On higher-power moments of $\Delta(x)$ (II)

by

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1. Introduction and main results

1.1. Notations. Throughout this paper, let $d(n)$ denote the Dirichlet divisor function, $r(n)$ the number of ways $n$ can be written as $n = x^2 + y^2$ for $x, y \in \mathbb{Z}$, and $a(n)$ the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group, $\tilde{a}(n) := a(n)n^{-\kappa/2+1/2}$. For short, we use $d, r, a, \tilde{a}$ to denote these functions, respectively. $\zeta(s)$ denotes the Riemann zeta-function.

Suppose $x, t > 0$. Define

\begin{align}
\Delta(x) &:= \sum_{n \leq x} d(n) - x \log x - (2\gamma - 1)x, \\
P(x) &:= \sum_{n \leq x} r(n) - \pi x, \\
A(x) &:= \sum_{n \leq x} a(n), \\
E(t) &:= \int_{0}^{t} |\zeta(1/2 + iu)|^2 \, du - t \log(t/2\pi) - (2\gamma - 1)t.
\end{align}

Suppose $f: \mathbb{N} \to \mathbb{R}$ is any function such that $f(n) \ll n^{\varepsilon}$, $k \geq 2$ is a fixed integer. Define

\begin{equation}
s_{k,l}(f) := \sum_{\sqrt{m_1} + \ldots + \sqrt{m_l} = \sqrt{n_1 + \ldots + n_k}} \frac{f(n_1) \ldots f(n_k)}{(n_1 \ldots n_k)^{3/4}} \quad (1 \leq l < k),
\end{equation}

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$$B_k(f) := \sum_{l=1}^{k-1} \binom{k-1}{l} s_{k;l}(f) \cos \frac{\pi(k-2l)}{4}. \quad (1.6)$$

We shall use $s_{k;l}(f)$ to denote both the series (1.5) and its value. We will prove the convergence of $s_{k;l}(f)$ in Section 3.

Suppose $A_0 > 2$ is a real number. Define

$$K_0 := \min\{n \in \mathbb{N} : n \geq A_0, 2 \mid n\}, \quad b(k) := 2^{k-2} + (k - 6)/4,$$

$$\sigma(k, A_0) := \begin{cases} \frac{1}{2} & \text{if } k - 1 < A_0/2; \\ \frac{A_0 - k}{2(A_0 - 2)} & \text{if } A_0/2 + 1 \leq k < A_0, \end{cases}$$

$$\delta_1(k, A_0) := \frac{\sigma(k, A_0)}{2b(K_0)}, \quad \delta_2(k, A_0) := \frac{\sigma(k, A_0)}{2b(k) + 2\sigma(k, A_0)}.$$  

$\mathbb{N}$ denotes the set of all natural numbers; $\varepsilon$ always denotes a sufficiently small positive constant which may be different at different places. We will use the inequality $d(n) << n^\varepsilon$ freely. $\text{SC}(\sum)$ denotes the summation condition of the sum $\sum$; $\mu(n)$ is the Möbius function.

1.2. Introduction. In this paper we shall study the higher-power moments of $\Delta(x)$, $P(x)$, $A(x)$ and $E(t)$.

We begin with the Dirichlet divisor problem. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{23/73} (\log x)^{315/146}, \quad (1.7)$$

which can be found in Huxley [6] (see also “Note added in proof”). It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \quad (1.8)$$

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x) \, dx = \frac{(\zeta(3/2))^4}{6\pi^2 \zeta(3)} T^{3/2} + O(T \log^5 T) \quad (1.9)$$

proved by Tong [17] and the upper bound estimate

$$\int_1^T |\Delta(x)|^{A_0} \, dx \ll T^{1+A_0/4+\varepsilon}, \quad (1.10)$$

where $A_0 > 2$ is a fixed real number. The estimate of type (1.10) can be found in Ivić [7, Thm. 13.9] with $A_0 = 35/4$ and Heath-Brown [5] with $A_0 = 28/3$. On the other hand, Voronoï [19] proved that

$$\int_1^T \Delta(x) \, dx = T/4 + O(T^{3/4}), \quad (1.11)$$
which in conjunction with (1.9) shows that $\Delta(x)$ has a lot of sign changes and cancellations between the positive and negative portions.

Tsang [18] first studied the third- and fourth-power moments of $\Delta(x)$. He proved that (with notations of Section 1.1)

$$
\int_1^T \Delta^3(x) \, dx = \frac{3s_{3:1}(d)}{28\pi^3} T^{7/4} + O(T^{7/4-1/14+\varepsilon}),
$$

(1.12)

$$
\int_1^T \Delta^4(x) \, dx = \frac{3s_{4:2}(d)}{64\pi^4} T^2 + O(T^{2-1/23+\varepsilon}).
$$

(1.13)

Heath-Brown [5] proved that for $k = 3, \ldots, 9$ the limit

$$
\lim_{T \to \infty} T^{1-k/4} \int_1^T \Delta(x)^k \, dx
$$

exists.

In [20] the author improved Tsang’s method and proved that

$$
\int_1^T \Delta^3(x) \, dx = \frac{3s_{3:1}(d)}{28\pi^3} T^{7/4} + O(T^{3/2+\varepsilon}),
$$

(1.14)

$$
\int_1^T \Delta^4(x) \, dx = \frac{3s_{4:2}(d)}{64\pi^4} T^2 + O(T^{2-2/41}),
$$

(1.15)

$$
\int_1^T \Delta^5(x) \, dx = \frac{5(2s_{5:2}(d) - s_{5:1}(d))}{288\pi^5} T^{9/4} + O(T^{9/4-5/816}).
$$

(1.16)

But the argument of [20] fails for $k \geq 6$.

1.3. New results on higher-power moments of $\Delta(x)$. In this paper we shall use a different approach to study the higher-power moments of $\Delta(x)$. This leads to the asymptotic formulas for the integral $\int_1^T \Delta^k(x) \, dx$ for $3 \leq k \leq 9$. Furthermore, if the estimate (1.8) is true, then our approach can give the asymptotic formulas for $\int_1^T \Delta^k(x) \, dx$ for any $k \geq 10$.

**Theorem 1.** Let $A_0 > 9$ be a real number such that (1.10) holds. Then for any integer $3 \leq k < A_0$, we have the asymptotic formula

$$
\int_1^T \Delta^k(x) \, dx = \frac{B_k(d)}{(1 + k/4)^{23k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).
$$

(1.17)

Remark 1.1. From Ivic’s argument [7, Thm. 13.9], we know that the value of $A_0$ for which (1.10) holds depends on the large-value estimate and the upper bound estimate of $\Delta(x)$. If we insert the estimate (1.7) into the argument of Ivic, we find that (1.10) holds with $A_0 = 184/19$. Hence for $k \in \{3, 4, 5, 6, 7, 8, 9\}$, we get the asymptotic formula (1.17). Moreover, if the
estimate $\Delta(x) \ll x^{5/16-\delta}$ holds for some small $\delta > 0$, then the asymptotic formula (1.17) holds for $k = 10$.

**Remark 1.2.** For $k \geq 10$, Theorem 1 is only a conditional result. However, it tells us that for any $k \geq 10$, the main term in the asymptotic formula for $\int_1^T \Delta^k(x) \, dx$ (if it exists) must have the form stated in (1.17).

**Remark 1.3.** We can state the following three conjectures about $\Delta(x)$:

**Conjecture 1.** The estimate (1.8) is true.

**Conjecture 2.** The estimate (1.10) is true for any $A_0 > 2$.

**Conjecture 3.** For any fixed $k \geq 3$, there exists a constant $\delta_k > 0$ such that the following asymptotic formula holds:

$$
\int_1^T \Delta^k(x) \, dx = \frac{B_k(d)}{(1 + k/4)2^{3k/2-1}\pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon}).
$$

It is well known that Conjectures 1 and 2 are equivalent. From Theorem 1 we know that actually the three conjectures are equivalent. It is easy to deduce Conjecture 2 from Conjecture 3. To deduce Conjecture 3 from Conjecture 2, we take $A_0 = 2(k-1)$ and $\delta_k = \delta_1(k, 2(k-1))$.

**Remark 1.4.** From (1.11) we know that the integral $\int_1^T \Delta(x) \, dx$ has many cancellations from the positive and negative portions of $\Delta(x)$. However, from (1.12) Tsang [18] observed that this is not so for $\int_1^T \Delta^3(x) \, dx$. From Theorem 1 we know that this is also not so for $\int_1^T \Delta^k(x) \, dx$ ($k = 5, 7, 9$) since numerical computation tells $B_k(d) > 0$ for $k = 5, 7, 9$. Maybe $B_k(d) > 0$ holds for any odd $k \geq 3$.

The constant $\delta_1(k, A_0)$ is small for $k$ small. If we combine Ivić’s argument with the proof of Theorem 1, we get the following Theorem 2 for $3 \leq k \leq 9$.

**Theorem 2.** For $3 \leq k \leq 9$, the asymptotic formula (1.17) holds with $\delta_1(k, A_0)$ replaced by $\delta_2(k, 184/19)$.

In particular, for $k = 5, 6, 7, 8, 9$, we have

$$
\int_1^T \Delta^5(x) \, dx = \frac{5(2s_5;2(d) - s_5;1(d))}{288\pi^5} T^{9/4} + O(T^{9/4-1/64+\varepsilon}),
$$

$$
\int_1^T \Delta^6(x) \, dx = \frac{5s_6;3(d) - 3s_6;1(d)}{320\pi^6} T^{5/2} + O(T^{5/2-35/4742+\varepsilon}),
$$

$$
\int_1^T \Delta^7(x) \, dx = \frac{7(5s_7;3(d) - 3s_7;2(d) - s_7;1(d))}{2816\pi^7} T^{11/4}
$$

$$
+ O(T^{11/4-17/6312+\varepsilon}).
$$
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\begin{align}
\int_1^T \Delta^8(x) \, dx &= \frac{7(588,4(d) - 488,2(d))}{6144\pi^8} T^3 + O(T^{3-8/9433+\varepsilon}), \\
\int_1^T \Delta^9(x) \, dx &= \frac{3(389,1(d) - 1289,2(d) - 2889,3(d) + 4289,4(d))}{26624\pi^9} T^{13/4} + O(T^{13/4-13/75216+\varepsilon}).
\end{align}

1.4. Higher-power moments of $P(x)$, $A(x)$ and $E(t)$. The method of proving Theorems 1 and 2 can also be applied to study the higher-power moments of $P(x), A(x)$ and $E(t)$.

The conjectured bound of $P(x)$ is

\begin{equation}
P(x) = O(x^{1/4+\varepsilon}),
\end{equation}

which is supported by

\begin{equation}
\int_2^T P^2(x) \, dx = \left( \frac{1}{3\pi^2} \sum_{n=1}^{\infty} r^2(n)n^{-3/2} \right) T^{3/2} + O(T \log^2 T)
\end{equation}

proved by Kátaï [14]. Tsang [18] also studied the third- and fourth-power moments of $P(x)$. His results were improved by the present author [20]. An asymptotic formula for the fifth-power moment of $P(x)$ was also obtained in [20], which is further improved by the following (for $k = 5$):

**Theorem 3.** Let $A_0 > 9$ be a real number such that

\begin{equation}
\int_1^T |P(x)|^{A_0} \, dx \ll T^{1+A_0/4+\varepsilon}.
\end{equation}

Then for any integer $3 \leq k < A_0$, the following asymptotic formula holds:

\begin{equation}
\int_1^T P^k(x) \, dx = \frac{(-1)^k B_k(r)}{(1+k/4)2^{k-1} \pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).
\end{equation}

In particular, for $3 \leq k \leq 9$, (1.26) holds with $\delta_1(k,A_0)$ replaced by $\delta_2(k,184/19)$.

**Remark 1.5.** Ivić [7, Thm. 13.12] proved that the estimate (1.25) holds for $A_0 = 35/4$. If we insert the estimate $P(x) = O(x^{23/73+\varepsilon})$ (see Huxley [6]) into his argument, we find that (1.25) holds for $A_0 = 184/19$.

It is well known that $A(x)$ has no main term and $A(x) \ll x^{\kappa/2-1/6+\varepsilon}$. From Deligné [4], we have $|\widetilde{a}(n)| \leq d(n)$.

The conjectured bound of $A(x)$ is $A(x) \ll x^{\kappa/2-1/4+\varepsilon}$. Ivić [9] proved that

\begin{equation}
\int_1^T A^2(x) \, dx = B_2 T^{\kappa+1/2} + O(T^\kappa \log^5 T),
\end{equation}
where
\[ B_2 = \frac{1}{4\kappa + 2} \sum_{n=1}^{\infty} a^2(n) n^{-\kappa - 1/2}. \]

Ivić [9] also proved that
\[
\int_1^T |A(x)|^{A_0} \, dx \ll T^{1 + A_0(2\kappa - 1)/4 + \varepsilon}
\]
for \( A_0 = 8 \). Cai [3] studied the third- and fourth-power moments of \( A(x) \). His results were improved in [20], where an asymptotic formula for the fifth-power moment of \( A(x) \) was also obtained, which is further improved by the case \( k = 5 \) of the following:

**Theorem 4.** Let \( A_0 \geq 8 \) be a real number such that (1.28) is true. Then for any \( 3 \leq k < A_0 \), we have the asymptotic formula
\[
\int_1^T A^k(x) \, dx = \frac{B_k(\bar{a})}{(1 + \frac{k(2\kappa - 1)}{4})2^{3k/2-1} \pi^k} T^{1 + \frac{k(2\kappa - 1)}{4}}
\]
\[ + O(T^{1 + \frac{k(2\kappa - 1)}{4} - \delta_1(k,A_0) + \varepsilon}). \]

In particular, for \( 3 \leq k \leq 7 \), (1.29) holds with \( \delta_1(k,A_0) \) replaced by \( \delta_2(k,8) \).

Many results for \( E(t) \) parallel to those for \( \Delta(x) \) have been obtained; see Ivić [8] for a survey. The conjectured bound for \( E(t) \) is \( E(t) \ll t^{1/4 + \varepsilon} \), which is supported by
\[
\int_2^T E^2(t) \, dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}} T^{3/2} + O(T \log^5 T),
\]
proved by Meurman [15]. It has been proved (see Huxley [6]) that
\[
E(t) \ll t^{72/227}(\log t)^{629/227}, \quad t > 2.
\]
Ivić [7, Thm. 15.7] proved that
\[
\int_1^T |E(t)|^{A_0} \, dt \ll T^{1 + A_0/4 + \varepsilon}
\]
for \( A_0 = 35/4 \). Inserting (1.31) into Ivić’s argument, we find that (1.32) is true for \( A_0 = 576/61 \).

Tsang [18] studied the third- and fourth-power moment of \( E(t) \) by using the Atkinson formula [1]. His results were further improved by Ivić [10] in a different way. The author [20] obtained new results on the third- and fourth-power moments of \( E(t) \). An asymptotic formula for the fifth-power moment
Higher-power moments of $E(t)$ was also obtained in [20], which is further improved by the case $k = 5$ of the following:

**Theorem 5.** Let $A_0 > 9$ be a real number such that the estimates (1.10) and (1.32) hold. Then for any $3 \leq k < A_0$, we have the asymptotic formula

$$
\int_1^T E^k(t) \, dt = \frac{B_k(d)}{(1 + k/4)2^{3k/4 - 1}\pi^{k/4}} T^{1+k/4} + O(T^{1+k/4 - \delta_1(k,A_0) + \varepsilon}).
$$

In particular, for $3 \leq k \leq 9$, (1.33) holds with $\delta_1(k,A_0)$ replaced by $\delta_2(k,576/61)$.

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**2. Some preliminary lemmas.** We need the following lemmas.

**Lemma 2.1.** The square roots of squarefree numbers are linearly independent over the integers.

**Proof.** This is a classical result of Besicovitch [2].

**Lemma 2.2.** Suppose $k \geq 3$ and $(i_1, \ldots, i_{k-1}) \in \{0,1\}^{k-1}$ are such that

$$\sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + \ldots + (-1)^{i_{k-1}} \sqrt{n_k} \neq 0.$$

Then

$$|\sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + \ldots + (-1)^{i_{k-1}} \sqrt{n_k}| \gg \max(n_1, \ldots, n_k)^{(2k-2-2^{-1})}.$$

**Proof.** The cases $k = 3, 4$ are Lemmas 1 and 2 of Tsang [18], respectively. The proof for the general case is the same as the proof of Lemma 1 of [18]. We note that Heath-Brown [5] stated a similar result for $k$ even.

**Lemma 2.3.** Suppose $A, B \in \mathbb{R}, A \neq 0$. Then

$$\int_1^T \cos(A\sqrt{t} + B) \, dt \ll T^{1/2}|A|^{-1}.$$

**Lemma 2.4.** Suppose $k \geq 3$, $(i_1, \ldots, i_{k-1}) \in \{0,1\}^{k-1}$, $(i_1, \ldots, i_{k-1}) \neq (0, \ldots, 0)$, $N_1, \ldots, N_k > 1$, $0 < \Delta \ll E^{1/2}$, $E = \max(N_1, \ldots, N_k)$. Let

$$\mathcal{A} = \mathcal{A}(N_1, \ldots, N_k; i_1, \ldots, i_{k-1}; \Delta)$$

denote the number of solutions of the inequality

$$|\sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + \ldots + (-1)^{i_{k-1}} \sqrt{n_k}| < \Delta$$
with \( N_j < n_j \leq 2N_j, 1 \leq j \leq k \). Then
\[
\mathcal{A} \ll \Delta E^{-1/2}N_1 \ldots N_k + E^{-1}N_1 \ldots N_k.
\]

**Proof.** Without loss of generality, suppose \( E = N_k \). If \((n_1, \ldots, n_k)\) satisfies (2.1), then
\[
\sqrt{n_1} + (-1)^i_1 \sqrt{n_2} + (-1)^i_2 \sqrt{n_3} + \ldots + (-1)^i_{k-2} \sqrt{n_{k-1}} = (-1)^{i_k+1} \sqrt{n_k} + \theta \Delta
\]
for some \(|\theta| < 1\), whence we get
\[
(\sqrt{n_1} + (-1)^i_1 \sqrt{n_2} + (-1)^i_2 \sqrt{n_3} + \ldots + (-1)^i_{k-2} \sqrt{n_{k-1}})^2 = n_k + O(\Delta N_k^{1/2}).
\]
Hence for fixed \((n_1, \ldots, n_{k-1})\), the number of \( n_k \) is \( \ll 1 + \Delta N_k^{1/2} \) and thus
\[
\mathcal{A} \ll \Delta N_k^{1/2}N_1 \ldots N_{k-1} + N_1 \ldots N_{k-1}.
\]

3. **On the series** \( s_{k;l}(d) \). Suppose \( y > 1 \) is a large parameter, and define
\[
s_{k;l}(d; y) := \sum_{\sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k}, \ n_1, \ldots, n_k \leq y} \frac{d(n_1) \ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}}, \quad 1 \leq l \leq k.
\]
We shall prove

**Lemma 3.1.** We have
\[
|s_{k;l}(d) - s_{k;l}(d; y)| \ll y^{-1/2+\varepsilon}, \quad 1 \leq l \leq k.
\]

**Remark.** Lemma 3.1 is still true if the divisor function \( d \) is replaced by any function \( f : \mathbb{N} \to \mathbb{R} \) with \( f(n) \ll n^\varepsilon \).

**Proof.** We shall prove Lemma 3.1 by induction in \( k \). The case \( k = 2 \) is easy. The case \( k = 3 \) is contained in [18, p. 70]. Later we suppose \( k \geq 4 \).

Since \( s_{k;l}(d) = s_{k;k-l}(d) \), we suppose \( l \leq k/2 \).

By symmetry, we get
\[
|s_{k;l}(d) - s_{k;l}(d; y)| \ll \sum_{\sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k}, \ n_1 > y} \frac{d(n_1) \ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}}
\]
\[
\ll U_1(d; y) + U_2(d; y),
\]
say, where
\[
U_1(d; y) := \sum_{j=l+1}^{k} \frac{d(n_1) \ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}},
\]
\[
U_2(d; y) := \sum_{\sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k}, \ n_1 > y, n_1 \neq n_j, l+1 \leq j \leq k} \frac{d(n_1) \ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}}.
\]
If \( l = 1 \), then obviously \( U_1(d; y) = 0 \). If \( l > 1 \), then by induction we get

\[
(3.2) \quad U_1(d; y) \leq \sum_{n > y} \frac{d^2(n)}{n^{3/2}} s_{k-2;l-1}(d) \leq y^{-1/2+\varepsilon}.
\]

Now we estimate \( U_2(d; y) \). Let \( I = \{1, \ldots, l\}, J = \{l+1, \ldots, k\} \). Suppose \((n_1, \ldots, n_k) \in \mathbb{N}^k\) are such that

\[
(\ast) \quad \sqrt{n_1} + \ldots + \sqrt{n_l} = \sqrt{n_{l+1}} + \ldots + \sqrt{n_k}, \quad n_1 \neq n_j, \quad l+1 \leq j \leq k.
\]

Then there exist two sets \( I_0 \subset I, J_0 \subset J \) with the following properties:

1. \( 1 \in I_0; \)
2. \( \sum_{i \in I_0} \sqrt{n_i} = \sum_{j \in J_0} \sqrt{n_j}; \)
3. For any real subset \( I'_0 \subset I_0, J'_0 \subset J_0 \), we have

\[
\sum_{i \in I'_0} \sqrt{n_i} \neq \sum_{j \in J'_0} \sqrt{n_j}.
\]

If \((I_0, J_0) = (I, J)\), then we say \((n_1, \ldots, n_k)\) is a primitive \((k, l)\)-point. Let \( \mathbb{N}_{k,l} \) denote the set of all points in \( \mathbb{N}^k \) which satisfy \((\ast)\) and \( \mathbb{N}^*_{k,l} \) the set of all primitive \((k, l)\)-points. Let \( G_{k,l} \) denote the set of all possible pairs \((I_0, J_0)\) when \((n_1, \ldots, n_k)\) runs through \( \mathbb{N}_{k,l} \). Note that if \( l = 1 \), then \( G_{k,1} = \{(I, J)\} \).

Suppose \((I_0, J_0) \in G_{k,l} \). Let \( l_1 = \#I_0, l_2 = l - l_1, k_1 = \#I_0 + \#J_0, k_2 = k - k_1 \). From \((\ast)\), we know that \( k_1 \geq 3 \). Define

\[
R_1^{(I_0, J_0)}(d; y) := \sum_{n_1 \geq \sqrt{n_{l+1}} \ldots \sqrt{n_k}, (n_1, \ldots, n_k) \in \mathbb{N}^*_{k,l}} \frac{d(n_1) \ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}}.
\]

If \((I_0, J_0) \neq (I, J)\), then \( l_1 < l, k_1 < k \) and we define

\[
R_2^{(I_0, J_0)}(d) := \sum_{n_1 \geq \sqrt{n_{l+1}} \ldots \sqrt{n_k}, (n_1, \ldots, n_k) \in \mathbb{N}^*_{k,l}} \frac{d(m_1) \ldots d(m_k)}{(m_1 \ldots m_k)^{3/4}}.
\]

By the induction assumption, \( R_2^{(I_0, J_0)}(d) \ll 1 \).

If \((n_1, \ldots, n_{k_1}) \in \mathbb{N}^*_{k_1,l_1} \), then by Lemma 2.1 we have

\[ n_j = s_j^2 h, \quad s_1 + \ldots + s_{l_1} = s_{l_1+1} + \ldots + s_{k_1}, \quad \mu(h) \neq 0. \]

Now \( n_1 > y \) implies that there exists at least one \( n_j \) \((l_1 + 1 \leq j \leq j_1)\) such that \( n_j \gg y \). We suppose \( n_{k_1} \gg y \). So we have

\[
R_1^{(I_0, J_0)}(d; y) \ll \sum_h \sum_{s_1 + \ldots + s_{l_1} = s_{l_1+1} + \ldots + s_{k_1}, s_1^2 h > y, s_{k_1}^2 h \gg y} \frac{d(s_1 h) \ldots d(s_{j_1} h)}{h^{3k_1/4}(s_1 \ldots s_{k_1})^{3/2}}.
\]
\[
\sum_h s_1 + \ldots + s_1 = s_{l+1} + \ldots + s_k \quad \text{for} \quad s_i^3 h > y, s_i^3 \gg y
\]
\[
\sum_h d^k(h) \sum_{s_1 > (y/h)^{1/2}} d^2(s_1) \sum_{s_k \gg (y/h)^{1/2}} d^2(s_k)
\]
\[
\sum_h d^k(h) \left(\frac{y}{h}\right)^{-1/2+\varepsilon} \ll y^{-1/2+\varepsilon}
\]
if we notice \(k \geq 3\).

If \(G_{k,l} = (I,J)\), we have

\[(3.3) \quad U_2(d; y) \ll R_1^{(I,J)}(d; y) \ll y^{-1/2+\varepsilon}.
\]

If \(G_{k,l} \neq (I,J)\), we have

\[(3.4) \quad U_2(d; y) \ll R_1^{(I,J)}(d; y) + \sum_{(I_0,J_0) \in G_{k,l}} R_1^{(I_0,J_0)}(d; y) R_2^{(I_0,J_0)}(d)
\]
\[\ll y^{-1/2+\varepsilon}.
\]

Now Lemma 3.1 follows from (3.1)–(3.4). \(\blacksquare\)

4. Proofs of Theorems 1 and 2. Suppose \(T \geq 10\) is a real number. It suffices to evaluate the integral \(\int_T^{2T} \Delta^k(x) \, dx\). Suppose \(y\) is a parameter such that \(T^\varepsilon < y \leq T^{1/3}\). For any \(T \leq x \leq 2T\), define

\[R_1 = R_1(x,y) := (\sqrt{2} \pi)^{-1} x^{1/4} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{xn} - \pi/4),
\]
\[R_2 = R_2(x,y) := \Delta(x) - R_1.
\]
We shall show that the higher-power moment of \(R_2\) is small and hence the integral \(\int_T^{2T} \Delta^k(x) \, dx\) can be well approximated by \(\int_T^{2T} R_1^k \, dx\), which is easy to evaluate.

4.1. Evaluation of the integral \(\int_T^{2T} R_1^k \, dx\). Suppose \(h \geq 3\) is any fixed integer. By the elementary formula

\[
\cos a_1 \ldots \cos a_h = \frac{1}{2^{h-1}} \sum_{(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}} \cos(a_1 + (-1)^{i_1} a_2 + (-1)^{i_2} a_3 + \ldots + (-1)^{i_{h-1}} a_h),
\]
we have
\[ R^h_1 = (\sqrt{2\pi})^{-h} x^{h/4} \sum_{n_1 \leq y} \ldots \sum_{n_h \leq y} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}} h \prod_{j=1}^h \cos(4\pi \sqrt{n_j x} - \pi/4) \]
\[ = \frac{x^{h/4}}{(\sqrt{2\pi})^{2h-1}} \sum_{(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}} \sum_{n_j \leq y} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}} \times \cos \left( 4\pi \sqrt{x} \alpha(n_1, \ldots, n_h; i_1, \ldots, i_{h-1}) - \frac{\pi}{4} \beta(i_1, \ldots, i_{h-1}) \right), \]
where
\[ \alpha(n_1, \ldots, n_h; i_1, \ldots, i_{h-1}) := \sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + \ldots + (-1)^{i_{h-1}} \sqrt{n_h}, \]
\[ \beta(i_1, \ldots, i_{h-1}) := 1 + (-1)^{i_1} + (-1)^{i_2} + \ldots + (-1)^{i_{h-1}}. \]
Thus we can write
\[ (4.1) \quad R^h_1 = \frac{1}{(\sqrt{2\pi})^{h} 2^{h-1}} (S_1(x) + S_2(x)), \]
where
\[ S_1(x) := x^{h/4} \sum_{(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}} \cos \left( -\frac{\pi \beta}{4} \right) \sum_{\alpha=0}^{n_j \leq y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}}, \]
\[ S_2(x) := x^{h/4} \sum_{(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}} \sum_{\alpha \neq 0} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}} \times \cos(4\pi \alpha \sqrt{x} - \pi \beta/4), \]
\[ \alpha := \alpha(n_1, \ldots, n_h; i_1, \ldots, i_{h-1}), \quad \beta := \beta(i_1, \ldots, i_{h-1}). \]

First consider the contribution of $S_1(x)$. We have
\[ (4.2) \quad \int_T^{2T} S_1(x) \, dx \]
\[ = \sum_{(i_1, \ldots, i_{h-1}) \in \{0,1\}^{h-1}} \cos \left( -\frac{\pi \beta}{4} \right) \sum_{\alpha=0}^{n_j \leq y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}} \int_T^{2T} x^{h/4} \, dx. \]
It is easily seen that if $\alpha = 0$, then $1 \in \{i_1, \ldots, i_{h-1}\}$. Let $l = i_1 + \ldots + i_{h-1}$. Then
\[ \sum_{\alpha=0}^{n_j \leq y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}} = s_{h;l}(d; y), \]
where $s_{h;l}(d; y)$ was defined in the last section.
By Lemma 3.1 we get

\[ (4.3) \quad \int_T S_1(x) \, dx = B_h^*(d) \int_T x^{h/4} \, dx + O(T^{1+\frac{h}{4}+\varepsilon} y^{-1/2}), \]

where

\[ B_h^*(d) := \sum_{(i_1, \ldots, i_{h-1}) \in \{0, 1\}^{h-1}} \cos \left( -\frac{\pi \beta}{4} \right) \sum_{(n_1, \ldots, n_h) \in \mathbb{N}^h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}}. \]

For any \((i_1, \ldots, i_{h-1}) \in \{0, 1\}^{h-1} \setminus \{(0, \ldots, 0)\},\) let

\[ S(d; i_1, \ldots, i_{h-1}) := \sum_{(n_1, \ldots, n_h) \in \mathbb{N}^h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}}, \]

\[ l(i_1, \ldots, i_{h-1}) := i_1 + \ldots + i_{h-1}. \]

It is easily seen that if \(l(i_1, \ldots, i_{h-1}) = l(i'_1, \ldots, i'_{h-1})\) or \(l(i_1, \ldots, i_{h-1}) + l(i'_1, \ldots, i'_{h-1}) = h,\) then

\[ S(d; i_1, \ldots, i_{h-1}) = S(d; i'_1, \ldots, i'_{h-1}) = s_{h; l(i_1, \ldots, i_{h-1})}(d). \]

From \((-1)^i = 1 - 2i\ (i = 0, 1)\) we also have

\[ \beta(i_1, \ldots, i_{h-1}) = h - 2l(i_1, \ldots, i_{h-1}). \]

So we get

\[ (4.4) \quad B_h^*(d) = \sum_{l=1}^{h-1} \sum_{l(i_1, \ldots, i_{h-1}) = l} \cos \left( -\frac{\pi \beta}{4} \right) S(d; i_1, \ldots, i_{h-1}) \]

\[ = \sum_{l=1}^{h-1} s_{h;l}(d) \cos \frac{\pi (h - 2l)}{4} \sum_{l(i_1, \ldots, i_{h-1}) = l} 1 \]

\[ = \sum_{l=1}^{h-1} \left( \frac{h-1}{l} \right) s_{h;l}(d) \cos \frac{\pi (h - 2l)}{4} = B_h(d). \]

Now we consider the contribution of \(S_2(x).\) By Lemma 2.3 we get

\[ (4.5) \quad \int_T S_2(x) \, dx \]

\[ \ll T^{1/2 + h/4} \sum_{(i_1, \ldots, i_{h-1}) \in \{0, 1\}^{h-1}} \sum_{n_j \leq y, 1 \leq j \leq h, \alpha \neq 0} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4} |\alpha|}. \]
Higher-power moments of $\Delta(x)$ (II)

It suffices to estimate the sum

$$\Sigma(y; i_1, \ldots, i_{h-1}) = \sum_{n_j \leq y, 1 \leq j \leq h, \alpha \neq 0} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}|\alpha|}$$

for fixed $(i_1, \ldots, i_{h-1}) \in \{0, 1\}^{h-1}$. If $(i_1, \ldots, i_{h-1}) = (0, \ldots, 0)$, then

$$\Sigma(y; 0, \ldots, 0) \ll \sum_{n_j \leq y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4} (\sqrt{n_1} + \ldots + \sqrt{n_h})} \ll \sum_{n_j \leq y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4 + 1/2h}} \ll y^{(h-2)/4} \log^h y,$$

where we used the estimates

$$\sum_{n \leq u} d(n) \ll u \log u, \quad x_1 + \ldots + x_h \gg (x_1 \ldots x_h)^{1/h}.$$

For $(i_1, \ldots, i_{h-1}) \neq (0, \ldots, 0)$, by a splitting argument we deduce that there exist a collection of numbers $1 < N_1, \ldots, N_h < y$ such that

$$\Sigma(y; i_1, \ldots, i_{h-1}) \ll \sum_{n_j < y, 1 \leq j \leq h} \frac{d(n_1) \ldots d(n_h)}{(n_1 \ldots n_h)^{3/4}|\alpha|},$$

where

$$\Sigma_1^* = \sum_{N_j < y, 1 \leq j \leq h} \frac{A(N_1, \ldots, N_h; i_1, \ldots, i_{h-1}; \Delta)}{(N_1 \ldots N_h)^{3/4} \Delta}.$$

Without loss of generality, we suppose $N_1 \leq \ldots \leq N_h \leq y$. By Lemma 2.2 we have $|\alpha| \gg N_h^{-(2h-2-2^{-1})}$. Then by a splitting argument and Lemma 2.4, for some $N_h^{-(2h-2-2^{-1})} \ll \Delta < y^{1/2}$ we get

$$\Sigma_1^* \ll \frac{y^e}{(N_1 \ldots N_h)^{3/4} \Delta} A(N_1, \ldots, N_h; i_1, \ldots, i_{h-1}; \Delta) \ll \frac{y^e}{(N_1 \ldots N_h)^{3/4} \Delta} (\Delta N_h^{1/2} N_1 \ldots N_{h-1} + N_1 \ldots N_{h-1}) \ll \frac{y^e}{(N_1 \ldots N_{h-1})^{1/4} + (N_1 \ldots N_{h-1})^{1/4} N_h^{3/4} \Delta} \ll y^e (N_h^{(h-2)/4} + N_h^{b(h)}) \ll y^{b(h)+e},$$

where $b(h)$ was defined in Section 1.1. Thus we get

$$\int_T^{2T} S_2(x) \, dx \ll T^{1/2 + h/4 + e} y^{b(h)}.$$
Hence from (4.1)–(4.6) we get

**Lemma 4.1.** For any fixed $h \geq 3$, we have

\[
\int T \mathcal{R}^h_1 dx = \frac{B_h(d)}{(\sqrt{2} \pi)^{h-1}} \int T x^{h/4} dx
\]

\[
+ O(T^{1+h/4+\varepsilon} y^{-1/2} + T^{1/2+h/4+\varepsilon} y^{b(h)}).
\]

**4.2. Higher-power moments of $\mathcal{R}_2$.** We first study the mean-square of $\mathcal{R}_2$. We begin with the truncated Voronoĭ formula \[9, (2.25)]

\[
\Delta(x) = (\pi \sqrt{2})^{-1} x^{1/4} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} \cos(4 \pi \sqrt{nx} - \pi/4)
\]

\[
+ O(x^{1/2+\varepsilon} N^{-1/2}),
\]

where $1 < N \ll x$. Taking $N = T$, we get

\[
\mathcal{R}_2 = (\pi \sqrt{2})^{-1} x^{1/4} \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} \cos(4 \pi \sqrt{nx} - \pi/4) + O(T^\varepsilon)
\]

\[
\ll \left| \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} e(2 \sqrt{nx}) \right| + T^\varepsilon,
\]

which implies

\[
\int T \mathcal{R}_2^2 dx \ll T^{1+\varepsilon} + \int T x^{1/4} \left| \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} e(2 \sqrt{nx}) \right|^2 dx
\]

\[
\ll T^{1+\varepsilon} + T^{3/2} \sum_{y < n \leq T} \frac{d^2(n)}{n^{3/2}}
\]

\[
+ T \sum_{y < m < n \leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})}
\]

\[
\ll T^{1+\varepsilon} + \frac{T^{3/2} \log^3 T}{y^{1/2}} \ll \frac{T^{3/2} \log^3 T}{y^{1/2}},
\]

where we used the estimates

\[
\sum_{n \leq u} d^2(n) \ll u \log^3 u, \quad \sum_{y < m < n \leq T} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n} - \sqrt{m})} \ll T^\varepsilon.
\]

Now suppose $y$ satisfies $y^{2b(K_0)} \leq T$. Hence from Lemma 4.1 we get

\[
\int T |\mathcal{R}_1|^K_0 dx \ll T^{1+K_0/4+\varepsilon},
\]
which implies

\( (4.10) \)

\[ \int_T^{2T} |\mathcal{R}_1|^{A_0} \, dx \ll T^{1+A_0/4+\varepsilon} \]

since \( A_0 \leq K_0 \). From (1.10) and (4.10) we get

\( (4.11) \)

\[ \int_T^{2T} |\mathcal{R}_2|^{A_0} \, dx \ll \int_T^{2T} (|\Delta(x)|^{A_0} + |\mathcal{R}_1|^{A_0}) \, dx \ll T^{1+A_0/4+\varepsilon}. \]

For any \( 2 < A < A_0 \), from (4.9), (4.11) and Hölder’s inequality we get

\( (4.12) \)

\[ \int_T^{2T} |\mathcal{R}_2|^A \, dx = \int_T^{2T} |\mathcal{R}_2|^{\frac{2(A_0-A)}{A_0-2}} \frac{A_0-(A-2)}{A_0-2} \, dx \]

\[ \ll \left( \int_T^{2T} \mathcal{R}_2^2 \, dx \right)^{\frac{A_0-A}{A_0-2}} \left( \int_T^{2T} |\mathcal{R}_2|^{A_0} \, dx \right)^{\frac{A_0-2}{A_0-2}} \]

\[ \ll T^{1+A/4+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}. \]

Thus, we have proved the following

**Lemma 4.2.** Suppose \( T^\varepsilon \leq y \leq T^{1/2b(K_0)} \), \( 2 < A < A_0 \). Then

\( (4.13) \)

\[ \int_T^{2T} |\mathcal{R}_2|^A \, dx \ll T^{1+A/4+\varepsilon} \frac{y}{y-(A_0-A)/2(A_0-2)}. \]

**4.3. Proof of Theorem 1.** Suppose \( 3 \leq k \leq K(A_0) \) and \( T^\varepsilon \leq y \leq T^{1/2b(K_0)} \). By the elementary formula \( (a+b)^k - a^k \ll |a^{k-1}b| + |b|^k \), we get

\( (4.14) \)

\[ \int_T^{2T} \Delta^k(x) \, dx = \int_T^{2T} \mathcal{R}_1^k \, dx + O\left( \int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2| \, dx \right) + O\left( \int_T^{2T} |\mathcal{R}_2|^k \, dx \right). \]

If \( k-1 < A_0/2 \), then from (4.9), (4.10) and Cauchy’s inequality we get

\[ \int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2| \, dx \ll \left( \int_T^{2T} |\mathcal{R}_1|^{2(k-1)} \, dx \right)^{1/2} \left( \int_T^{2T} |\mathcal{R}_2|^2 \, dx \right)^{1/2} \ll T^{1+k/4+\varepsilon} y^{-1/4}. \]

If \( k-1 \geq A_0/2 \), then from (4.10), Lemma 4.2 and Hölder’s inequality we get

\[ \int_T^{2T} |\mathcal{R}_1^{k-1}\mathcal{R}_2| \, dx \ll \left( \int_T^{2T} |\mathcal{R}_1|^{A_0} \, dx \right)^{(k-1)/A_0} \left( \int_T^{2T} |\mathcal{R}_2|^{A_0/(A_0-k+1)} \, dx \right)^{(A_0-k+1)/A_0} \]

\[ \ll T^{1+k/4+\varepsilon} y^{-(A_0-k)/2(A_0-2)}. \]
Thus we have
\begin{equation}
2 \int_T |\mathcal{R}_1^{k-1} \mathcal{R}_2| \, dx + \int_T |\mathcal{R}_2|^k \, dx \ll T^{1+k/4+\varepsilon} y^{-\sigma(k,A_0)},
\end{equation}
where $\sigma(k,A_0)$ was defined in Section 1.1.

From (4.14) and (4.15) we get
\begin{equation}
2 \int_T \Delta^k(x) \, dx = \int_T \mathcal{R}_1^k \, dx + O(T^{1+k/4+\varepsilon} y^{-\sigma(k,A_0)}).
\end{equation}

Now take $y = T^{1/2b(K_0)}$. From Lemma 4.1 and (4.16) we get
\begin{equation}
2 \int_T \Delta^k(x) \, dx = \frac{B_k(d)}{(\sqrt{2 \pi})^{k^2 k - 1}} \int_T x^{k/4} \, dx + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon}).
\end{equation}

Theorem 1 follows from (4.17) immediately.

**4.4. Proof of Theorem 2.** Suppose $T^\varepsilon \leq y \leq T^{1/3}$. By the truncated Voronoï formula (4.8), we have
\begin{align*}
\mathcal{R}_2 &= (\sqrt{2 \pi})^{-1} x^{1/4} \sum_{y < n \leq N} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}),
\end{align*}
where $y < N \ll T$. Using Ivić’s large-value technique directly to $\mathcal{R}_2$ without modifications, we get the estimate
\begin{equation}
\int_T |\mathcal{R}_2|^{A_0} \, dx \ll T^{1+A_0/4+\varepsilon}
\end{equation}
with $A_0 = 184/19, T^\varepsilon \leq y \leq T^{1/3}$. We omit the details since the argument is completely the same as that of Ivić. Combining (4.18) and (1.10) we get
\begin{equation}
\int_T |\mathcal{R}_1|^{A_0} \, dx \ll T^{1+A_0/4+\varepsilon}
\end{equation}
with $A_0 = 184/19, T^\varepsilon \leq y \leq T^{1/3}$.

By the same argument as in the last subsection, we deduce that for $T^\varepsilon \leq y \leq T^{1/3}$,
\begin{equation}
\int_T \Delta^k(x) \, dx = \int_T \mathcal{R}_1^k \, dx + O(T^{1+k/4+\varepsilon} y^{-\sigma(k,184/19)}).
\end{equation}
Higher-power moments of $\Delta(x)$ (II)

Take $y = T^{1/(2b(k)+2\sigma(k,184/19))}$. From Lemma 4.1 again we get

\begin{align*}
(4.21) \int_T^{2T} \Delta^k(x) \, dx &= \frac{B_k(d)}{(\sqrt{2 \pi} k^{2k-1})} \int_T^{2T} x^{k/4} \, dx + O(T^{1+k/4 - \frac{\sigma(k,184/19)}{2b(k)+2\sigma(k,184/19)} + \varepsilon}) \\
&= \frac{B_k(d)}{(\sqrt{2 \pi} k^{2k-1})} \int_T^{2T} x^{k/4} \, dx + O(T^{1+k/4 - \delta_2(k,184/19) + \varepsilon}),
\end{align*}

and Theorem 2 follows.

5. Proofs of other theorems. $P(x)$ has the following truncated Voronoï formula:

\begin{align*}
(5.1) \quad P(x) &= -\frac{1}{\pi} \sum_{n \leq N} r(n)n^{-3/4}x^{1/4}\cos(4\pi \sqrt{n}x + \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2})
\end{align*}

for $1 \leq N \ll x$, which follows from Lemma 3 of Müller [16]. Moreover, $A(x)$ has the following truncated Voronoï formula:

\begin{align*}
(5.2) \quad A(x) &= \frac{1}{\pi \sqrt{2}} x^{\kappa/2-1/4} \sum_{n \leq N} a(n)n^{-\kappa/2-1/4}\cos(4\pi \sqrt{n}x - \pi/4) \\
&\quad + O(x^{\kappa/2+\varepsilon}N^{-1/2})
\end{align*}

for $1 \leq N \ll x$, which is a special case of Theorem 1.1 of Jutila [13]. So in the same way as in the last section, we get Theorems 3 and 4.

Now we prove Theorem 5. We shall follow Ivić [10]. Define

$$\Delta^*(x) := \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1), \quad x > 0.$$ 

Jutila [12] proved that

\begin{align*}
(5.3) \quad \int_0^T \left( E(t) - 2\pi \Delta^* \left( \frac{t}{2\pi} \right) \right)^2 \, dt \ll T^{4/3} \log^3 T,
\end{align*}

which means that $E(t)$ is well approximated by $2\pi \Delta^*(t/2\pi)$ at least in the mean square sense.

Suppose $A_0 > 9$ is a real number such that both (1.10) and (1.32) hold. Since (see Jutila [11])

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2} \Delta(4x),$$

from (1.10) we get

\begin{align*}
(5.4) \quad \int_0^T |\Delta^*(t)|^{A_0} \, dt \ll T^{1+A_0/4+\varepsilon}.
\end{align*}
Then from (1.32), (5.3), (5.4) and Hölder’s inequality, for any $3 \leq k < A_0$ we get

$$\int_{0}^{T} E^k(t) \, dt - (2\pi)^{k+1} \int_{0}^{T/2\pi} (\Delta^*(t))^{k} \, dt$$

$$= \int_{0}^{T} \left( E^k(t) - \left( 2\pi \Delta^* \left( \frac{t}{2\pi} \right) \right)^k \right) \, dt$$

$$\ll \int_{0}^{T} \left| E(t) - 2\pi \Delta^* \left( \frac{t}{2\pi} \right) \right| \left( |E(t)|^{k-1} + \left| \Delta^* \left( \frac{t}{2\pi} \right) \right|^{k-1} \right) \, dt$$

$$\ll T^{1+k/4-\sigma(k,A_0)/3+\varepsilon},$$

where $\sigma(k,A_0)$ was defined in Section 1.1. By (5.5) the problem is reduced to evaluating the integral $\int_{0}^{T} (\Delta^*(t))^{k} \, dt$. For $1 \ll N \ll x$, we have [10, (7)]

$$\Delta^*(x) = \frac{1}{\pi \sqrt{2}} \sum_{n \leq N} (-1)^n d(n) n^{-3/4} x^{1/4} \cos(4\pi \sqrt{n x} - \pi/4)$$

$$+ O(x^{1/2+\varepsilon} N^{-1/2}),$$

which is similar to (4.8). Let $d^*(n) = (-1)^n d(n)$. Then in the same way as in the proof of Theorem 1, we get the asymptotic formula

$$\int_{1}^{T} (\Delta^*(t))^{k} \, dt = \frac{B_k(d^*)}{(1 + k/4) 2^{3k/2 - 1} \pi^k} T^{1+k/4} + O(T^{1+k/4-\delta_1(k,A_0)+\varepsilon})$$

for any $3 \leq k < A_0$.

We shall use

**Lemma 5.1.** Suppose $1 \leq l < k$ are fixed integers and $(n_1, \ldots, n_k) \in \mathbb{N}^k$. If

$$\sqrt{n_1} + \cdots + \sqrt{n_l} = \sqrt{n_{l+1}} + \cdots + \sqrt{n_k},$$

then $2 \mid (n_1 + \cdots + n_k)$.

**Proof.** For any $n \in \mathbb{N}$, let $h(n)$ denote the squarefree part of $n$. Let $S = \{h(n_1), \ldots, h(n_k)\} \cap \mathbb{N}$ and $s = \#S$. For convenience, write

$$S = \{h_1, \ldots, h_s\}, \quad I = \{1, \ldots, l\}, \quad J = \{l+1, \ldots, k\}.$$

From Lemma 2.1 we can write $I = \bigcup_{e=1}^{s} I_e$, $J = \bigcup_{e=1}^{s} J_e$ so that for each $1 \leq e \leq s$,

$$\sum_{i \in I_e} \sqrt{n_i} = \sum_{j \in J_e} \sqrt{n_j}$$

and all $n_i$ ($i \in I_e$) and $n_j$ ($j \in J_e$) have the same squarefree part $h_e$. Namely
we have \((1 \leq e \leq s)\)
\[
n_i = m_i^2 h_e \quad (i \in I_e), \quad n_j = m_j^2 h_e \quad (j \in J_e), \quad \sum_{i \in I_e} m_i = \sum_{j \in J_e} m_j.
\]

Thus we get
\[
n_1 + \ldots + n_k = \sum_{e=1}^{s} \left( \sum_{i \in I_e} n_i + \sum_{j \in J_e} n_j \right)
\]
\[
= \sum_{e=1}^{s} \left( \sum_{i \in I_e} m_i^2 h_e + \sum_{j \in J_e} m_j^2 h_e \right) = \sum_{e=1}^{s} \left( \sum_{i \in I_e} m_i + \sum_{j \in J_e} m_j \right) h_e
\]
\[
= 2 \sum_{e=1}^{s} h_e \sum_{i \in I_e} m_i \equiv 0 \pmod{2},
\]
where we used the simple congruence \(n^2 \equiv n \pmod{2}\).

From Lemma 5.1, for any \(1 \leq l < k\) we get
\[
s_{k;l}(d^*) = \sum_{\sqrt{n_1+\ldots+n_{l-1}}=\sqrt{m_{l+1}+\ldots+n_k}} (-1)^{n_1+\ldots+n_k} \frac{d(n_1)\ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}}
\]
\[
= \sum_{\sqrt{n_1+\ldots+n_{l-1}}=\sqrt{m_{l+1}+\ldots+n_k}} \frac{d(n_1)\ldots d(n_k)}{(n_1 \ldots n_k)^{3/4}} = s_{k;l}(d).
\]

Hence we conclude that
\[
(5.8) \quad b_k(d^*) = b_k(d).
\]

From (5.5), (5.7) and (5.8) we get (1.33).

Similarly to Theorem 2, we can prove the asymptotic formula
\[
(5.9) \quad \int_{1}^{T} (\Delta^*(t))^k dt = \frac{b_k(d)}{(1 + k/4)^{2^{3k/2 - 1}} \pi^k} T^{1+k/4} + O(T^{1+k/4 - \delta_2(k; 576/61) + \varepsilon})
\]
for any \(3 \leq k \leq 9\), which combined with (5.5) yields the second part of Theorem 3.

\[
\Delta(x) \ll x^{131/416} (\log x)^{2^{6957/8320}},
\]
which implies that the exponent 184/19 for which the formula (1.10) holds can be improved to \(A_0 = 262/27\). Correspondingly, the exponent \(\delta_2(k, 184/19)\) in Theorem 2 can be improved to \(\delta_2(k, 262/27)\) for \(k = 6, 7, 8, 9\). The author deeply thanks Professor A. Schinzel for informing him about M. N. Huxley’s new result.
References


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