ACTA ARITHMETICA
114.1 (2004)

Uniform bounds for Fourier coefficients of theta-series with arithmetic applications

by

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1. Introduction. An integer n is said to be *squareful* if $p \mid n$ implies $p^2 \mid n$ for all primes p. Erdős and Ivić conjectured that every sufficiently large number can be written as a sum of three squareful numbers. Heath-Brown [11] showed slightly more, namely that it is enough to take one squareful number and two squares. We shall investigate the number of such representations; to be precise, we consider

$$\begin{aligned} R_3(n) &:= \#\{(s_1, s_2, s_3) \in \mathcal{S}^3 \mid n = s_1 + s_2 + s_3\}, \\ R_1(n) &:= \#\{(s_1, t_2, t_3) \in \mathcal{S} \times \mathcal{T}^2 \mid n = s_1 + t_2 + t_3\}, \\ R_3^*(n) &:= \#\{(s_1, s_2, s_3) \in (\mathcal{S} \setminus \mathcal{T})^3 \mid n = s_1 + s_2 + s_3\} \end{aligned}$$

where $\mathcal{S}$ is the set of squareful numbers and $\mathcal{T}$ is the set of perfect squares. It is natural to regard this as a problem of the representation of integers by certain positive definite ternary quadratic forms since a squareful number n can uniquely be written as $n = a^3b^2$, $\mu^2(a) = 1$. Thus we have for example

$$\begin{aligned} R_1(n) = \sum_{\mu^2(d)=1} &\#\{(x, y, z) \in \mathbb{N} \times \mathbb{N}_0^2 \mid d^3x^2 + y^2 + z^2 = n\} \\ &+ \#\{(y, z) \in \mathbb{N}_0^2 \mid y^2 + z^2 = n\}. \end{aligned} \tag{1.1}$$

For a positive k-ary quadratic form f let

$$r(f, n) := \#\{\mathbf{x} \in \mathbb{Z}^k \mid f(\mathbf{x}) = n\}.$$

The corresponding theta-series

$$\theta(f, z) := \sum_{n=0}^{\infty} r(f, n)e(nz) = \sum_{\mathbf{x} \in \mathbb{Z}^k} e(f(\mathbf{x})z)$$

is a modular form of weight $k/2$. A good approximation for $r(f, n)$ in the

2000 *Mathematics Subject Classification*: 11E25, 11F11, 11E20, 11N25.

Key words and phrases: squareful numbers, ternary quadratic forms, Fourier coefficients of modular forms of half-integral weight, Shimura lifting, asymptotic behaviour.

case $k = 3$ is given by the weighted mean $r(\operatorname{spn} f, n)$ (see Section 2 for the definitions); in many cases this coincides with Siegel's mean $r(\operatorname{gen} f, n)$ which is the product of all local densities. To control the error term, we thus have to estimate the Fourier coefficients of $\theta(f, z) - \theta(g, z)$ for f, g in the same spinor genus. This difference is a cusp form with the additional property that also its Shimura lifts are cusp forms ([21]). The generalized Ramanujan–Petersson conjecture predicts an upper bound for the order of Fourier coefficients of cusp forms, but has not been proved so far for forms of half-integral weight. By the celebrated results of Iwaniec [12] and their extension by Duke [5], however, there exist sufficiently strong estimates to detect $r(\operatorname{spn} f, n)$ as a main term, at least for squarefree n. Results by Schulze-Pillot [7, 21, 22] indicate how to extend this to numbers containing a square factor. The key ingredients are (a) Shimura's lift and Deligne's Theorem to deal with square factors prime to the level of the form, and (b) the observation that (for forms coming from theta-series) square factors dividing the level can to large extent be eliminated ([22, Lemmata 3–5]). In fact, Theorem 3 of [7] states

$$r(\operatorname{spn} f, n) - r(f, n) \ll_{\varepsilon,f} n^{1/2-1/28+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$. Unfortunately the proof of this important result is somewhat sketchy and some arguments are missing for numbers n divisible by a large power of 2. In addition, many applications require an explicit dependence on f. The aim of this paper is to extend the result (1.2) by making the dependence on the form f explicit, and to supply a more detailed proof. We shall show that the implied constant increases in the level N of the form polynomially at most:

THEOREM 1. *There is an effective constant A with the following property: Let f be a positive definite ternary quadratic form of level N and let the representation functions $r(\operatorname{spn} f, n)$ and $r(f, n)$ be defined as in Section 2 below. Then for $\varepsilon > 0$ and $N \le n^{1/2}$ we have*

$$r(\operatorname{spn} f, n) - r(f, n) \ll_{\varepsilon} N^A n^{13/28+\varepsilon} \tag{1.3}$$

where the implied constant depends on ε alone. If we restrict ourselves to squarefree n, we have

$$r(\operatorname{spn} f, n) - r(f, n) \ll_{\varepsilon} N n^{13/28+\varepsilon}. \tag{1.4}$$

For all $n = 2^{e_2} \prod_{p \ge 3} p^{e_p}$ we have

$$r(\operatorname{spn} f, n) - r(f, n) \ll_{\varepsilon,e_2} N^{45/28} n^{13/28+\varepsilon}. \tag{1.5}$$

A similar, but weaker result has been obtained in a forthcoming paper by Duke [6]; in fact, he shows (1.4) with $\Delta^{11/2}$ instead of N (Δ being the discriminant).

There are a number of applications of Theorem 1, some of which will be considered elsewhere. Here, with the help of (1.3), we want to deduce estimates for the representation numbers $R_1(n)$, $R_3(n)$ and $R_3^*(n)$. Let

$$l(n,m) := \max_{1\le d\le m} (L(1,\chi_{-4nd}))$$

which is bounded by $O(\log\log n)$ for $m \le n$ if the Generalized Riemann Hypothesis holds. (Here χ_D is the Kronecker symbol, see below.)

THEOREM 2. *For $\varepsilon > 0$ and sufficiently large n we have*

$$n^{1/2}(\log n)^{-1/2-\varepsilon} \ll_\varepsilon R_1(n) \ll n^{1/2}(\log\log n)l(n,(\log n)^3).$$

The lower bound holds also for $R_3^*(n)$.

Probably (i.e. if GRH is true) the lower bound in Theorem 2 is in general up to a power $(\log n)^\varepsilon$ best possible, as our next theorem shows. In particular, there are (probably) infinitely many integers n having exceptionally small representation numbers $R_1(n)$.

THEOREM 3. *There is an infinite set $\mathcal{N}$ of integers such that for any $\varepsilon > 0$ and for all $n \in \mathcal{N}$,*

$$R_1(n) \ll_\varepsilon n^{1/2}(\log n)^{-1/2+\varepsilon} l(n,(\log n)^3).$$

The upper bound in Theorem 2 holds essentially also for $R_3(n)$ up to a thin set of exceptions: There exists a $\delta > 0$ such that

$$\#\{n \le x \mid R_3(n) \ge n^{1/2}(\log\log n)l(n,(\log n)^7)\xi(n)\} \ll_\xi x^{1-\delta}$$

for all functions ξ with $\xi(x) \to \infty$ for $x \to \infty$.

Although the second part of Theorem 3 is most likely far from being best possible, it seems very difficult to obtain better results. We remark that all implied constants in Theorems 1–3 can be made effective.

Notation. For a real number x let $\lceil x\rceil := \min(r \in \mathbb{Z} \mid r \ge x)$, $\lfloor x\rfloor := \max(r \in \mathbb{Z} \mid r \le x)$, $e(x) = \exp(2\pi i x)$. The letter p is reserved for (positive) prime numbers. $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p$ the ring of p-adic integers; $\mathrm{ord}_p n$ denotes the exponent of p in the factorization of n. As usual the value of ε may change during a calculation. The (extended) Jacobi–Kronecker symbol χ_Δ is the completely multiplicative function given by

$$\chi_\Delta(p) = \left(\frac{\Delta}{p}\right) \quad \text{for odd } p,$$

$$\chi_\Delta(2) = \begin{cases} 1 & \text{for } \Delta \equiv 1 \pmod 8, \\ -1 & \text{for } \Delta \equiv 5 \pmod 8, \\ 0 & \text{if } 2 \mid \Delta, \end{cases}$$

$$\chi_\Delta(-1) = \begin{cases} 1 & \text{for } \Delta \ge 0, \\ -1 & \text{for } \Delta < 0. \end{cases}$$

Acknowledgements. I would like to thank Prof. R. Schulze-Pillot for suggesting a way to fill a gap in the paper [7] and Dr. R. Dietmann for useful discussions.

2. Quadratic forms and modular forms. For convenience we compile some classical definitions and results on quadratic forms and modular forms which are dispersed over the literature.

Let $(\mathbb{Q}^k, f)$ be a regular positive definite quadratic space. We have three equivalence relations on the set of $\mathbb{Z}$-lattices which yield a partition into classes, spinor genera and genera (see e.g. [15]). For a lattice L with $f(L) \subseteq \mathbb{Z}$ and a $\mathbb{Z}$-basis $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of L we obtain the symmetric matrix $A = A_L = (B(\mathbf{v}_i, \mathbf{v}_j))_{i,j} \in \mathrm{GL}_k(\mathbb{Z})$ where $B(\mathbf{x}, \mathbf{y}) = f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})$ is the bilinear form corresponding to f. (Here the trouble with the prime 2 begins because several authors insert a factor $\frac{1}{2}$ into the definition of B.) A change of base in L yields a conjugate matrix $\widetilde{A}$.

Most of the time we shall only speak of quadratic forms $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{t}}A\mathbf{x}$ having symmetric matrices $A \in \mathrm{GL}_k(\mathbb{Z})$ with even diagonal elements (i.e. our quadratic space is $(\mathbb{Q}^k, f)$ with lattice $\mathbb{Z}^k$). Two forms are in the same class (spinor genus, genus) if they have matrices obtained—with respect to any basis—from lattices in the same class (spinor genus, genus) in a suitable quadratic space. Two forms in the same genus with matrices A_1, A_2 are everywhere locally equivalent, i.e. for all p there are $T_p \in \mathrm{GL}_k(\mathbb{Z}_p)$ with $T_p^{\mathrm{t}} A_1 T_p = A_2$. In this case we shall write $A_1 \cong A_2$ over $\mathbb{Z}_p$.

The following invariants are the products of their local components and therefore the same within an entire genus (of forms or lattices): The determinant $\Delta \neq 0$ of the matrix A, the *norm* $\mathfrak{n}$ which is the positive number generating the ideal $f(L)\mathbb{Z}$, and the *level* N which is the smallest number N such that NA^{-1} is integral with even diagonal elements, i.e. $N = \mathfrak{n}(L^{\#})^{-1}$ where $L^{\#}$ is the dual lattice with respect to the bilinear form B. A lattice is called *maximal* if there is no larger lattice with the same norm. (One has to take some care since Eichler's [8] definition of the norm differs from ours by a factor 2.)

A form f over $\mathbb{Z}_p$ is for odd p equivalent to a diagonal form; for $p = 2$ there may be binary summands $2^\nu x_1 x_2$ or $2^\nu(x_1^2 + x_1x_2 + x_2^2)$ with $\nu \geq 0$ (cf. [13, Ch. 4]). This implies $2 \mid \Delta$ and $4 \mid N$ for odd k. A form f over a ring R is called *isotropic* if there is a solution $\mathbf{x} \in R^k \setminus \{0\}$ of $f(\mathbf{x}) = 0$, otherwise it is *anisotropic*. A binary primitive isotropic form over $\mathbb{Z}_p$ is for odd p equivalent to a hyperbolic plane x_1x_2. An application of Hensel's Lemma shows:

Lemma 2.1. *Let p be odd and $f \cong f_1 + pf_2$ over $\mathbb{Q}_p$ where f_1, f_2 are* (*possibly empty*) *diagonal forms and the coefficients of f_1 and f_2 are not* *divisible by p. Then f is anisotropic over $\mathbb{Z}_p$ if and only if f_1 and f_2 are anisotropic over $\mathbb{Z}/p\mathbb{Z}$.*

For $p=2$ note that $2^\nu x_1x_2 + a_3x_3^2$ is isotropic, and $2^\nu(x_1^2+x_1x_2+x_2^2) + a_3x_3^2$ is isotropic if and only if $\operatorname{ord}_2 a_3 \equiv \nu \pmod 2$. For diagonal forms one may use [13, p. 36].

For the positive definite ternary quadratic space $V=(\mathbb{Q}^3,f)$ the second Clifford algebra $C_2(V)$ (see [8]) is a definite quaternion algebra over $\mathbb{Q}$ that ramifies at a prime $p \in \mathbb{P}\cup\{\infty\}$ if and only if f is anisotropic over $\mathbb{Q}_p$ (see [15, (57:9)]). $C_2(V)$ becomes a positive definite quadratic space with the reduced norm nr. For every order $\mathfrak{O} \subseteq C_2(V)$ with $\mathbb{Z}$-basis $(\mathbf{v}_1,\dots,\mathbf{v}_4)$ we define the complement $\mathfrak{O}^\# = \bigoplus_{i=1}^4 \mathbf{w}_i\mathbb{Z}$ where $\operatorname{tr}(\overline{\mathbf{v}}_i\mathbf{w}_j) = \overline{\mathbf{v}}_i\mathbf{w}_j + \overline{\mathbf{w}}_j\mathbf{v}_i = \delta_{ij}$. The level $q(\mathfrak{O})$ ("Grundideal") is defined as $\mathfrak{n}(\mathfrak{O}^\#)^{-1}$, where we consider $\mathfrak{O}^\#$ as a lattice on the quadratic space $(C_2(V),\mathrm{nr})$, thus it is the level of the associated normform.

To every lattice L on V there corresponds by Satz 14.1 of [8] a certain order $\mathfrak{O}$ in $C_2(L)$ with (see [16, Satz 7])

$$q(\mathfrak{O}) = \frac{1}{2}\,\Delta(L)\mathfrak{n}(L)^{-3}. \tag{2.1}$$

Let $L=L_1,\dots,L_h$ be a set of representatives of the classes in the genus of L and $\mathfrak{O}_1,\dots,\mathfrak{O}_h$ the associated orders; they form a complete set of representatives of the types of locally conjugate orders (cf. [16, Sätze 1, 8, 4]). Let $e_j = |\mathfrak{O}_j^*|$ be the (finite) number of units of $\mathfrak{O}_j$ and let $\mathfrak{I}_1,\dots,\mathfrak{I}_H$ be a set of representatives of the classes of left $\mathfrak{O}_1$-ideals. For $1 \le i,j \le H$ and $n \in \mathbb{N}$ let $b_{ij}(n)$ be the number of integral left $\mathfrak{O}_i$-ideals of norm n equivalent to $\mathfrak{I}_i^{-1}\mathfrak{I}_j$. These are exactly the ideals $\mathfrak{I}_i^{-1}\mathfrak{I}_j a$ with norm n and $a \in \mathfrak{I}_j^{-1}\mathfrak{I}_i$. Thus $b_{ij}(n)$ is $1/e_j$ times the number of representations of n by the lattice $\frac{1}{\mathfrak{n}(\mathfrak{I}_j^{-1}\mathfrak{I}_i)}\mathfrak{I}_j^{-1}\mathfrak{I}_i$ in the quadratic space $(C_2(V),\mathrm{nr})$. The *reduced Brandt matrix* $\overline{B}(n)$ is defined as follows (cf. [18, 22]): The right order of $\mathfrak{I}_i$ $(1 \le i \le H)$ is one of the $\mathfrak{O}_j$. This gives a surjective map $\lambda : \{1,\dots,H\} \to \{1,\dots,h\}$. $\overline{B}(n)$ is the $h\times h$-matrix with entries $\overline{b}_{ij}(n) = \sum b_{kl}(n)$ where k is an arbitrary fixed index with $\lambda(k)=i$ and the sum is over all l with $\lambda(l)=j$.

Let $r(f,n) := \#\{\mathbf{x}\in\mathbb{Z}^3 \mid f(\mathbf{x}) = n\}$, let $o(f) = \#\{T \in \mathrm{SL}_3(\mathbb{Z}) \mid T^{\mathrm{t}}AT = A\}$ be the (finite) number of automorphs of f, and define the weighted means

$$r(\operatorname{gen} f,n) := \Bigg(\sum_{\widetilde{f}\in\operatorname{gen} f} \frac{1}{o(\widetilde{f})}\Bigg)^{-1} \sum_{\widetilde{f}\in\operatorname{gen} f} \frac{r(\widetilde{f},n)}{o(\widetilde{f})},$$

$$r(\operatorname{spn} f,n) := \Bigg(\sum_{\widetilde{f}\in\operatorname{spn} f} \frac{1}{o(\widetilde{f})}\Bigg)^{-1} \sum_{\widetilde{f}\in\operatorname{spn} f} \frac{r(\widetilde{f},n)}{o(\widetilde{f})}$$

where the summations are taken over a set of representatives of all classes in the genus and spinor genus of f respectively. By a well known result of Siegel [25], $r(\operatorname{gen} f, n)$ can be obtained by local computations:

PROPOSITION 2.2 (Siegel). *Let f be a positive definite ternary quadratic form and*

$$r_p(f,n) := \lim_{\nu\to\infty} p^{-2\nu} \#\{\mathbf{x} \in (\mathbb{Z}/p^\nu\mathbb{Z})^3 \mid f(\mathbf{x}) \equiv n \pmod{p^\nu}\}$$

the p-adic densities. Then

$$r(\operatorname{gen} f, n) = 2\pi\sqrt{\frac{8n}{\Delta}} \prod_p r_p(f,n).$$

Let

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}), \qquad \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\},$$
$$\Gamma(N) = \{\gamma \in \Gamma \mid \gamma \equiv I \pmod{N}\},$$

and

$$\begin{aligned} \mu &:= [\Gamma : \Gamma_0(N)] = N \prod_{p\mid N}\left(1 + \frac{1}{p}\right), \\ \widetilde{\mu} &:= [\Gamma : \Gamma(N)] = N^3 \prod_{p\mid N}\left(1 - \frac{1}{p^2}\right). \end{aligned} \tag{2.2}$$

We write

$$\Gamma = \bigcup_{j=1}^{\mu} \Gamma_0(N)\sigma_j = \bigcup_{j=1}^{\widetilde{\mu}} \Gamma(N)\tau_j \quad \text{with } \sigma_j, \tau_j \in \Gamma.$$

If $F := \{z \in \mathbb{C} \mid |z| \ge 1, |\Re z| \le 1/2\}$ is the standard fundamental domain for Γ, then $F_0(N) = \bigcup_{j=1}^{\mu} \sigma_j F$ and $F(N) = \bigcup_{j=1}^{\widetilde{\mu}} \tau_j F$ are fundamental domains for $\Gamma_0(N), \Gamma(N)$, respectively. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ we define the θ-multiplier by

$$j(\gamma, z) = \left(\frac{c}{d}\right)\left(\frac{-1}{d}\right)^{-1/2} (cz+d)^{1/2}$$

where $\left(\frac{c}{d}\right)$ is the extended Kronecker symbol. We always take the branch of the square root having argument in $(-\pi/2, \pi/2]$. For a holomorphic function on the upper half-plane $\phi : H \to \mathbb{C}$ and $\gamma \in \Gamma$ we write

$$\phi(z) \mid [\gamma]_k := \begin{cases} (cz+d)^{-k}\phi(\gamma z) & \text{if } k \in \mathbb{N}, \\ j(\gamma,z)^{-2k}\phi(\gamma z) & \text{if } k \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N} \text{ and } \gamma \in \Gamma_0(4). \end{cases}$$

For a character $\chi \pmod{N}$ and a positive integer or half-integer $k \ge 3/2$ we denote by $M_k(N,\chi)$ and $S_k(N,\chi)$ the spaces of modular forms and cusp forms of weight k for $\Gamma_0(N)$ (where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$) and character χ

respectively. The spaces $S_k(N,\chi)$ become Hilbert spaces via the Petersson scalar product for the group $\Gamma_0(N)$,

$$\langle \phi_1, \phi_2 \rangle = \int\limits_{F_0(N)} \phi_1(z)\overline{\phi}_2(z) y^k \,\frac{dx\,dy}{y^2}.$$

We remark that some authors (e.g. [19]) insert a factor $1/\mu$ into this definition. By [19, Theorem 4.2.1], we have

$$\dim_{\mathbb{C}} S_k(N,\chi) \le 1 + \frac{k\mu}{12} \ll_k N \log\log N. \tag{2.3}$$

For $(n,N)=1$ let $T(n): M_k(N,\chi) \to M_k(N,\chi)$ be the Hecke operator (see [19, 24]). By Deligne's celebrated proof of the Ramanujan–Petersson conjecture ([3, p. 302]) for $k \in \mathbb{N}$, $k \ge 2$, we have

Proposition 2.3 (Deligne). *Let λ_n be an eigenvalue for $T(n)|_{S_k(N,\chi)}$. Then $|\lambda_n| \le d(n) n^{(k-1)/2}$.*

Finally, we need Shimura's correspondence: Let $k \ge 3$ be odd, N a multiple of 4 and $\varepsilon = (-1)^{(k-1)/2}$. If $\phi = \sum a(n)e(nz) \in S_{k/2}(N,\chi)$ and t is squarefree, we define $A_t(n)$ by the formal identity

$$\sum_{n=1}^{\infty} A_t(n) n^{-s} = \Big(\sum_{n=1}^{\infty} a(tn^2) n^{-s}\Big) L(s-k/2+3/2, \chi_{4\varepsilon t}\chi).$$

Then $\Phi_t(z) = \sum A_t(n)e(nz)$ is called the *t-Shimura lift* of ϕ, and we have a mapping

$$S_{k/2}(N,\chi) \xrightarrow{t\text{-Shimura}} M_{k-1}(N/2,\chi^2)$$

that commutes with the Hecke operators (see [2, 24]). In particular, if ϕ is an eigenform for $T(p^2)$ with eigenvalue λ_p, then Φ_t (if not 0) is an eigenform for $T(p)$ with the same eigenvalue. For $k \ge 5$, $M_{k-1}(N/2,\chi^2)$ can be replaced with $S_{k-1}(N/2,\chi^2)$. For $k=3$ (which concerns us here) we denote by $V(N,\chi)$ the subspace of $S_{3/2}(N,\chi)$ that is mapped into $S_2(N/2,\chi^2)$ under all t-Shimura lifts. $V(N,\chi)$ is invariant under all $T(p^2)$ and has a base of eigenforms for all $T(p^2)$ with $p \nmid N$. If $\phi = \sum a(n)e(nz) \in S_{k/2}(N,\chi)$ is an eigenform of all $T(p^2)$ $(p \nmid N)$ with eigenvalue λ_p, and t is an integer having no square factor (different from 1) prime to N, then by [24, p. 452] and the properties of the λ_n we have the formal identity

$$\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s} = \sum_{p|n \Rightarrow p|N} \frac{a(tn^2)}{n^s} \Big(\sum_{(n,N)=1} \frac{\lambda_n}{n^s}\Big) (L(s-k/2+3/2, \chi_{4\varepsilon t}\chi))^{-1}. \tag{2.4}$$

Let

$$\theta(f,z) := \sum_{n=0}^{\infty} r(f,n)e(nz) = \sum_{\mathbf{x}\in\mathbb{Z}^k} e(f(\mathbf{x})z) = \sum_{\mathbf{x}\in\mathbb{Z}^k} \exp(\pi i \mathbf{x}^{\mathrm{t}} A\mathbf{x} z)$$

be the theta-series of a k-ary positive definite quadratic form f. Then $\theta(f,z) \in M_{k/2}(N,\chi)$ for the character $\chi = \chi_{(-1)^{k/2}\Delta}$ if k is even and $\chi = \chi_{2\Delta}$ if k is odd (cf. [8, p. 142] and [24, p. 456]). If f and g are two forms of level N in the same genus, then $\theta(f,z) - \theta(g,z) \in S_{k/2}(N,\chi)$ (see [25, p. 577], the calculation there being unchanged for odd k). If f and g are in the same spinor genus and $k = 3$, then $\theta(f,z) - \theta(g,z) \in V(N,\chi_{2\Delta})$ (see [21, Satz 4]).

Let $\overline{B}(n)$ be the reduced Brandt matrix, $\theta_{ij}(z) = \sum_{n=1}^{\infty} \overline{b}_{ij}(n)e(nz)$ the corresponding theta-series, and let L_i, $L_{i'}$ be two lattices in the same spinor genus. By the Proposition in [22] and (2.1), for $1 \le k \le h$ we have

$$\theta_{ik} - \theta_{i'k} \in S_2\left(\frac{1}{2}\,\Delta(L_i)\mathfrak{n}(L_i)^{-3}, 1\right). \tag{2.5}$$

3. Local computations

Lemma 3.1 ([22, Lemma 3]). *Let f be a ternary form of level N, $p \mid N$, $\mathbf{x} \in \mathbb{Z}^3$ with $f(\mathbf{x}) = n$, and assume* $\operatorname{ord}_p n > \operatorname{ord}_p(N/2)$.

(a) *If f is anisotropic over $\mathbb{Z}_p$, then $\mathbf{x} \in p\mathbb{Z}^3$, hence $r(f,n) = r(f,n/p^2)$.*

(b) *If f is isotropic over $\mathbb{Z}_p$, there is a form $\widetilde{f} \in \mathbb{Z}[x_1,x_2,x_3]$ such that*

$$r(\widetilde{f},n) = r(f,n), \qquad A_f \cong A_{f'} \quad \textit{over } \mathbb{Z}_{p'} \textit{ for all } p' \ne p,$$

$$A_{\widetilde{f}} \cong \begin{pmatrix} 0 & p^{s_1} \\ p^{s_1} & 0 \end{pmatrix} \perp (2up^{s_2}) \quad \textit{over } \mathbb{Z}_p$$

with a unit $u \in \mathbb{Z}_p^*$, and $0 \le s_1 \le s_2 \le \operatorname{ord}_p(N) + 1$.

Proof. (a) It is enough to consider f over $\mathbb{Z}_p$. Without loss of generality we may assume $\mathfrak{n}(\mathbb{Z}^3) = 1$ (i.e. f is primitive). For odd p we may diagonalize f; then the assertion is easily seen by divisibility considerations and Lemma 2.1. For $p = 2$ we use Satz 9.4 and 9.6 of [8]: Clearly there is a maximal lattice L with $\mathfrak{n}(L) = 1$ and $\mathbb{Z}_2^3 \subseteq L \subseteq 2^{-c}\mathbb{Z}_2^3$ where $c = \left\lfloor \frac{1}{2} \operatorname{ord}_2(N/4) \right\rfloor$. If M denotes the maximal lattice consisting of all $\mathbf{x}$ with $\operatorname{ord}_2 f(\mathbf{x}) > \operatorname{ord}_2(N/2)$, then $\mathfrak{n}(M) \ge 2^{\operatorname{ord}_2(N/2)+1} \ge 2^{2(c+1)}$, hence $\mathbf{x} \in M \subseteq 2^{c+1}L \subseteq 2\mathbb{Z}_2^3$.

(b) We shall show that $\mathbf{x}$ is in a certain sublattice $L \subseteq \mathbb{Z}_p^3$ such that

$$A_L = \begin{pmatrix} 0 & p^{s_1} \\ p^{s_1} & 0 \end{pmatrix} \perp (2up^{s_2}).$$

Then, by [15, (81:14)], we can find an $\widetilde{f}$ with the desired properties. The lattice L is constructed as follows: If p is odd, we may diagonalize f over $\mathbb{Z}_p$. By obvious divisibility considerations and Lemma 2.1 we see that the x_i's must be divisible by p^{α_i} for some suitable $\alpha_i \in \mathbb{N}_0$ $(1 \le i \le 3)$, and we just take $L := p^{\alpha_1}\mathbb{Z}_p\mathbf{e}_1 + p^{\alpha_2}\mathbb{Z}_p\mathbf{e}_2 + p^{\alpha_3}\mathbb{Z}_p\mathbf{e}_3$.

If $p = 2$, we have to investigate a number of cases. If $2 \parallel \Delta$, we use Satz 9.5 of [8], since $\mathbb{Z}_2^3$ is a maximal isotropic lattice. The same argument can be used for all cases where f contains a binary summand $2^\nu(x_1^2 + x_1x_2 + x_2^2)$. If f is diagonal with unit coefficients (in $\mathbb{Z}_2$), then it is up to units equivalent to $x_1^2 + x_2^2 + 3x_3^2$ (mod 4). Now $f(\mathbf{x}) \equiv 0$ (mod 4) implies $x_1 + x_2 + x_3 \equiv 0$ (mod 2), hence we may take $L = 2\mathbb{Z}_2\mathbf{e}_1 + \mathbb{Z}_2(\mathbf{e}_1 + \mathbf{e}_2) + \mathbb{Z}_2(\mathbf{e}_1 + \mathbf{e}_3)$, obtaining

$$\widetilde{f}(\mathbf{x}) = 4x_1^2 + 4x_1x_2 + 2x_2^2 + 4x_1x_3 + 2x_2x_3 + 4x_3^2 \cong 4x_1^2 + 2x_2x_3$$

(cf. [13, p. 85]). All other diagonal cases are treated similarly.

LEMMA 3.2. *For* $\mathbf{d} \in \mathbb{N}^3$ *define* $f_{\mathbf{d}}(\mathbf{x}) := d_1^3x_1^2 + d_2^3x_2^2 + d_3^3x_3^2$ *with* $\Delta = 8(d_1d_2d_3)^3$. *For brevity we write* f_d *for* $f_{(d,1,1)}$.

(a) *If* $p \nmid n\Delta$, *we have* $r_p(f_{\mathbf{d}}, n) = 1 + \chi_{-2n\Delta}(p)/p$. *If* $p \nmid \Delta$, *then* $1 - 1/p \le r_p(f_{\mathbf{d}}, n) \le 1 + 1/p$.

(b) *Let* p *be odd*, $\operatorname{ord}_p(d_1)$ *odd*, *and* $p \nmid d_2d_3$. *If* $f_{\mathbf{d}}$ *is isotropic over* $\mathbb{Z}_p$ *or if* $p \nmid n$, *then* $r_p(f_{\mathbf{d}}, n) \ge 1 - 1/p$. *Furthermore*, $r_2(f_{\mathbf{d}}, n) \ge 1/4$ *for* $\mathbf{d} \equiv (7, 1, 1)$ (mod 8).

(c) *Let* d_1, d_2, d_3 *be squarefree*, *and* $1 \le \nu := \operatorname{ord}_p(d_1d_2d_3) \le 3$, *then* $r_p(f_{\mathbf{d}}, n) \ll p^{3(\nu-1)/2}$.

Proof. (a) See [25, Hilfssätze 12 and 16].

(b) If $f_{\mathbf{d}}$ is isotropic over $\mathbb{Z}_p$, then $f_{\mathbf{d}}$ is equivalent to $\widetilde{f}(x_1, x_2, x_3) = d_1^3x_1^2 + x_2x_3$ over $\mathbb{Z}_p$. Choosing x_1 arbitrarily and x_2 invertible, one always finds x_3 to solve $\widetilde{f} \equiv n$ (mod p^ν), hence $r_p(f_{\mathbf{d}}, n) \ge 1 - p^{-1}$. If $p \nmid n$, the assertion follows from [25, Hilfssatz 13]. Furthermore, for $\mathbf{d} \equiv (7, 1, 1)$ (mod 8), the form $f_{\mathbf{d}}$ is equivalent to $\widetilde{f}(\mathbf{x}) = -x_1^2 + x_2^2 + x_3^2$ over $\mathbb{Z}_2$. Choosing x_3 with opposite parity as n and t odd, one finds $2^{2\nu-2}$ solutions

$$\left(\frac{-t + (n - x_3^2)t^{-1}}{2}, \frac{t + (n - x_3^2)t^{-1}}{2}, x_3\right)$$

of $\widetilde{f}(\mathbf{x}) \equiv n$ (mod 2^ν).

(c) This is a standard calculation with Gauss sums (cf. e.g. [25, §3]). Let

$$G(h, p^k) := \sum_{x=1}^{p^k} e\left(\frac{hx^2}{p^k}\right).$$

For $p \nmid h$ we have $|G(h,p^k)| = p^{k/2}$ if p is odd and $|G(h,2^k)| = 2^{(k+1)/2}$. If $d_1 = p\widetilde{d}_1$, $p \nmid \widetilde{d}_1 d_2 d_3$, we obtain with $P := p^\nu$ for ν large

$$\begin{aligned} r_p(f_{\mathbf{d}}, n) &= \frac{1}{P^3}\sum_{h=1}^{P}\sum_{x_1,x_2,x_3=1}^{P} e\bigg(h\,\frac{p^3\widetilde{d}_1^3x_1^2}{P}\bigg)e\bigg(h\,\frac{d_2^3x_2^2}{P}\bigg)e\bigg(h\,\frac{d_3^3x_3^2}{P}\bigg)e\bigg(-h\,\frac{n}{P}\bigg)\\ &= \sum_{k=0}^{\nu}\frac{1}{p^{3k}}\sum_{\substack{h=1\\(p,h)=1}}^{p^k} G(p^3h\widetilde{d}_1^3,p^k)G(hd_2^3,p^k)G(hd_3^3,p^k)e\bigg(-h\,\frac{n}{p^k}\bigg)\\ &\ll 1+1+1+\sum_{k=3}^{\infty}\frac{p^{(k+3)/2}p^k\phi(p^k)}{p^{3k}} = O(1). \end{aligned}$$

Analogously we see that $r_p(f_{\mathbf{d}}, n) = O(p^{3/2})$ if $p^2 \,\|\, d_1d_2d_3$, and $r_p(f_{\mathbf{d}}, n) = O(p^3)$ if $p^3 \,\|\, d_1d_2d_3$.

From Lemma 3.2(a) and Proposition 2.2 we infer that

$$r(\mathrm{gen}\, f_{\mathbf{d}}, n) = c(n)n^{1/2}\varDelta^{-1/2}L(1,\chi_{-2n\varDelta})\prod_{p\mid\varDelta} r_p(f_{\mathbf{d}}, n) \tag{3.1}$$

with $(\log\log n)^{-1} \ll c(n) \ll \log\log n$. By Lemma 3.2(c) the series

$$\sum_{\substack{d_1,d_2,d_3\\ \text{squarefree}}} \frac{\prod_{p\mid 2d_1d_2d_3} r_p(f_{\mathbf{d}}, n)}{(d_1d_2d_3)^{3/2}} = \prod_p\bigg(1+\frac{O(1)}{p^{3/2}}\bigg) \tag{3.2}$$

is convergent.

4. Fourier coefficients of theta-series. Let $\phi = \sum a(n)e(nz) \in V(N,\chi)$ (see Section 2) be an eigenform of all $T(p^2)$ $(p \nmid N)$ with eigenvalue λ_p, and t be an integer having no square factor (different from 1) prime to N. Then by (2.4), the Möbius inversion formula and Proposition 2.3 we have

$$|a(tn_0^2)| = \bigg|a(t)\sum_{m\mid n_0}\mu(m)\chi_{4\varepsilon t}\chi(m)\lambda_{n_0/m}\bigg| \le |a(t)|d(n_0)^2n_0^{1/2} \tag{4.1}$$

for $(n_0, N) = 1$. From [10] we cite the uniform bound

$$a(n) \ll \|\phi\| d(n) n^{1/2} \tag{4.2}$$

for the Fourier coefficients of a cusp form $\phi = \sum a(n)e(nz) \in S_2(N,\chi)$ and $(n, N) = 1$.

LEMMA 4.1. (a) *Let f be an integral positive ternary quadratic form. Then $r(f,n) \ll_\varepsilon n^{1/2+\varepsilon}$ for any $\varepsilon > 0$ where the implied constant is absolute.*

(b) *Let f be an integral positive k-ary quadratic form. Then*

$$r(f,n) \ll_k \frac{n^{k/2}}{\sqrt{\Delta}} + n^{(k-1)/2}$$

where the implied constant is independent of f.

Proof. Both statements are essentially known.

(a) We may assume that $f(\mathbf{x}) = ax_1^2 + bx_2^2 + cx_3^2 + rx_2x_3 + sx_1x_3 + tx_1x_2$ is Eisenstein-reduced (see [13, p. 188]), in particular

$$\begin{gathered} 0 \le \max(|s|,|t|) \le a \le b \le c, \qquad |r| \le b, \\ r,s,t \text{ are all positive or all non-positive,} \\ \text{if } r,s,t \le 0 \text{ and } a = -t, \text{ then } s = 0. \end{gathered} \tag{4.3}$$

Then no x_3 can be found to solve $f(\mathbf{x}) = n$ unless

$$4c(ax_1^2 + bx_2^2 + tx_1x_2) - (rx_1 + sx_2)^2 \le 4cn.$$

Since $|x_1x_2| \le (x_1^2 + x_2^2)/2$, we must therefore have

$$(4ac - s^2 - |2ct - rs|)x_1^2 + (4bc - r^2 - |2ct - rs|)x_2^2 \le 4cn.$$

By (4.3) it is easily seen that both coefficients on the left-hand side are at least c, hence $x_1 \ll n^{1/2}$. But if x_1 is fixed, the number of solutions of the remaining binary problem is bounded by n^{ε}.

(b) This is by induction on k. There is nothing to show for $k \le 3$. Assume that $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{t}}A\mathbf{x}$ with $A = (a_{ij})$ is Minkowski-reduced (see [1, Ch. 12]). Let $M_1, \ldots, M_k$ be the successive minima of f ([1, p. 262]). Then $a_{jj} \ll_k M_j$ for $1 \le j \le k$ and $|a_{ij}| \ll a_{ii}$. By a criterion of Gerschgorin the maximal eigenvalue $\lambda_{\max}$ of A is of order $O_k(M_k)$. If $n < M_k$, then the vectors representing n come from a $(k-1)$-dimensional subspace spanned by the vectors representing $M_1, \ldots, M_{k-1}$, and we are done by the induction hypothesis. If $n \ge M_k$, then

$$r(f,n) \le \#\{\mathbf{x} \in \mathbb{Z}^k \mid f(\mathbf{x}) \le n\} \ll_k \frac{n^{k/2}}{\Delta^{1/2}} + O_k\left(\frac{n^{(k-1)/2}}{\Delta^{1/2}\lambda_{\max}^{-1/2}}\right) \ll_k \frac{n^{k/2}}{\sqrt{\Delta}}$$

where the O-term is an upper bound for the surface of the ellipsoid.

Lemma 4.2. *Let f,g be two positive definite k-ary quadratic forms of level N in the same genus. Write $\theta(z) := \theta(f,z) - \theta(g,z)$. Then $\|\theta\| \ll_{\varepsilon} N^{1+\varepsilon}$ for $k = 3$ and $\|\theta\| \ll_k N^{k/2+\varepsilon} + (N\Delta)^{1/2+\varepsilon}$ for $k > 3$ where the norm is induced by the Petersson scalar product for the group $\Gamma_0(N)$. The implied constants are independent of f and g.*

Proof. Let A be the matrix of f. With the notation as in Section 2 we have

$$\|\theta\|^2 = \frac{\mu}{\widetilde{\mu}} \int\limits_{F(N)} |\theta(z)|^2 y^{k/2} \, \frac{dx\, dy}{y^2} = \frac{\mu}{\widetilde{\mu}} \sum_{j=1}^{\widetilde{\mu}} \int\limits_{F} |\theta(z) \mid [\tau_j]_{k/2}|^2 y^{k/2} \, \frac{dx\, dy}{y^2}.$$

On the set $\{\tau_1, \ldots, \tau_{\widetilde{\mu}}\}$ we define an equivalence relation by $\tau_i \sim \tau_j \Leftrightarrow \tau_i \in \Gamma(N)\tau_j T$ where T is the set of translations

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & N-1 \\ 0 & 1 \end{pmatrix} \right\},$$

and choose a subset $\{\varrho_1, \ldots, \varrho_{\widetilde{\mu}/N}\}$ of representatives. Then we have

$$\|\theta\|^2 = \frac{\mu}{\widetilde{\mu}} \sum_{j=1}^{\widetilde{\mu}/N} \int\limits_{\bigcup_{t\in T} tF} |\theta(z) \mid [\varrho_j]_{k/2}|^2 y^{k/2} \, \frac{dx\, dy}{y^2}. \tag{4.4}$$

Let $\varrho_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$. By construction, the $\widetilde{\mu}/N$ cusps a_j/c_j (with the convention $1/0 = \infty$) are a complete set of representatives of cusps for $\Gamma(N)$. It is easy to see ([23, Lemma 1.42]) that two rational numbers a/c, a'/c' are $\Gamma(N)$-equivalent cusps if and only if $a \equiv a' \pmod N$ and $c \equiv c' \pmod N$. In particular, this implies $(c, N) = (c', N)$. Thus for $d \mid N$ exactly

$$\phi\left(\frac{N}{d}\right) \frac{\phi(d)}{d} N$$

of the c_j satisfy $(c_j, N) = d$.

We estimate first the contribution of the $\phi(N)$ matrices ϱ_j with $c_j = 0$. In this case we have $|\theta(z) \mid [\varrho_j]_{k/2}|^2 = |\theta(z)|^2$, so that by (2.2) they contribute at most

$$\frac{\mu}{\widetilde{\mu}} \phi(N) \int\limits_{1/2}^{\infty} \int\limits_{-1/2+iy}^{N-1/2+iy} \Big| \sum_{n=1}^{\infty} (r(f,n) - r(g,n)) e(nz) \Big|^2 dx \, y^{k/2} \, \frac{dy}{y^2} \ll_k 1$$

to (4.4).

If $c_j \neq 0$, then by the transformation formula for the theta-series and the factorization

$$\varrho_j = \begin{pmatrix} 1 & a_j/c_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c_j \\ c_j & 0 \end{pmatrix} \begin{pmatrix} 1 & d_j/c_j \\ 0 & 1 \end{pmatrix}$$

(cf. also [25, p. 575]) we have

$$\theta(f, z) \mid [\varrho_j]_{k/2} = \sum_{\mathbf{x} \in \mathbb{Z}^k} \alpha(\mathbf{x}, A, \varrho_j) e\left(\frac{1}{2} z \mathbf{x}^{\mathrm{t}} A^{-1} \mathbf{x} \right)$$

where

$$\alpha(\mathbf{x}, A, \varrho_j) = \frac{\omega}{c_j^{k/2} \Delta^{1/2}} \sum_{\mathbf{x}_1 \in (\mathbb{Z}/c_j\mathbb{Z})^k} e\left(\frac{d_j}{2c_j} \mathbf{x}^{\mathrm{t}} A^{-1} \mathbf{x} + \frac{1}{c_j} \mathbf{x}_1^{\mathrm{t}} \mathbf{x} + \frac{a_j}{2c_j} \mathbf{x}_1^{\mathrm{t}} A \mathbf{x}_1 \right)$$

with $|\omega| = 1$. We have

$$\Big|\sum_{\mathbf{x}_1}\Big|^2 = \sum_{\mathbf{x}_1}\sum_{\mathbf{x}_2} e\bigg(\frac{a_j}{2c_j}(\mathbf{x}_1^{\mathrm{t}}A\mathbf{x}_1 - \mathbf{x}_2^{\mathrm{t}}A\mathbf{x}_2) + \frac{1}{c_j}(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{t}}\mathbf{x}\bigg)$$
$$= \sum_{\mathbf{x}_3} e\bigg(\frac{a_j}{2c_j}\,\mathbf{x}_3^{\mathrm{t}}A\mathbf{x}_3 + \frac{1}{c_j}\,\mathbf{x}_3^{\mathrm{t}}\mathbf{x}\bigg)\sum_{\mathbf{x}_2} e\bigg(\frac{a_j}{c_j}\,\mathbf{x}_2^{\mathrm{t}}A\mathbf{x}_3\bigg) \ll c_j^k(\Delta, c_j),$$

hence $|\alpha(\mathbf{x}, A, \varrho_j)| \ll_k (\Delta, c_j)^{1/2}\Delta^{-1/2}$. This yields

$$\theta(f, z) \,|\, [\varrho_j]_{k/2} = \frac{(\Delta, c_j)^{1/2}}{\Delta^{1/2}}\sum_{n=0}^{\infty}\beta_n(A, \varrho_j)e\bigg(\frac{zn}{N}\bigg)$$

where $\beta_n(A, \varrho_j)$ is bounded by the number of solutions to $\frac{1}{2}\,\mathbf{x}^{\mathrm{t}}(NA^{-1})\mathbf{x} = n$, which can be estimated with Lemma 4.1. Since θ is a cusp form, we have

$$\theta(z) \,|\, [\varrho_j]_{k/2} = \frac{(\Delta, c_j)^{1/2}}{\Delta^{1/2}}\sum_{n=1}^{\infty}\widetilde{\beta}_n(\varrho_j)e(zn/N)$$

where the first summation index is 1 and $\widetilde{\beta}_n$ is bounded by twice the number of solutions to $\frac{1}{2}\,\mathbf{x}^{\mathrm{t}}(NA^{-1})\mathbf{x} = n$. Using

$$\int_{-1/2+iy}^{N-1/2+iy} |\theta(z) \,|\, [\varrho_j]_{k/2}|^2\,dx = N\sum_{n=1}^{\infty}|\widetilde{\beta}_n(\varrho_j)|^2\exp\bigg(-\frac{4\pi yn}{N}\bigg)$$

for $y > 0$, for $k = 3$ by (4.4) and Lemma 4.1 we obtain

$$\|\theta\|^2 \ll \frac{\mu}{\widetilde{\mu}}\sum_{j=1}^{\widetilde{\mu}/N}\int_{1/2}^{\infty} N\sum_{n=1}^{\infty}\frac{(\Delta, c_j)}{\Delta}\,n^{1+\varepsilon}\exp\bigg(-\frac{4\pi yn}{N}\bigg)y^{-1/2}\,dy.$$

Since

$$\sum_{j=1}^{\widetilde{\mu}/N}\frac{(\Delta, c_j)}{\Delta} \ll \sum_{j=1}^{\widetilde{\mu}/N}\frac{(N, c_j)}{N} = \sum_{d\mid N}\phi\bigg(\frac{N}{d}\bigg)\phi(d) \ll N^{1+\varepsilon},$$

by (2.2) we obtain

$$\|\theta\|^2 \ll_\varepsilon \sum_{n=1}^{\infty} n^{1+\varepsilon}\,\frac{N^{1+\varepsilon}}{n}\exp\bigg(-\frac{n}{N}\bigg) \ll_\varepsilon N^{2+\varepsilon}.$$

Since NA^{-1} has discriminant $N^k\Delta^{-1}$ we find similarly, using Lemma 4.1(b),

$$\|\theta\|^2 \ll_k N^{k+\varepsilon} + N^{1+\varepsilon}\Delta$$

for $k > 3$.

REMARK. Lemma 4.2 is probably not best possible, but enough for our needs. A similar estimate in the case $k = 4$ was claimed by Fomenko ([9]).

However, note that the estimate from Lemma 4.2(b) has to be used in order to make his equation (3) correct.

From (4.2) we infer

COROLLARY 4.3. *Let f, g be two quaternary forms of level N and discriminant Δ in the same genus. Then*

$$r(f,n) - r(g,n) \ll_\varepsilon (N^2 + \sqrt{N\Delta})n^{1/2}(nN)^\varepsilon$$

for $(n,N)=1$ where the implied constant is absolute.

LEMMA 4.4. *Let (a_n) be the Fourier coefficients of a cusp form $\phi \in S_{3/2}(N,\chi)$. Write $n = tv^2$ with squarefree t. Then*

$$a_n \ll_\varepsilon \|\phi\|(n^{1/4}v + n^{13/28}v^{3/14})n^\varepsilon$$

for any $\varepsilon > 0$ where the implied constant depends on ε alone.

Proof. This is a slight modification of Iwaniec's result [12] on the estimation of Fourier coefficients for indices n containing a square factor v^2. In the following we shall give a short account of the necessary changes in his proof. Our notation and numbering of lemmata, theorems, sections and equations refers to [12].

The trivial bound (4.2) has to be used for all c with $nv^{-2} \mid c$, hence (4.3) changes to

$$|K_{[nv^{-2},Q]}(x)| \ll v^2 \frac{(nv^{-2},Q)\tau(nv^{-2}Q)\tau(n)}{n^{1/2}Q} x(\log x).$$

The remaining sum then can be estimated as in [12], in particular Δ_1, defined in Section 5, is a squarefree number different from 1. The proof of Lemma 7 shows that we may assume $p^2 \mid r$ for all primes p satisfying $p \mid n$, $p \nmid v$. To obtain (6.1)–(6.3), we may therefore use $(n,r) \le v^2 r^{1/2}$. For the proof of a modified version of Theorem 3 in [12] we use (5.2) or (5.3) if A or B is

$$\le \left(1 + \frac{n}{y}\right)^{-1/4} n^{-1/4} r^{-7/8} y^{1/2} P^{-1/2} v^{1/2}.$$

This yields

$$\sum_{Q\in\mathfrak{Q}} |K_Q(x)| \ll_\varepsilon (xP^{-1/2} + xn^{-1/2}v^2 + (x+n)^{5/8}(x^{1/4}P^{3/8} + n^{1/8}P^{1/4}v^{3/4}))(nx)^\varepsilon$$

whence for $k \ge 5/2$, $k \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$ uniformly in N,

$$\widehat{P}(n,k,\Gamma_0(N)) \ll_{\varepsilon,k} P + (n^{1/2}P^{-1/2} + v^2 + (nv^2P)^{3/8})(nP)^\varepsilon.$$

We now appeal to Proskurin's generalization of the Kuznetsov sum formula (cf. Theorem 2 in [5]) exactly as in [5, Section 5], with the changes as in the proof of Lemma 2 in [7]. In view of (2.5) in [5] the proof of Theorem 5 in [5] with $k = 3/2$, $\lambda = 3/16$ and $D = -4$ yields, for $N \equiv 0 \pmod 4$,

$$a_n \ll_\varepsilon \|\phi\| n^{1/4+\varepsilon} P^\varepsilon (P + (n^{1/2}P^{-1/2} + v^2 + (nv^2P)^{3/8}))^{1/2}.$$

On choosing $P = n^{1/7}v^{-6/7}$ the lemma follows.

We are now prepared for the *proof of Theorem 1*. Using the method sketched in Lemmata 2–4 in [7] we shall estimate $r(g,n) - r(f,n)$ for any two forms f, g lying in the same spinor genus. We write $n = tw^2v^2$ with squarefree t, $(w,N) = 1$, $v \mid N^\infty$.

For squarefree n the assertion of Theorem 1 follows directly from Lemmata 4.2 and 4.4. Equation (1.5) for $v^2 \mid N$, $w = 1$ follows in the same way, even with 31/28 instead of 45/28 in the exponent. The case $w > 1$ and $v^2 \mid N$ can be tackled using (4.1). Let $\{\phi_j : j = 1, \ldots, J\}$ be an orthonormal basis of $V(N, \chi_{2\Delta})$ (see Section 2) consisting of eigenforms for all the Hecke operators $T(p^2)$, $p \nmid N$. By (2.3) we have $J \ll_\varepsilon N^{1+\varepsilon}$. If we write $\theta(f,z) - \theta(g,z) = \sum \alpha_j \phi_j(z)$ with $\alpha_j \in \mathbb{C}$, then by Cauchy's inequality $\sum |\alpha_j| \le J^{1/2} \|\theta(f,\cdot) - \theta(g,\cdot)\|$, whence by (4.1) and Lemma 4.4,

$$r(g,n) - r(f,n) \ll_\varepsilon J^{1/2} N^{31/28+\varepsilon} (tv^2)^{13/28+\varepsilon} w^{1/2+\varepsilon}. \tag{4.5}$$

For the remaining square factors we proceed by multiplicative induction. To this end, we use the fact that $r(f, mp^\nu)$ ($p \mid N$, $p \nmid m$, $\nu > \operatorname{ord}_p N$) can be expressed by a linear combination of some $r(\widetilde{f}, mp^{\widetilde{\nu}})$ for which we can apply the above result.

If f and g are anisotropic over $\mathbb{Q}_p$, then

$$r(g, mp^\nu) - r(f, mp^\nu) = r(g, mp^c) - r(f, mp^c)$$

by Lemma 3.1(a) where $c = \nu - 2\lfloor (\nu - \operatorname{ord}_p(N/2) + 1)/2 \rfloor \le \operatorname{ord}_p N$. Thus we are back in the case $v^2 \mid N$.

If f and g are isotropic over $\mathbb{Q}_p$, we use the ideas of Lemmata 3–5 in [22]: We start by replacing f and g with forms $\widetilde{f}$, $\widetilde{g}$ as in Lemma 3.1(b). Since f and g are equivalent over $\mathbb{Z}_p$, we have $\widetilde{f} \cong \widetilde{g}$ over $\mathbb{Z}_p$. By Lemmata 4 and 5 in [22] there are $s_2 - s_1 + 1$ forms f_i, g_i with matrices A_i, B_i, and $s_2 - s_1$ forms f_i^*, g_i^* with matrices A_i^*, B_i^* such that

- for all i the corresponding forms f_i and g_i as well as f_i^* and g_i^* are in the same spinor genus,
- we have
$$A_i \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp (2u) \cong B_i, \qquad A_i^* \cong p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp (2u) \cong B_i^* \quad \text{over } \mathbb{Z}_p$$
with $u \in \mathbb{Z}_p^*$,
- $f_i \cong f$ over $\mathbb{Z}_{p'}$ for all $p' \neq p$, and analogously for g_i, f_i^*, g_i^*, and

- we have

$$r(\widetilde{f}, mp^\nu) = \sum_{i=0}^{s_2-s_1} \sum_{j=0}^{\lfloor(\nu-s_2)/2\rfloor} \gamma_{ij} r(f_i, mp^{\nu-s_2-2j})$$
$$- (-1)^{\lfloor(\nu-s_2)/2\rfloor} \sum_{i=1}^{s_2-s_1} r(f_i^*, mp^{\nu-s_2-2\lfloor(\nu-s_2)/2\rfloor})$$

  with $|\gamma_{ij}| \le 2$; the same holds with g instead of f.

The differences

$$r(f_i, mp^{\nu-s_2-2j}) - r(g_i, mp^{\nu-s_2-2j}) \tag{4.6}$$

for odd p and

$$r(f_i^*, mp^{\nu-s_2-2\lfloor(\nu-s_2)/2\rfloor}) - r(g_i^*, mp^{\nu-s_2-2\lfloor(\nu-s_2)/2\rfloor})$$

for all p do not cause any problems since in the first case $p \nmid N_{f_i}$ and in the second case $p^2 \nmid mp^{\nu-s_2-2\lfloor(\nu-s_2)/2\rfloor}$. In particular, the p-part of the levels of the f_i, f_i^*, is not larger than the p-part of the level of f, thus we may apply the induction hypothesis. The final linear combination consists of at most

$$\prod_{p\mid(n,N)} (2(\operatorname{ord}_p N + 2))(\operatorname{ord}_p n + 1) \ll_\varepsilon (Nn)^\varepsilon$$

terms (due to the γ_{ij}), which is harmless. So far we have shown (1.4) and (1.5).

Unfortunately this proof only works for $p = 2$ if the Shimura lift of theta-series for forms with $2 \parallel \Delta$ has odd level (so that we can apply (4.1)), but we did not find this in general in the literature. To estimate (4.6) for $p = 2$ and obtain the uniform result (1.3), we use the following approach which was communicated to the author by R. Schulze-Pillot.

The main idea is to replace the Shimura lifting by the related Brandt matrix lifting which is given purely arithmetically. If $\overline{B}(2^\nu) = (\overline{b}_{ij}(2^\nu))_{ij}$ is the reduced Brandt matrix as in Section 2, by Lemma 2 in [22] for forms f with $2 \parallel \Delta$ we have

$$r(f, 2^{2\nu}m) = \sum_{j=1}^{h} (\overline{b}_{1j}(2^\nu)) r(f_j, m)$$

(if $4 \nmid m$) where $f = f_1, \dots, f_h$ is a set of representatives of the classes in the genus of f. Therefore,

$$r(f_1, 2^{2\nu}m) - r(f_2, 2^{2\nu}m) = \sum_{j=1}^{h} (\overline{b}_{1j}(2^\nu) - \overline{b}_{2j}(2^\nu)) r(f_j, m).$$

If f_1, f_2 are in the same spinor genus, we see, as in the proof of the proposition in [22],

$$\sum_{j=1}^{h} \overline{b}_{1j}(2^\nu) = \sum_{j=1}^{h} \overline{b}_{2j}(2^\nu)$$

so that

$$r(f_1, 2^{2\nu}m) - r(f_2, 2^{2\nu}m) = \sum_{j=1}^{h} (\overline{b}_{1j}(2^\nu) - \overline{b}_{2j}(2^\nu))(r(f_j, m) - r(f_1, m)).$$

By the above procedure we may assume that $m = tv^2w^2$ with t squarefree, $v^2 \mid N$, v odd, $(w, N) = 1$. By (2.5), Corollary 4.3 and [16, Satz 7] we obtain

$$\bar{b}_{1j}(2^\nu) - \bar{b}_{2j}(2^\nu) \ll N^{2+\varepsilon} 2^{\nu/2+\varepsilon}. \tag{4.7}$$

If $2^\nu \le m^{3/8}$, we estimate the contribution of $2^{2\nu}$ trivially by Lemma 4.4, getting as in (4.5)

$$\begin{aligned} &r(f_1, 2^{2\nu}m) - r(f_2, 2^{2\nu}m) \\ &\quad \ll J^{1/2} N (2^\nu v (2^{2\nu} v^2 t)^{1/4} + (2^\nu v)^{3/14} (2^{2\nu} v^2 t)^{13/28}) w^{1/2} (2^\nu m N)^\varepsilon \\ &\quad \ll N^{2+\varepsilon} (2^{2\nu} m^{1/4} + (2^\nu)^{3/14} w^{-3/7} (2^{2\nu} m)^{13/28}) \ll N^{2+\varepsilon} (2^{2\nu} m)^{13/28+\varepsilon}. \end{aligned}$$

If $2^\nu \ge m^{1/12}$, then by (4.7) and Lemma 4.1(a) we have

$$r(f_1, 2^{2\nu}m) - r(f_2, 2^{2\nu}m) \ll H N^{2+\varepsilon} 2^{\nu/2+\varepsilon} m^{1/2+\varepsilon} \ll H N^{2+\varepsilon} (2^{2\nu} m)^{13/28+\varepsilon}$$

if H denotes the ideal class number of any order satisfying (2.1). Explicit formulae for H are known in many special cases (e.g. [17]). At any rate it is easy to see that H increases in N polynomially at most, e.g. by analysing the standard proof ([4, p. 90]) of the finiteness of the class number by means of the Minkowski lattice point theorem. This yields (1.3), completing the proof of Theorem 1.

5. Sums of three squareful integers. With the following very strong result (whose proof is elementary) due to Heath-Brown we can exploit the fact that we have a lot of forms contributing to $R_1(n)$, so we can avoid Siegel's ineffective lower bound for $L(1, \chi)$.

LEMMA 5.1. *Let $\varepsilon > 0$ be given and let S be a set of odd primes with $\#S > (1 + 2/\varepsilon)^4$. Then there is a prime $p \in S$ with*

$$L(1, \chi_{-4np}) \ge (\log np)^{-\varepsilon}$$

for all $n > n_0(\varepsilon)$.

Proof. See [11, Theorem 3].

Lemma 5.2. (a) *Let f be a diagonal form with odd coefficients such that every $p \geq 3$ divides at most one coefficient. Then the genus of f contains only one spinor genus.*

(b) *For any form f we have*

$$r(\operatorname{spn} f, n) \leq 2r(\operatorname{gen} f, n). \tag{5.1}$$

Proof. For the first part see [15, (102:10)]. Equation (5.1) is well known (e.g. [21, Satz 3]).

Remark. The condition $(2\Delta, n) = 1$, as claimed by Moroz ([14]), is *not* sufficient to ensure $r(\operatorname{gen} f, n) = r(\operatorname{spn} f, n)$. It may fail if n is a square as the example $f(\mathbf{x}) = 3x_1^2 + 4x_2^2 + 9x_3^2$, $n = p^2$, $p \equiv 1 \pmod 3$ shows (see [20] for some more examples of this type).

Proof of Theorem 2. Let

$$Q := \{q \text{ prime} \mid q \equiv 7 \ (\operatorname{mod} 8),\ q \nmid n,\ q \leq (\log n)^2\},$$

hence Q contains all primes $\equiv 7 \pmod 8$ not exceeding $(\log n)^2$ with the exception of at most $O\big(\frac{\log n}{\log\log n}\big)$. Let $\varepsilon > 0$ be given and let S be the set of the first $\lceil (1+2/\varepsilon)^4 \rceil$ primes $p \equiv 1 \ (\operatorname{mod} 8)$. By Lemma 5.1 we can, for each $q \in Q$, choose a $p = p_q \in S$ with $L(1, \chi_{-4nqp}) \gg_\varepsilon (\log n)^{-\varepsilon}$ if n is large enough. The forms $f(\mathbf{x}) = p^3q^3x_1^2 + x_2^2 + x_3^2$ are isotropic over $\mathbb{Z}_p$. Thus by Lemma 3.2(a), (b) we obtain

$$r(\operatorname{gen} f, n) \gg_\varepsilon n^{1/2} q^{-3/2} (\log n)^{-\varepsilon},$$

and by Lemma 5.2 and Theorem 1,

$$r(f, n) \gg_\varepsilon n^{1/2} q^{-3/2} (\log n)^{-\varepsilon} + O_\varepsilon(n^{1/2-1/28+\varepsilon} q^{3A})$$

where A is the same constant as in Theorem 1. The same lower bound holds for $\#\{\mathbf{x} \in \mathbb{N} \times \mathbb{N}_0^2 \mid f(\mathbf{x}) = n\}$. Observing (1.1) and summing over $q \in Q$ (by partial summation together with the Prime Number Theorem) yields the lower bound of Theorem 2.

The same proof goes through for R_3^* with forms $f(\mathbf{x}) = p^3q^3x_1^2 + p_1^3x_2^2 + p_2^3x_3^2$ with primes $p_1, p_2 \equiv 1 \pmod 8$ (p, p_1, p_2 all different) such that f is isotropic over $\mathbb{Z}_p$, $\mathbb{Z}_{p_1}$ and $\mathbb{Z}_{p_2}$, i.e.

$$\left(\frac{p_1p_2}{p}\right) = \left(\frac{pqp_2}{p_1}\right) = \left(\frac{pqp_1}{p_2}\right) = 1.$$

This can easily be ensured by imposing additional congruence conditions on the sets Q and S.

The upper bound follows by straightforward calculation: Let $f_d(\mathbf{x}) = d^3x_1^2 + x_2^2 + x_3^2$ as in Lemma 3.2. By (3.1), (3.2), (5.1) and Theorem 1 we have

$$\sum_{\substack{1\le d\le(\log n)^3\\ d\,\mathrm{squarefree}}} r(f_d,n) \ll n^{1/2}(\log\log n)l(n,(\log n)^3).$$

Using the estimates $L(1,\chi_D)\ll\log|D|$, $O(1)^{\omega(d)}\ll_\varepsilon d^\varepsilon$ for any $\varepsilon>0$ and $\sum_{d\le n^B}O(d^{3A})\ll n^{1/50}$ for $B=(50(3A+1))^{-1}$, we obtain

$$\begin{aligned}\text{(5.2)}\qquad \sum_{\substack{(\log n)^3\le d\le n^B\\ d\,\mathrm{squarefree}}} r(f_d,n) &\ll_\varepsilon n^{1/2}(\log\log n)(\log n)\sum_{(\log n)^3\le d\le n^B} d^{-3/2+\varepsilon}\\ &\ll_\varepsilon n^{1/2}(\log n)^{-1/2+\varepsilon}.\end{aligned}$$

The trivial estimation finally yields

$$\text{(5.3)}\qquad \sum_{d\ge n^B} r(f_d,n)\ll n^\varepsilon\sum_{d\ge n^B} n^{1/2}d^{-3/2}\ll_\varepsilon n^{1/2-B/2+\varepsilon}.$$

Note that $l(n,(\log n)^3)\gg_\varepsilon(\log n)^{-\varepsilon}$ for any $\varepsilon>0$ by Lemma 5.1. Combining these results completes the proof of Theorem 2.

Proof of Theorem 3. For real $t>10$ let

$$n=\prod_{p\le t}p^{e_p+4}\quad\text{with}\quad e_p=\left\lceil 10\,\frac{\log t}{\log p}\right\rceil.$$

Let $\mathcal{N}$ be the set of all integers n obtained in this way. We have $\log n\ll t$. We first consider the number of representations of n by forms $f_d(\mathbf{x})=d^3x^2+y^2+z^2$ with $d\le t$ and $\mu^2(d)=1$. Since a positive definite form is anisotropic over the reals, it must be anisotropic over at least one non-archimedian completion (cf. [13, p. 36]), i.e. f_d is anisotropic over $\mathbb{Z}_p$ for at least one prime $p\le t$. By Lemma 3.1(a) we see similarly to the preceding proof,

$$\begin{aligned}\sum_{\substack{d\le t\\ \mu^2(d)=1}} r(f_d,n) &\ll n^{1/2}(\log\log n)(\log n)^{-5}\sum_{d\le t}L(1,\chi_{-4nd})d^{-4/3}\\ &\ll n^{1/2}(\log n)^{-3}\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{t\le d\le(\log n)^3\\ \mu^2(d)=1}} r(f_d,n) &\ll_\varepsilon n^{1/2}(\log\log n)\sum_{t\le d\le(\log n)^3}L(1,\chi_{-4nd})d^{-3/2+\varepsilon}\\ &\ll_\varepsilon n^{1/2}(\log n)^{-1/2+\varepsilon}l(n,(\log n)^3).\end{aligned}$$

Together with (5.2) and (5.3) this completes the first part of the proof.

For the second part we note that as in the proof of Theorem 2 (again using $L(1,\chi_D) \ll \log|D|$)

$$\sum_{\substack{1\le d_1d_2d_3\le(\log n)^7\\ d_1,d_2,d_3 \text{ squarefree}}} r(f_{\mathbf{d}},n) \ll n^{1/2}(\log\log n)l(n,(\log n)^7),$$

$$\sum_{\substack{(\log n)^7\le d_1d_2d_3\le n^B\\ d_1,d_2,d_3 \text{ squarefree}}} r(f_{\mathbf{d}},n) \le \sum_{\substack{d_1d_2d_3\le n^B\\ d_1,d_2,d_3 \text{ squarefree}}} \left(\frac{d_1d_2d_3}{(\log n)^7}\right)^{1/7} r(f_{\mathbf{d}},n)$$
$$= n^{1/2}(\log\log n)\prod_p \left(1+\frac{O(1)}{p^{3/2-1/7}}+\frac{O(p^{3/2})}{p^{3-2/7}}+\frac{O(p^3)}{p^{9/2-3/7}}\right);$$

thus all exceptions come from forms with large discriminants. But the trivial estimation shows

$$\sum_{n\le x}\sum_{\substack{d_1d_2d_3\ge n^B\\ d_1,d_2,d_3 \text{ squarefree}}} r(f_{\mathbf{d}},n) \ll \sum_{d_1d_2d_3\ge n^B} \frac{x^{3/2}}{(d_1d_2d_3)^{3/2}} \ll_\varepsilon x^{3/2-B/2+\varepsilon},$$

and the assertion follows by a standard argument.

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Received on 29.7.2002
and in revised form on 15.10.2003 (4336)