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The simultaneous Pell equations $y^2 - Dz^2 = 1$ and $x^2 - 2Dz^2 = 1$

by

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1. Introduction. Throughout this paper we let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and positive integers respectively, and let $D \in \mathbb{N}$ be square-free. In 1983, Cao [3] studied the system of simultaneous Pell equations

$$\begin{cases} y^2 - Dz^2 = 1, \\ x^2 - 2Dz^2 = 1. \end{cases} \tag{1.1}$$

Solutions of (1.1) lead to solutions of the Diophantine equation $x^4 - Dy^2 = 1$.

We recall Cao's result as

THEOREM 1.1. *Let D have at most four odd prime factors. Then the only positive integer solutions of* (1.1) *are*:

$$\begin{array}{ll} D = 2\cdot 3, & (x,y,z) = (7,5,2), \\ D = 2\cdot 3\cdot 5\cdot 7, & (x,y,z) = (41,29,2), \\ D = 3\cdot 5\cdot 7\cdot 17, & (x,y,z) = (239,169,4), \\ D = 3\cdot 17\cdot 29\cdot 41, & (x,y,z) = (1393,985,4), \\ D = 2\cdot 5\cdot 7\cdot 11\cdot 239, & (x,y,z) = (47321,33461,78), \\ D = 2\cdot 3\cdot 17\cdot 239\cdot 577, & (x,y,z) = (275807,195025,52). \end{array}$$

As the solutions (x, y, z) to (1.1) imply the existence of rational points of infinite order on the elliptic curve $Y^2 = X(X + D)(X + 2D)$, that led Ono [11] to study the solutions of (1.1). He proved

THEOREM 1.2. *If the number of representations of D in the form $2a^2 + b^2 + 8c^2$ equals twice the number of representations of D in the form $2a^2 +$*

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$b^2 + 32c^2$, *where* $a, b, c \in \mathbb{Z}$, *then the system of simultaneous Pell equations* (1.1) *has no solutions in positive integers* x, y, z.

Walsh [12] pointed out that heuristics shows that the condition in Theorem 1.2 only applies to a set of integers which has asymptotic density less than 1/2. So, he looked for other conditions. Walsh's result is: *If* D *is a product of fewer than five primes*, *then* (1.1) *has only the following positive integer solutions*:

$$D = 2 \cdot 3,\ (x, y, z) = (7, 5, 2);\ D = 2 \cdot 3 \cdot 5 \cdot 7,\ (x, y, z) = (41, 29, 2);$$
$$D = 3 \cdot 5 \cdot 7 \cdot 17,\ (x, y, z) = (239, 169, 4);$$
$$D = 3 \cdot 17 \cdot 29 \cdot 41,\ (x, y, z) = (1393, 985, 4).$$

But this result is included in Theorem 1.1.

In this paper, we shall improve Theorem 1.1 to

THEOREM 1.3. *If* D *is a product of fewer than seven primes*, *then the only positive integer solutions of* (1.1) *are*:

$$\begin{array}{ll}
D = 2 \cdot 3, & (x, y, z) = (7, 5, 2), \\
D = 2 \cdot 3 \cdot 5 \cdot 7, & (x, y, z) = (41, 29, 2), \\
D = 3 \cdot 5 \cdot 7 \cdot 17, & (x, y, z) = (239, 169, 4), \\
D = 3 \cdot 17 \cdot 29 \cdot 41, & (x, y, z) = (1393, 985, 4), \\
D = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 239, & (x, y, z) = (47321, 33461, 78), \\
D = 2 \cdot 3 \cdot 17 \cdot 239 \cdot 577, & (x, y, z) = (275807, 195025, 52), \\
D = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 29 \cdot 41, & (x, y, z) = (8119, 5741, 6).
\end{array}$$

By Cohn's work [7] on the Diophantine equation $x^4 - Dy^2 = 1$ we know that (1.1) has at most one positive integer solution (see also [5, pp. 50–53] or [12]). Bennett [1] proved that the general system of Pell equations

$$\begin{cases} y^2 - az^2 = 1, \\ x^2 - bz^2 = 1, \end{cases} \tag{1.2}$$

where a and b are distinct positive integers, has at most three positive integer solutions. In 2002, Yuan [13] proved that if $\max\{a, b\} > 1.4 \cdot 10^{57}$, then (1.2) has at most two positive integer solutions. Recently, Bennett *et al.* [2] based on Yuan's work proved that (1.2) has at most two positive integer solutions for any distinct positive integers a and b.

2. Lemmas. We shall need the following lemmas to prove Theorem 1.3.

LEMMA 2.1. *The only integer solutions of the equation* $x^4 - 2y^2 = 1$ *are* $(x, y) = (\pm 1, 0)$.

Proof. See [4, p. 19 or p. 27]. In fact, $x^4 - 2y^2 = 1$ is equivalent to $x^4 + y^4 = (y^2+1)^2$, which from Fermat's famous result [10, p. 16] yields $y = 0$. ∎

LEMMA 2.2. *The only integer solutions of the equation* $x^4 - 2y^2 = -1$ *are* $(x, y) = (\pm 1, \pm 1)$.

Proof. This is a special case of the equation $x^4 + z^4 = 2y^2$. For its proof we refer to [10, p. 18] or [4, p. 28]. ∎

LEMMA 2.3. *The only integer solutions of the equation* $x^2 - 2y^4 = 1$ *are* $(x, y) = (\pm 1, 0)$.

Proof. See [4, p. 26] or [10, p. 269]. ∎

LEMMA 2.4. *The only integer solutions of the equation* $x^2 - 2y^4 = -1$ *are* $(x, y) = (\pm 1, \pm 1)$ *and* $(\pm 239, \pm 13)$.

Proof. See [8]. ∎

LEMMA 2.5. *The only integer solutions of the equation* $x^2 - 8y^4 = 1$ *are* $(x, y) = (\pm 1, 0)$ *and* $(\pm 3, \pm 1)$.

Proof. See [4, p. 25]. Indeed, from $x^2 - 8y^4 = 1$ one has $x \pm 1 = 2y_1^4$, $x \mp 1 = 4y_2^4$, $y = y_1 y_2$ and hence $y_1^4 - 2y_2^4 = \pm 1$. Thus the lemma follows from Lemmas 2.1 and 2.2. ∎

LEMMA 2.6. *The only positive integer solution of the system*

$$\begin{cases} z = x^2 + (x+1)^2, \\ z^2 = y^2 + (y+1)^2, \end{cases} \tag{2.1}$$

is $(x, y, z) = (1, 3, 5)$.

Proof. See [9]. ∎

LEMMA 2.7. *The only positive integer solutions of the equation*

$$y^2 - 2\left(\frac{x^2+1}{2}\right)^2 = -1 \tag{2.2}$$

are $(x, y) = (1, 1)$ *and* $(3, 7)$.

Proof. Clearly $(x, y) = (1, 1)$ is a positive integer solution of (2.2). Suppose (x, y) is another positive integer solution. Then x and y must be odd. Let $x = 2x_1 + 1$, $y = 2y_1 + 1$, $x_1, y_1 \in \mathbb{N}$. Put $z_1 = (x^2+1)/2$. Then

$$z_1 = \frac{1}{2}\left((2x_1+1)^2 + 1\right) = x_1^2 + (x_1+1)^2.$$

Since x, y satisfy (2.2),

$$z_1^2 = \left(\frac{x^2+1}{2}\right)^2 = \frac{y^2+1}{2} = y_1^2 + (y_1+1)^2.$$

Hence the positive integers x_1, y_1 and z_1 satisfy the system (2.1). By Lemma 2.6 we have $x_1 = 1$, $y_1 = 3$, $z_1 = 5$. This implies that $(x, y) = (3, 7)$. ∎

REMARK 2.1. In [6], Cao *et al.* used an elementary method to prove that the only positive integer solutions of $y^2 - 2\left(\frac{x^2-1}{4}\right)^2 = 1$ are $(x, y) = (1, 1)$, $(3, 3)$ and $(7, 17)$. Since Ljunggren's proof of Lemma 2.6 is not elementary, we wonder if an elementary proof of Lemma 2.7 can be provided.

3. Proof of the main theorem. Eliminating Dz^2 from (1.1) we get

$$x^2 - 2y^2 = -1. \tag{3.1}$$

It is well known [4, 5, 10] that all positive integer solutions of the Pell equation (3.1) are given by

$$x + y\sqrt{2} = (1 + \sqrt{2})^{2n-1}, \qquad n \in \mathbb{N}. \tag{3.2}$$

Let $\varrho = 1 + \sqrt{2}$ and $\overline{\varrho} = 1 - \sqrt{2}$. Then (3.2) yields

$$x = \frac{1}{2}\,(\varrho^{2n-1} + \overline{\varrho}^{2n-1}), \qquad y = \frac{1}{2\sqrt{2}}\,(\varrho^{2n-1} - \overline{\varrho}^{2n-1}), \qquad n \in \mathbb{N}.$$

Let sequences $\{\xi_n\}$ and $\{\eta_n\}$ be such that $\xi_n + \eta_n\sqrt{2} = \varrho^n$, $n \in \mathbb{Z}$. Then

$$\xi_n^2 - 2\eta_n^2 = (-1)^n, \tag{3.3}$$

$$\xi_{2n} = \xi_n^2 + 2\eta_n^2 = 2\xi_n^2 - (-1)^n = 4\eta_n^2 + (-1)^n, \tag{3.4}$$

$$\eta_{2n} = 2\xi_n\eta_n. \tag{3.5}$$

One can easily check that $\{\xi_n\}$ and $\{\eta_n\}$ satisfy

$$\xi_{n+2} = 2\xi_{n+1} + \xi_n, \qquad \xi_0 = \xi_1 = 1,$$
$$\eta_{n+2} = 2\eta_{n+1} + \eta_n, \qquad \eta_0 = 0, \quad \eta_1 = 1.$$

Thus we have the following table:

Table 1

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	⋯
ξ_n	1	1	3	7	17	41	99	239	577	1393	3363	8119	19601	47321	⋯
η_n	0	1	2	5	12	29	70	169	408	985	2378	5741	13860	33461	⋯

Direct verification leads to

$$\xi_{2n-1}^2 - 1 = 8\xi_{n-1}\xi_n\eta_{n-1}\eta_n. \tag{3.6}$$

Also for any $n \in \mathbb{Z}$, ξ_{n-1}, ξ_n, η_{n-1}, η_n are pairwise coprime (see [3, 12]). By (3.2) we have $x = \xi_{2n-1}$, $n \in \mathbb{N}$. Therefore, from (1.1) and (3.6) we have

$$Dz^2 = \frac{1}{2}\,(x^2 - 1) = \frac{1}{2}\,(\xi_{2n-1}^2 - 1) = 4\xi_{n-1}\xi_n\eta_{n-1}\eta_n, \qquad 1 < n \in \mathbb{N}. \tag{3.7}$$

Now consider the following four cases.

CASE 1: $n \equiv 0 \pmod 4$. Let $n = 4m$ for some $m \in \mathbb{N}$. By (3.7) and (3.5) we have

$$Dz^2 = 4\xi_{4m-1}\xi_{4m}\eta_{4m-1}\eta_{4m} = 4^2\xi_{4m-1}\xi_{4m}\eta_{4m-1}\xi_{2m}\xi_m\eta_m, \tag{3.8}$$

where $\xi_{4m-1}, \xi_{4m}, \eta_{4m-1}, \xi_{2m}, \xi_m, \eta_m$ are pairwise coprime.

(a) If $\xi_k \in \{\xi_{4m-1}, \xi_{4m}, \xi_{2m}, \xi_m\}$ is such that $\xi_k = u^2$ is a perfect square for some $u \in \mathbb{N}$, then by (3.3) we have

$$u^4 - 2\eta_k^2 = (-1)^k, \quad k \in \{4m-1, 4m, 2m, m\}. \tag{3.9}$$

By Lemmas 2.1 and 2.2 we have $\xi_k = u^2 = 1$. Thus $k = 0$ or 1 and hence $m = 1$. Using Table 1 and (3.8) we find that $D = 3 \cdot 5 \cdot 7 \cdot 17$. The corresponding positive integer solution is $(x, y, z) = (239, 169, 4)$.

(b) Suppose $\eta_k \in \{\eta_{4m-1}, \eta_m\}$. If $\eta_k = v^2$ is a perfect square for some $v \in \mathbb{N}$, then by (3.3) we get

$$\xi_k^2 - 2v^4 = (-1)^k, \quad k \in \{4m-1, m\}. \tag{3.10}$$

If k is even, then from Lemma 2.3 we see that (3.10) is impossible since $v \in \mathbb{N}$. If k is odd, then Lemma 2.4 shows that positive integer solutions to (3.10) are $\xi_k = 1$ and 239. This implies that $k = 1, 7$. But as $k \in \{4m-1, m\}$ for $m \in \mathbb{N}$, we have $m = 1, 2$ or 7. For $m = 1$, $D = 3 \cdot 5 \cdot 7 \cdot 17$; this solution has already been computed in (a). For $m = 2$, $D = 2 \cdot 3 \cdot 17 \cdot 239 \cdot 577$. The corresponding positive integer solution is $(x, y, z) = (275807, 195025, 52)$.

Similarly, putting $m = 7$ into (3.8) we have

$$Dz^2 = (4 \cdot 13)^2 \cdot 239\xi_{27}\xi_{28}\eta_{27}\xi_{14}. \tag{3.11}$$

From (3.4) we have $\xi_{14} = 4\eta_7^2 - 1 = (2\eta_7 - 1)(2\eta_7 + 1) = 337 \cdot 339 = 3 \cdot 113 \cdot 337$. Since ξ_{27}, ξ_{28}, η_{27} are not perfect squares, by (3.11), D has at least seven prime factors, contrary to assumption.

(c) If none of the $\xi_{4m-1}, \xi_{4m}, \eta_{4m-1}, \xi_{2m}, \xi_m, \eta_m$ is a perfect square, then D in (3.8) has at least six prime factors. Hence D has exactly six prime factors since D is a product of fewer than seven primes. This implies that each of ξ_{4m-1}, ξ_{4m}, η_{4m-1}, ξ_{2m}, ξ_m, η_m is a product of a prime and a square.

Suppose m is even. Then from (3.5) we have $\eta_m = 2\xi_{m/2}\eta_{m/2}$. Since η_m is a product of a prime and a square, either $\xi_{m/2}$ is a square or $\eta_{m/2}$ is twice a square. If $\xi_{m/2}$ is a square, then from (a), $m = 2$. This is already computed in (b). Suppose $\eta_{m/2} = 2t^2$ for some $t \in \mathbb{N}$. Then from (3.3), $\xi_{m/2}^2 - 8t^4 = 1$. By Lemma 2.5, we have $\xi_{m/2} = 3$. Hence $m = 4$. Thus (3.8) becomes $Dz^2 = 4^3 \cdot 3 \cdot 17 \cdot 577\xi_{15}\xi_{16}\eta_{15}$. One can check that $\xi_{15} = 7 \cdot 41 \cdot 31^2$ and none of ξ_{16} and η_{15} is a square. Hence D has at least seven prime factors, which is not the case.

Suppose m is odd. From (3.4) we have $\xi_{2m} = 4\eta_m^2 - 1 = (2\eta_m - 1)(2\eta_m + 1)$. Since ξ_{2m} is a product of a prime and a square, we have $2\eta_m - 1 = s^2$ or

$2\eta_m+1 = s^2$ for some $s \in \mathbb{N}$. Since m is odd, η_m is odd. Taking residues modulo 4 implies that we can only have $2\eta_m - 1 = s^2$, that is, $\eta_m = (s^2+1)/2$. By (3.3),

$$\xi_m^2 - 2\left(\frac{s^2+1}{2}\right)^2 = -1.$$

By Lemma 2.7, $\xi_m = 1, 7$. Thus $m = 1, 3$. We only have to compute the case $m = 3$. Substituting $m = 3$ into (3.8) and then using Table 1, we obtain

$$\begin{aligned} Dz^2 &= 4^2\xi_{11}\xi_{12}\eta_{11}\xi_6\xi_3\eta_3 = 4^2 \cdot 8119 \cdot 19601 \cdot 5741 \cdot 99 \cdot 7 \cdot 5 \\ &= 12^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot 353 \cdot 1153 \cdot 5741. \end{aligned}$$

Hence D has eight prime factors, contrary to assumption.

CASE 2: $n \equiv 1 \pmod 4$. Let $n = 4m+1$ for some $m \in \mathbb{N}$. By (3.5) and (3.7) we have

$$Dz^2 = 4^2\xi_{4m}\xi_{4m+1}\eta_{4m+1}\xi_{2m}\xi_m\eta_m, \tag{3.12}$$

where ξ_{4m}, ξ_{4m+1}, η_{4m+1}, ξ_{2m}, ξ_m, η_m are pairwise coprime. The same arguments as in Case 1 lead to the following results.

If $\xi_k \in \{\xi_{4m}, \xi_{4m+1}, \xi_{2m}, \xi_m\}$ is a square, then $m = 1$.

If $\eta_k \in \{\eta_{4m+1}, \eta_m\}$ is a square, then $m = 1, 7$.

If no member of $\{\xi_{4m}, \xi_{4m+1}, \eta_{4m+1}, \xi_{2m}, \xi_m, \eta_m\}$ is a square, then D in (3.12) has at least six prime factors. Hence D has exactly six prime factors. Therefore $m = 1, 2, 3, 4$.

For $m = 1$, by Table 1, (3.12) yields $Dz^2 = 4^2\xi_4\xi_5\eta_5\xi_2\xi_1\eta_1 = 4^2 \cdot 3 \cdot 17 \cdot 29 \cdot 41$. Thus $D = 3 \cdot 17 \cdot 29 \cdot 41$. Hence the positive integer solution is $(x, y, z) = (1393, 985, 4)$.

When $m = 2$, we have $Dz^2 = 4^2\xi_8\xi_9\eta_9\xi_4\xi_2\eta_2 = 4^2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 197 \cdot 199 \cdot 577$. Thus D has eight prime factors, which cannot happen.

For $m = 3$, we have $Dz^2 = 4^2\xi_{12}\xi_{13}\eta_{13}\xi_6\xi_3\eta_3 = 4^2\xi_{13}\eta_{13} \cdot 19601 \cdot 99 \cdot 7 \cdot 5 = 12^2\xi_{13}\eta_{13} \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 1153$. Therefore D has at least seven prime factors.

If $m = 4$, then

$$\begin{aligned} Dz^2 &= 4^2\xi_{16}\xi_{17}\eta_{17}\xi_8\xi_4\eta_4 = 4^3\xi_{16}\xi_{17}\eta_{17} \cdot 3 \cdot 17 \cdot 577 \\ &= 4^3 \cdot 3 \cdot 17 \cdot 103 \cdot 577 \cdot 15607 \cdot 665857\eta_{17}. \end{aligned}$$

This implies that D has at least seven prime factors.

Finally, for $m = 7$, we have

$$Dz^2 = (4 \cdot 13)^2 \cdot 239\xi_{28}\xi_{29}\eta_{29}\xi_{14} = (4 \cdot 13)^2 \cdot 3 \cdot 113 \cdot 239 \cdot 337\xi_{28}\xi_{29}\eta_{29}.$$

Since ξ_{28}, ξ_{29} and η_{29} are not squares this implies that D has at least seven prime factors. Again this contradicts the hypothesis.

Case 3: $n \equiv 2 \pmod 4$. Let $n = 4m - 2$ for $m \in \mathbb{N}$. Again from (3.5) and (3.7) we have

$$Dz^2 = 4\xi_{4m-3}\xi_{4m-2}\eta_{4m-3}\eta_{4m-2} = 2^3\xi_{4m-3}\xi_{4m-2}\eta_{4m-3}\xi_{2m-1}\eta_{2m-1}, \tag{3.13}$$

where any two of ξ_{4m-3}, ξ_{4m-2}, η_{4m-3}, ξ_{2m-1} and η_{2m-1} are coprime. Exactly the same argument as above shows that if $\xi_k \in \{\xi_{4m-3}, \xi_{4m-2}, \xi_{2m-1}\}$ is a square, then $m = 1$; if $\eta_k \in \{\eta_{4m-3}, \eta_{2m-1}\}$ is a square, then $m = 1$ or 4; if neither ξ_{4m-3}, ξ_{4m-2}, η_{4m-3}, ξ_{2m-1} nor η_{2m-1} is a square, then since ξ_{4m-3}, ξ_{4m-2}, η_{4m-3}, ξ_{2m-1} and η_{2m-1} are all odd, D in (3.13) has at least six, hence exactly six prime factors. Since $\xi_{4m-2} = 4\eta_{2m-1}^2 - 1$, we get $m = 1$ or 2.

For $m = 1$, from Table 1 and (3.13) we have $Dz^2 = 2^3\xi_1\xi_2\eta_1\xi_1\eta_1 = 2^3 \cdot 3$. Hence $D = 2 \cdot 3$ and the corresponding positive integer solution is $(x, y, z) = (7, 5, 2)$.

For $m = 2$, we have $Dz^2 = 2^3\xi_5\xi_6\eta_5\xi_3\eta_3 = 6^2 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \cdot 29 \cdot 41$. So, $D = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 29 \cdot 41$ and the corresponding positive integer solution is $(x, y, z) = (8119, 5741, 6)$.

Substituting $m = 4$ into (3.2) we have $Dz^2 = 2^3\xi_{13}\xi_{14}\eta_{13}\xi_7\eta_7 = 26^2 \cdot 2 \cdot 3 \cdot 79 \cdot 113 \cdot 239 \cdot 337 \cdot 599 \cdot 33461$. This implies that D has eight prime factors.

Case 4: $n \equiv 3 \pmod 4$. Put $n = 4m - 1$ for $m \in \mathbb{N}$. In this case we infer from (3.5) and (3.7) that

$$Dz^2 = 2^2\xi_{4m-2}\xi_{4m-1}\eta_{4m-2}\eta_{4m-1} = 2^3\xi_{4m-2}\xi_{4m-1}\eta_{4m-1}\xi_{2m-1}\eta_{2m-1}, \tag{3.14}$$

where $\xi_{4m-2}, \xi_{4m-1}, \eta_{4m-1}, \xi_{2m-1}, \eta_{2m-1}$ are pairwise coprime.

Again, arguing as before we deduce that: if $\xi_k \in \{\xi_{4m-2}, \xi_{4m-1}, \xi_{2m-1}\}$ is a square, then $m = 1$; if $\eta_k \in \{\eta_{4m-1}, \eta_{2m-1}\}$ is a square, then $m = 1, 2, 4$; if none of ξ_{4m-2}, ξ_{4m-1}, η_{4m-1}, ξ_{2m-1}, η_{2m-1} is a square, then since ξ_{4m-2}, ξ_{4m-1}, η_{4m-1}, ξ_{2m-1} and η_{2m-1} are all odd, D in (3.14) must have at least six, has exactly six, prime factors. Hence from $\xi_{4m-2} = 4\eta_{2m-1}^2 - 1$ we get $m = 1, 2$.

When $m = 1$, D satisfies $Dz^2 = 2^3\xi_2\xi_3\eta_3\xi_1\eta_1 = 2^3 \cdot 3 \cdot 5 \cdot 7$. Thus $D = 2 \cdot 3 \cdot 5 \cdot 7$ and hence the positive integer solution is $(x, y, z) = (41, 29, 2)$.

When $m = 2$, we see from (3.14) that $Dz^2 = 2^3\xi_6\xi_7\eta_7\xi_3\eta_3 = 78^2 \cdot 2 \cdot 5 \cdot 7 \cdot 11 \cdot 239$. This forces $D = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 239$. The corresponding positive integer solution is $(x, y, z) = (47321, 33461, 78)$.

For $m = 4$, we get $Dz^2 = 2^3\xi_{14}\xi_{15}\eta_{15}\xi_7\eta_7 = 26^2 \cdot 2 \cdot 3 \cdot 113 \cdot 239 \cdot 237\xi_{15}\eta_{15}$. This implies that D has at least seven prime factors. Hence (1.1) has no solution in this case.

Having exhausted all the possible cases, we have thus completed the proof of the theorem. ∎

4. Further discussion. Let $\omega(D)$ denote the number of distinct prime factors of D. Theorem 1.3 provides all positive integer solutions of (1.1) for all D with $\omega(D) \le 6$. The question arises whether all positive integer solutions can be found for all D with $\omega(D) = 7$. But this seems difficult to decide by using the proof of Theorem 1.3 without additional conditions. In particular, we do not know whether there are finitely many D with $\omega(D) = 7$ such that (1.1) has a positive integer solution. If some suitable conditions are imposed, then all positive integer solutions of (1.1) with $\omega(D) = 7$ may be easily obtained. For example, we may assume that D does not contain a prime factor of the form $8k + 3$, nor of the form $8k + 1$.

We would like to pose the following conjecture:

There are finitely many D with $\omega(D) = 7$ such that (1.1) *has a positive integer solution.*

A natural question is:

For any constant $k \ge 7$, are there finitely many D with $\omega(D) \le k$ such that (1.1) *has a positive integer solution?*

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