$ the field of formal Laurent series. For $f \in \mathbb{F}_q((T^{-1}))$ let $|f| = q^{\deg f}$ be the valuation induced by the generalized degree function. Set

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) : |f| < 1\}.$$

Then, with the restriction of $|\cdot|$ to $\mathbb{L}$, $\mathbb{L}$ is a compact topological group. Hence, there exists a (unique) translation-invariant probability measure, which will be denoted by m.

We are interested in the Diophantine approximation problem

$$\left|f - \frac{P}{Q}\right| < \frac{1}{q^{n+l_n}}, \qquad \deg Q = n,\ Q \text{ monic},\ (P,Q) = 1, \tag{1}$$

where $f \in \mathbb{L}$, $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$, and l_n is a sequence of non-negative integers (subsequently, we will use $(\cdot,\cdot)$ to denote the gcd, whereas $\langle \cdot,\cdot \rangle$ will be used for pairs).

Concerning the number of solutions of (1), Inoue and Nakada [5] proved the following 0-1 law: the number of solutions is either finite or infinite for almost all $f \in \mathbb{L}$, the latter holding if and only if

$$\sum_{n=0}^{\infty} q^{n-l_n} = \infty.$$

Moreover, the method of proof in [5] also gives a quantitative result under one additional assumption on l_n: if $l_n \geq n$, then the number of solutions of (1) with $\deg Q \leq N$ is given by

$$(1-q^{-1})\Psi(N) + \mathcal{O}(\Psi(N)^{1/2}(\log \Psi(N))^{3/2+\epsilon}),$$

where $\epsilon > 0$ is an arbitrary small constant and $\Psi(N) := \sum_{n \leq N} q^{n-l_n}$.

2010 *Mathematics Subject Classification*: 11J61, 11J83, 11K60.
Key words and phrases: formal Laurent series, simultaneous Diophantine approximation, 0-1 law, strong law of large numbers.

DOI: 10.4064/aa147-2-5

The purpose of this note is to prove generalizations of the above two results to multidimensional Diophantine approximation. Therefore, consider

$$\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l_n^{(j)}}}, \quad \deg Q = n,\ Q \text{ monic},\ (P_j, Q) = 1,\ j = 1, \dots, d, \tag{2}$$

where $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$, $P_j, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$, $j = 1, \dots, d$, and $l_n^{(j)}$, $j = 1, \dots, d$, are sequences of non-negative integers. Moreover, set $l_n := \sum_{j=1}^{d} l_n^{(j)}$.

Then the first result above has the following extension to the multidimensional setting.

Theorem 1. *The number of solutions of* (2) *is either finite or infinite for almost all* $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$, *the latter holding if and only if*

$$\sum_{n=0}^{\infty} q^{n-l_n} = \infty. \tag{3}$$

Moreover, also the second result admits an extension to higher dimensions.

Theorem 2. *Assume that* $l_n \geq n$. *Then, for almost all* $(f_1, \dots, f_d)$, *the number of solutions of* (2) *with* $\deg Q \leq N$ *is given by*

$$c_0 \Psi(N) + \mathcal{O}(\Psi(N)^{1/2+\epsilon}),$$

where $\epsilon > 0$ *is an arbitrarily small constant,* $\Psi(N) := \sum_{n \leq N} q^{n-l_n}$, *and*

$$c_0 := \sum_{Q_1 \text{ monic}} \cdots \sum_{Q_d \text{ monic}} \frac{\mu(Q_1)}{|Q_1|} \cdots \frac{\mu(Q_d)}{|Q_d|} \frac{1}{|\mathrm{lcm}(Q_1, \dots, Q_d)|} > 0.$$

Here, $\mu(\cdot)$ *denotes the Möbius function on* $\mathbb{F}_q[T]$.

Remark 1. The constant c_0 will already appear in the proof of Theorem 1. In particular, we will show the claim about the positivity already in the next section (see the proof of Lemma 1 below).

Remark 2. Observe that the error term in the above result for $d = 1$ is weaker than the corresponding one in the result of Inoue and Nakada. The reason for this is that our method is completely different from the approach used by the latter two authors (it is not obvious how to generalize their approach to higher dimensions).

Notation. We will use $[D_1, \dots, D_d]$ to denote the lcm of the polynomials $D_1, \dots, D_d$. All sums will be over monic polynomials. Logarithms in this paper just take on values ≥ 1, i.e. $\log_a x$ should be interpreted as $\max\{\log_a x, 1\}$. We will use both Landau's notation $f(x) = \mathcal{O}(g(x))$ and Vinogradov's notation $f(x) \ll g(x)$. Finally, ϵ will denote an arbitrarily small positive number whose value might change from one appearance to the next.

2. Proof of Theorem 1. First, note that the necessity of (3) for the number of solutions of (2) being infinite follows from a standard application of the Borel–Cantelli lemma. Hence, we only have to focus on the sufficiency part. For this purpose, we use a slight extension of the d-dimensional Duffin–Schaeffer theorem for formal Laurent series due to Inoue [4].

THEOREM 3 (Inoue). *Consider*

$$\left|f_j - \frac{P_j}{Q}\right| < \frac{1}{q^{n+l_Q^{(j)}}}, \quad \deg Q = n,\ Q \text{ monic},\ (P_j, Q) = 1,\ j = 1, \dots, d, \tag{4}$$

where $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$, P_j, $j = 1, \dots, d$, Q *with* $Q \neq 0$, *and* $l_Q^{(j)}$, $j = 1, \dots, d$, *are sequences of non-negative integers. Assume that*

$$\sum_Q q^{-l_Q^{(1)} - \dots - l_Q^{(j)}} = \infty$$

and that for infinitely many N,

$$\sum_{\deg Q \le N} q^{-l_Q^{(1)} - \dots - l_Q^{(j)}} < C \sum_{\deg Q \le N} q^{-l_Q^{(1)} - \dots - l_Q^{(j)}} \varphi(Q)^d / |Q|^d,$$

where C *is some positive constant. Then* (4) *has infinitely many solutions for almost all* $(f_1, \dots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}$.

REMARK 3. Note that the result in [4] is just stated for the special case $l_Q^{(1)} = \cdots = l_Q^{(d)}$. An inspection of the proof, however, shows that the result continues to hold for different approximation functions in every coordinate.

Before we can apply this result, we need a technical lemma.

LEMMA 1. *We have*

$$\sum_{\deg Q = n} \varphi(Q)^d = c_0 q^{n(d+1)} + \mathcal{O}(q^{n(d+\epsilon)}),$$

where c_0 *is as in Theorem 2 and* $\varphi(\cdot)$ *is Euler's totient function.*

Proof. Note that

$$\begin{aligned}
\sum_{\deg Q = n} \varphi(Q)^d &= q^{nd} \sum_{\deg Q = n} \left(\sum_{D|Q} \frac{\mu(D)}{|D|} \right)^d \\
&= q^{nd} \sum_{\deg Q = n} \sum_{D_1|Q} \cdots \sum_{D_d|Q} \frac{\mu(D_1)}{|D_1|} \cdots \frac{\mu(D_d)}{|D_d|} \\
&= q^{nd} \sum_{\deg D_1 \le n} \cdots \sum_{\deg D_d \le n} \frac{\mu(D_1)}{|D_1|} \cdots \frac{\mu(D_d)}{|D_d|} \sum_{[D_1, \dots, D_d] | Q,\, \deg Q = n} 1.
\end{aligned}$$

The last sum becomes

$$\sum_{[D_1,\ldots,D_d]|Q,\,\deg Q=n} 1=\begin{cases} q^n/|[D_1,\ldots,D_d]| & \text{if } \deg[D_1,\ldots,D_d]\le n,\\ 0 & \text{otherwise.}\end{cases}$$

Consequently,

$$\begin{aligned}(5)\qquad &\sum_{\deg Q=n}\varphi(Q)^d\\ &=q^{n(d+1)}\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\,\frac{1}{|[D_1,\ldots,D_d]|}\\ &\quad+\mathcal{O}\biggl(q^{nd}\biggl(\sum_{\deg D\le n}\frac{1}{|D|}\biggr)^d\biggr)\\ &=q^{n(d+1)}\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\,\frac{1}{|[D_1,\ldots,D_d]|}+\mathcal{O}(n^dq^{nd}).\end{aligned}$$

Next, observe that

$$\begin{aligned}&\biggl|\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\,\frac{1}{|[D_1,\ldots,D_d]|}-c_0\biggr|\\ &\qquad\qquad\le\sum_{\deg D>n}\sum_{[D_1,\ldots,D_d]=D}\frac{1}{|D_1\cdots D_d|\cdot|D|}\le\sum_{\deg D>n}\frac{\omega(D)^d}{|D|^2},\end{aligned}$$

where $\omega(D)$ denotes the number of monic divisors of D. Since $\omega(D)=\mathcal{O}(|D|^{\epsilon})$ for arbitrarily small $\epsilon>0$ (this is proved as for integers; see page 296 in [1]), we obtain

$$\sum_{\deg D>n}\frac{\omega(D)^d}{|D|^2}\ll\sum_{l=n+1}^{\infty}q^{l(\epsilon d-1)}\ll q^{n(\epsilon d-1)}.$$

So, we have

$$\sum_{\deg D_1\le n}\cdots\sum_{\deg D_d\le n}\frac{\mu(D_1)}{|D_1|}\cdots\frac{\mu(D_d)}{|D_d|}\,\frac{1}{|[D_1,\ldots,D_d]|}=c_0+\mathcal{O}(q^{n(-1+\epsilon)}).$$

Plugging this into (5) yields the claimed expansion.

What is left to show is that $c_0>0$. Therefore, observe that

$$\sum_{\deg Q=n}\varphi(Q)^d\ge\sum_{\deg I=n}\varphi(I)^d=(q^n-1)^d\sum_{\deg I=n}1\gg(q^n-1)^dq^n/n,$$

where the second and third sum runs over all irreducible polynomials and the last bound is well-known. Hence, $c_0>0$ as claimed. ■

Remark 4. For $d = 1$, note that

$$c_0 = \sum_Q \frac{\mu(Q)}{|Q|^2} = \prod_I \Big(1 - \frac{1}{|I|^2}\Big) = \Big(\sum_Q \frac{1}{|Q|^2}\Big)^{-1} = 1 - \frac{1}{q}.$$

In this situation even more is known, namely,

$$\sum_{\deg Q = n} \varphi(Q) = \Big(1 - \frac{1}{q}\Big) q^{2n}.$$

For a proof of the latter claim see e.g. [5].

Now, we can prove our first main result.

Proof of Theorem 1. As already mentioned before, we only have to show that (3) is sufficient for the number of solutions of (2) to be infinite. For this purpose, we just have to check the two conditions in Inoue's result. First, note that since

$$l_Q^{(1)} + \cdots + l_Q^{(d)} = l_{\deg Q},$$

we have

$$\sum_{\deg Q \le N} q^{-l_{\deg Q}} = \sum_{n \le N} q^{n - l_n}$$

and

$$\begin{aligned}\sum_{\deg Q \le N} q^{-l_{\deg Q}} \varphi(Q)^d / |Q|^d &= \sum_{n \le N} q^{-nd - l_n} \sum_{\deg Q = n} \varphi(Q)^d \\ &= c_0 \sum_{n \le N} q^{n - l_n} + \mathcal{O}\Big(\sum_{n \le N} q^{\epsilon n - l_n}\Big).\end{aligned}$$

Moreover, by Cauchy's inequality

$$\sum_{n \le N} q^{\epsilon n - l_n} \ll \Big(\sum_{n \le N} q^{n - l_n}\Big)^{1/2}.$$

Hence, both conditions are satisfied and our result follows from Inoue's result. ∎

3. Proof of Theorem 2. We start with a technical lemma.

Lemma 2. *We have*

$$\sum_{\deg(D_1), \ldots, \deg(D_d) \le n} \frac{1}{|[D_1, \ldots, D_d]|} \ll q^{n\epsilon}.$$

Proof. First note that

$$\sum_{\deg(D_1),\ldots,\deg(D_d)\le n} \frac{1}{|[D_1,\ldots,D_d]|} \le \sum_{\deg(D_1),\ldots,\deg(D_d)\le n} \frac{1}{|[D_1,\ldots,D_d]|^{1-\epsilon}} \le \sum_{\deg(D_1),\ldots,\deg(D_d)\le n} \frac{|([D_1,\ldots,D_{d-1}],D_d)|^{1-\epsilon}}{|[D_1,\ldots,D_{d-1}]|^{1-\epsilon}\cdot |D_d|^{1-\epsilon}}.$$

Next we change the order of summation to obtain

$$\begin{aligned}
&\sum_{\deg(D_1),\ldots,\deg(D_d)\le n} \frac{1}{|[D_1,\ldots,D_d]|^{1-\epsilon}}\\
&\le \sum_{\deg D\le n}\ \sum_{D|[D_1,\ldots,D_{d-1}],\,\deg D_i\le n} \frac{1}{|[D_1,\ldots,D_{d-1}]|^{1-\epsilon}} \sum_{D|D_d,\,\deg D_d\le n} \left(\frac{|D|}{|D_d|}\right)^{1-\epsilon}\\
&\le \sum_{\deg D\le n}\ \sum_{D|[D_1,\ldots,D_{d-1}],\,\deg D_i\le n} \frac{1}{|[D_1,\ldots,D_{d-1}]|^{1-\epsilon}} \sum_{\deg Q\le n} \frac{1}{|Q|^{1-\epsilon}}\\
&\ll q^{n\epsilon} \sum_{\deg(D_1),\ldots,\deg(D_{d-1})\le n} \frac{1}{|[D_1,\ldots,D_{d-1}]|^{1-\epsilon}} \sum_{D|[D_1,\ldots,D_{d-1}]} 1\\
&= q^{n\epsilon} \sum_{\deg(D_1),\ldots,\deg(D_{d-1})\le n} \frac{\omega([D_1,\ldots,D_{d-1}])}{|[D_1,\ldots,D_{d-1}]|^{1-\epsilon}}.
\end{aligned}$$

Now, as before, we use the estimate $\omega(D)=\mathcal{O}(|D|^{\epsilon})$ for all sufficiently small ϵ. Hence,

$$\sum_{\deg(D_1),\ldots,\deg(D_d)\le n} \frac{1}{|[D_1,\ldots,D_d]|^{1-\epsilon}} \ll q^{n\epsilon} \sum_{\deg(D_1),\ldots,\deg(D_{d-1})\le n} \frac{1}{|[D_1,\ldots,D_d]|^{1-2\epsilon}}.$$

Iterating this result proves the claim. ∎

Now, we turn to the proof of Theorem 2. For this purpose, we extend an approach due to Harman (see proof of Theorem 4.4 starting on page 109 in [3]) to higher dimensions.

We first need some notation. Let $\Gamma_1(N)=\lfloor \log_q \Psi(N)^2\rfloor$ and $\Gamma_2(N)=\lfloor \log_q \Psi(N)^4\rfloor$. Moreover, consider the following approximation problem:

$$\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l_n^{(j)}}}, \qquad \deg Q = n,\ Q \text{ monic}, \tag{6}$$
$$D_j \mid (P_j, Q), \qquad \deg(P_j, Q) \le \Gamma_2(N), \qquad j = 1, \ldots, d,$$

where $D_1, \ldots, D_d$ are fixed monic polynomials. For fixed $(f_1, \ldots, f_d)$ and Q denote by $s(Q; D_1, \ldots, D_d)$ the number of solutions of (6).

We gather some properties of $s(Q; D_1, \ldots, D_d)$ needed below.

Lemma 3. *We have*

$$\mathbb{E}\Big(\sum_{M_1 < n \le M_2} \sum_{\deg Q = n,\, [D_1, \ldots, D_d] \mid Q} s(Q; D_1, \ldots, D_d) \Big) \ll \frac{1}{|D_1 \cdots D_d| \cdot |[D_1, \ldots, D_d]|} \sum_{M_1 < n \le M_2} q^{n - l_n}$$

and

$$\mathbb{E}\Big(\sum_{M_1 < n \le M_2} \sum_{\deg Q = n,\, [D_1, \ldots, D_d] \mid Q} \Big(s(Q; D_1, \ldots, D_d) - \frac{1}{|D_1 \cdots D_d|} \cdot \frac{1}{q^{l_n}} \Big) \Big)^2 \ll \frac{\Gamma_2(N)}{|D_1 \cdots D_d| \cdot |[D_1, \ldots, D_d]|} \sum_{M_1 < n \le M_2} q^{n - l_n}$$

for all $M_1 \le M_2$.

Proof. Both properties are easy extensions of the corresponding properties from the case $d = 1$ (see Propositions 3 and 4 in [2]). For the reader's convenience, we recall the proof of the first property.

Therefore, observe that $s(Q; D_1, \ldots, D_d) \le s^*(Q; D_1, \ldots, D_d)$ where the latter denotes the number of solutions of (6) with the upper bound on the gcd removed. Of course, $s^*(Q; D_1, \ldots, D_d) = 0$ if $[D_1, \ldots, D_d] \nmid Q$.

Now, for $[D_1, \ldots, D_d] \mid Q$, note that $s^*(Q; D_1, \ldots, D_d) = \mathbf{1}_A$ ($\mathbf{1}_A$ denotes an indicator random variable) with

$$A = \bigcup_{P_j \mid D_j,\, \deg P_j < n,\, 1 \le j \le d} B(P_1/Q; q^{-n-l_n^{(1)}}) \times \cdots \times B(P_d/Q; q^{-n-l_n^{(d)}}),$$

where $B(f; q^{-n})$ denotes the open ball with center f and radius q^{-n} and the above union is disjoint. Since

$$(m \times \cdots \times m)(B(P_1/Q; q^{-n-l_n^{(1)}}) \times \cdots \times B(P_d/Q; q^{-n-l_n^{(d)}})) = q^{-dn-l_n}$$

and consequently

$$m(A) = \frac{1}{|D_1 \cdots D_d|} q^{-l_n},$$

the result follows from elementary properties of the mean. ∎

Next, we prove the following proposition for the number of solutions of (6).

Proposition 1. *For almost all* $(f_1, \ldots, f_d)$, *the number of solutions of* (6) *with* $\deg Q \le N$ *is given by*

$$\frac{1}{|D_1 \cdots D_d| \cdot |[D_1, \ldots, D_d]|} \Psi(N) + E(N; D_1, \ldots, D_d),$$

where the second term satisfies

$$\sum_{\deg(D_1), \ldots, \deg(D_d) \le \Gamma_1(N)} E(N; D_1, \ldots, D_d) = \mathcal{O}(\Psi(N)^{1/2+\epsilon})$$

with $\epsilon > 0$ *an arbitrarily small constant.*

Proof. First note that it suffices to prove our claim for the case where $\Psi(N) \to \infty$ as $N \to \infty$ (otherwise, the result is an easy consequence of the Borel–Cantelli lemma). Next, denote by N_k the largest integer with $\Psi(N_k) < k$. It is easy to see that we only have to prove the result for the subsequence N_k.

We are going to need some notation. First, put

$$k = \sum_{j=0}^{l} a_j 2^j, \qquad a_l \neq 0,\ a_j \in \{0,1\}\ \forall j.$$

Define

$$S(k) = \Big\{(i,m) : a_i = 1,\ m = \sum_{j=i+1}^{l} a_j 2^{j-i}\Big\}.$$

Moreover, let

$$u_t = u_t(i,m) = \max\{n \in \mathbb{N} : \Psi(n) < (m+t)2^i\},$$

where $t \in \{0,1\}$. Finally, with the notation of Lemma 3, put

$$\begin{aligned} &E(i,m; D_1, \ldots, D_d) \\ &\quad = \sum_{u_0 < n \le u_1} \sum_{\deg Q = n,\, [D_1, \ldots, D_d] \mid Q} \bigg(s(Q; D_1, \ldots, D_d) - \frac{1}{|D_1 \cdots D_d|} \cdot \frac{1}{q^{l_n}}\bigg). \end{aligned}$$

Then we obviously have

$$E(N_k; D_1, \ldots, D_d) = \sum_{(i,m) \in S(k)} E(i,m; D_1, \ldots, D_d).$$

Now, set

$$E(l) := \sum_{\deg(D_1),\ldots,\deg(D_d)\le \Gamma_1(N_{2^{l+1}})} |D_1\cdots D_d| \times \sum_{0\le i\le l,\, m<2^{l-i+1}} E(i,m;D_1,\ldots,D_d)^2.$$

Then, with the estimate of Lemma 3,

$$\mathbb{E}E(i,m;D_1,\ldots,D_d)^2 \ll \frac{\Gamma_2(N_{2^{l+1}})}{|D_1\cdots D_d|\cdot|[D_1,\ldots,D_d]|}\sum_{u_0<n\le u_1} q^{n-l_n},$$

we obtain

$$E(l) \ll 2^l l^2 \sum_{\deg(D_1),\ldots,\deg(D_d)\le \Gamma_1(N_{2^{l+1}})} \frac{1}{|[D_1,\ldots,D_d]|} \ll 2^{l(1+\bar\epsilon)},$$

where the last step follows from Lemma 2 and $\bar\epsilon$ will be chosen below. This in turn implies that

$$P(E(l)\ge 2^{l(1+\epsilon)}) \ll \frac{1}{2^{l(\epsilon-\bar\epsilon)}},$$

where we choose $\bar\epsilon<\epsilon$. Hence, the Borel–Cantelli lemma yields

$$E(l) < 2^{l(1+\epsilon)} \qquad \text{a.s.}$$

for l large enough.

Finally consider

$$\begin{aligned}&\sum_{\deg(D_1),\ldots,\deg(D_d)\le \Gamma_1(N_k)} E(N_k;D_1,\ldots,D_d)\\ &\quad\le \Bigg(\sum_{\deg(D_1),\ldots,\deg(D_d)\le \Gamma_1(N_k)} \frac{1}{|D_1\cdots D_d|}\sum_{(i,m)\in S(k)} 1\Bigg)^{1/2}\cdot (E(r))^{1/2}\\ &\quad\ll 2^{l(1/2+\epsilon)} l^{d+1} \ll 2^{l(1/2+\epsilon)}.\end{aligned}$$

From this the assertion follows. ∎

Now, we can prove our second main result.

Proof of Theorem 2. As in the proof of the proposition, we can assume that $\Psi(N)\to\infty$ as $N\to\infty$. Then we again choose N_k as the largest integer with $\Psi(N_k)<k$. As before, it is easy to see that it suffices to prove our claim for the sequence N_k.

Next, we introduce the notation $S(N_k;D_1,\ldots,D_d)$ for the number of solutions of (6) with $\deg Q\le N_k$ (here, $(f_1,\ldots,f_d)$ is fixed). Then, by an inclusion-exclusion argument, the number of solutions of (2) with $\deg Q\le N_k$

is given by

$$\sum_{\deg(D_1),\dots,\deg(D_d)\le \Gamma_2(N_k)} \mu(D_1)\cdots\mu(D_d)S(N_k;D_1,\dots,D_d),$$

where $\mu(\cdot)$ denotes the Möbius function. We split the sum into two parts A and B according to whether there is a D_i with $\deg D_i > \Gamma_1(N_k)$ or not, respectively.

First, we will consider A. Note that

$$\begin{aligned}
\mathbb{E}|A| &\le \sum_{\substack{\deg(D_1),\dots,\deg(D_d)\le\Gamma_2(N_k)\\ \deg D_i>\Gamma_1(N_k)\text{ for some } i}} \mathbb{E}S(N_k;D_1,\dots,D_d)\\
&\ll \Psi(N_k)\sum_{\substack{\deg(D_1),\dots,\deg(D_d)\le\Gamma_2(N_k)\\ \deg D_i>\Gamma_1(N_k)\text{ for some } i}} \frac{1}{|D_1\cdots D_d|}\,\frac{1}{|[D_1,\dots,D_d]|}\\
&\ll \Psi(N_k)\Bigg(\sum_{\deg D_1>\Gamma_1(N_k)}\frac{1}{|D_1|^2}\Bigg)\\
&\quad\times\Bigg(\sum_{\deg D_2\le\Gamma_2(N_k)}\frac{1}{|D_2|}\Bigg)\cdots\Bigg(\sum_{\deg D_d\le\Gamma_2(N_k)}\frac{1}{|D_d|}\Bigg)\\
&\ll \frac{(\log\Psi(N_k))^{d-1}}{\Psi(N_k)},
\end{aligned}$$

where we have used Lemma 3. Consequently,

$$P(|A|>(\log\Psi(N_k))^{d+1}) \ll \frac{1}{\Psi(N_k)(\log\Psi(N_k))^2} \ll \frac{1}{k(\log k)^2}.$$

Hence, the Borel–Cantelli lemma implies that for almost all $(f_1,\dots,f_d)$,

$$A=\mathcal{O}((\log\Psi(N_k))^{d+1}).$$

So, in view of our claimed result, the main contribution will come from B. Here, we can use the above proposition to obtain

$$\begin{aligned}
B=\Psi(N_k)&\sum_{\deg(D_1),\dots,\deg(D_d)\le\Gamma_1(N_k)}\frac{\mu(D_1)\cdots\mu(D_d)}{|D_1\cdots D_d|\cdot|[D_1,\dots,D_d]|}\\
&+\mathcal{O}(\Psi(N_k)^{1/2+\epsilon}).
\end{aligned}$$

Now, as in the proof of Lemma 1,

$$\sum_{\deg(D_1),\dots,\deg(D_d)\le\Gamma_1(N_k)}\frac{\mu(D_1)\cdots\mu(D_d)}{|D_1\cdots D_d|\cdot|[D_1,\dots,D_d]|}=c_0+\Psi(N_k)^{\epsilon-2}.$$

Combining all the estimates proves the claimed result. ■

Acknowledgments. The author acknowledges partial support by National Science Council under the grant NSC-98-2115-M-009-009. Moreover, the author thanks the anonymous referee for a careful reading.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] M. Fuchs, *Metrical theorems for inhomogeneous Diophantine approximation in positive characteristic*, Acta Arith. 141 (2010), 191–208.

[3] G. Harman, *Metric Number Theory*, London Math. Soc. Monogr. (N.S.) 18, Oxford Univ. Press, New York, 1998.

[4] K. Inoue, *The metric simultaneous Diophantine approximations over formal power series*, J. Théor. Nombres Bordeaux 15 (2003), 151–161.

[5] K. Inoue and H. Nakada, *On metric Diophantine approximation in positive characteristic*, Acta Arith. 110 (2003), 205–218.

Michael Fuchs
Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsue Road
Hsinchu, 300, Taiwan
E-mail: mfuchs@math.nctu.edu.tw

Received on 22.7.2009
and in revised form on 27.9.2010 (6093)