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A note on the fourth moment of Dirichlet L-functions

by

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1. Introduction. For χ a Dirichlet character (mod q), the moments of $L(s,\chi)$ have many applications, for example to the distribution of primes in the arithmetic progressions to modulus q. The asymptotic formula of the fourth power moment in the q-aspect has been obtained by Heath-Brown [1], for q prime, and more recently by Soundararajan [5] for general q. Following Soundararajan's work, Young [7] pushed the result much further by computing the fourth moment for prime moduli q with a power saving in the error term. The problem essentially reduces to the analysis of a particular divisor sum. To this end, Young used various techniques to estimate the off-diagonal terms.

In the case that the t-aspect is also included, a result of Montgomery [2] states that

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_0^T |L(1/2+it,\chi)|^4\,dt \ll \varphi(q)T(\log qT)^4$$

for $q,T\geq 2$, where $\sum^*_{\chi\,(\mathrm{mod}\,q)}$ indicates that the sum is restricted to the primitive characters modulo q. As we shall see, the upper bound is too large by a factor $(q/\varphi(q))^5$. A second result of relevance is due to Rane [4]. After correcting a misprint it states that

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_T^{2T} |L(1/2+it,\chi)|^4\,dt = \frac{\varphi^*(q)T}{2\pi^2}\prod_{p\mid q}\frac{(1-p^{-1})^3}{1+p^{-1}}(\log qT)^4 + O(2^{\omega(q)}\varphi^*(q)T(\log qT)^3(\log\log 3q)^5),$$

where $\varphi^*(q)$ is the number of primitive characters modulo q and $\omega(q)$ is the number of distinct prime factors of q. This can only give an asymptotic relation when $2^{\omega(q)}\leq \log q$, which holds for some values of q, but not others.

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Finally, we mention the work of Wang [6], where an asymptotic formula is proved for $q \le T^{1-\delta}$, for any fixed $\delta > 0$.

The goal of the present note is to establish an asymptotic formula, valid for all $q, T \ge 2$, as soon as $q \to \infty$.

THEOREM 1. *For $q, T \ge 2$ we have, in the notation above,*

$$\sum_{\chi \,(\mathrm{mod}\, q)}^{*} \int_0^T |L(1/2+it,\chi)|^4\,dt$$
$$= \left(1+O\left(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\right)\right)\frac{\varphi^*(q)T}{2\pi^2}\prod_{p\mid q}\frac{(1-p^{-1})^3}{1+p^{-1}}(\log qT)^4 + O(qT(\log qT)^{7/2}).$$

Our proof uses ideas from the works of Heath-Brown [1] and Soundararajan [5], but there is extra work to do to handle the integration over t.

REMARK 1. It is possible, with only a little more effort, to extend the range to cover all $T > 0$. In this case the term $\varphi^*(q)T$ in the main term remains the same, as does the factor qT in the error term, but one must replace $\log qT$ by $\log q(T+2)$ both in the main term and in the error term.

REMARK 2. One may readily verify that our result provides an asymptotic formula, as soon as $q \to \infty$, with an error term which saves at least a factor $O((\log\log q)^{-1/2})$.

REMARK 3. The literature appears not to contain a precise analogue of this for the second moment. However, Motohashi [3] has considered a uniform mean value in t-aspect. He proved that if χ is a primitive character modulo a prime q, then

$$\int_0^T |L(1/2+it,\chi)|^2\,dt = \frac{\varphi(q)T}{q}\left(\log\frac{qT}{2\pi} + 2\gamma + 2\sum_{p\mid q}\frac{\log p}{p-1}\right) + O((q^{1/3}T^{1/3} + q^{1/2})(\log qT)^4)$$

for $T \ge 2$. This provides an asymptotic formula when $q \le T^{2-\delta}$ for any fixed $\delta > 0$. Our theorem does not give a power saving in the error term, but it yields an asymptotic formula without any restrictions on q and T.

2. Auxiliary lemmas

LEMMA 1. *Let χ be a primitive character* (mod q) *such that $\chi(-1) = (-1)^{\mathfrak{a}}$ with $\mathfrak{a} = 0$ or 1. Then*

$$|L(1/2+it,\chi)|^2 = 2\sum_{a,b\ge 1}\frac{\chi(a)\,\overline{\chi(b)}}{\sqrt{ab}}\left(\frac{a}{b}\right)^{-it} W_{\mathfrak{a}}\left(\frac{\pi ab}{q};t\right),$$

where

$$W_{\mathfrak{a}}(x;t) = \frac{1}{2\pi i}\int\limits_{(2)} \frac{\Gamma(1/4+it/2+z/2+\mathfrak{a}/2)\Gamma(1/4-it/2+z/2+\mathfrak{a}/2)}{|\Gamma(1/4+it/2+\mathfrak{a}/2)|^2}\, e^{z^2}x^{-z}\frac{dz}{z}.$$

Proof. Let

$$I := \frac{1}{2\pi i}\int\limits_{(2)} \frac{\Lambda(1/2+it+z,\chi)\Lambda(1/2-it+z,\overline{\chi})}{|\Gamma(1/4+it/2+\mathfrak{a}/2)|^2}\, e^{z^2}\frac{dz}{z},$$

where

$$\Lambda(1/2+s,\chi) = \left(\frac{q}{\pi}\right)^{s/2}\Gamma\left(\frac{1}{4}+\frac{s}{2}+\frac{\mathfrak{a}}{2}\right)L(1/2+s,\chi).$$

We recall the functional equation

$$\Lambda(1/2+s,\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}\,\Lambda(1/2-s,\overline{\chi}).$$

Hence, moving the line of integration to $\Re z = -2$ and applying Cauchy's theorem, we obtain $|L(1/2+it,\chi)|^2 = 2I$. Finally, expanding $L(1/2+it+z,\chi)L(1/2-it+z,\overline{\chi})$ in a Dirichlet series and integrating termwise we obtain the lemma. ∎

We decompose $|L(1/2+it,\chi)|^2$ as $2(A(t,\chi)+B(t,\chi))$, where

$$A(t,\chi) = \sum_{ab\le Z}\frac{\chi(a)\,\overline{\chi(b)}}{\sqrt{ab}}\left(\frac{a}{b}\right)^{-it} W_{\mathfrak{a}}\left(\frac{\pi ab}{q};t\right),$$

$$B(t,\chi) = \sum_{ab> Z}\frac{\chi(a)\,\overline{\chi(b)}}{\sqrt{ab}}\left(\frac{a}{b}\right)^{-it} W_{\mathfrak{a}}\left(\frac{\pi ab}{q};t\right),$$

with $Z = qT/2^{\omega(q)}$. In the next two sections, we evaluate the second moments of $A(t,\chi)$ and $B(t,\chi)$, after which our theorem will be an easy consequence.

The function $W_{\mathfrak{a}}(x;t)$ approximates the characteristic function of the interval $[0,|t|]$. Indeed, we have the following.

LEMMA 2. *The function* $W_{\mathfrak{a}}(x;t)$ *satisfies*

$$W_{\mathfrak{a}}(x;t) = \begin{cases} O((\tau/x)^2) & \text{for } x\ge\tau,\\ 1+O((x/\tau)^{1/4}) & \text{for } 0<x<\tau,\end{cases}$$

and

$$\frac{\partial}{\partial t}W_{\mathfrak{a}}(x;t) \ll \begin{cases} \tau^{-1}(\tau/x)^2 & \text{for } x\ge\tau,\\ \tau^{-1}(x/\tau)^{1/4} & \text{for } 0<x<\tau,\end{cases}$$

where $\tau = |t|+2$.

Proof. The first estimate is a direct consequence of Stirling's formula, while for the second one merely shifts the line of integration to $\Re z = -1/4$ before employing Stirling's formula. To handle the derivative one proceeds as before, differentiates under the integral sign and uses the estimate

$$\Gamma'(w)/\Gamma(w) = \log w + O(|w|^{-1}),$$

which holds for $1/8 \le \Re w \le 2$. ∎

The next lemma concerns the orthogonality of primitive Dirichlet characters.

LEMMA 3. *For $(mn, q) = 1$, we have*

$$\sum_{\chi\,(\mathrm{mod}\,q)}^{*} \chi(m)\overline{\chi}(n) = \sum_{k\mid(q,m-n)} \varphi(k)\mu(q/k).$$

Moreover,

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi(-1)=(-1)^{\mathfrak{a}}}}^{*} \chi(m)\overline{\chi}(n) = \frac{1}{2}\sum_{k\mid(q,m-n)} \varphi(k)\mu(q/k) + \frac{(-1)^{\mathfrak{a}}}{2}\sum_{k\mid(q,m+n)} \varphi(k)\mu(q/k).$$

Proof. This follows from [1, p. 27]. ∎

To handle the off-diagonal term we shall use the following bounds.

LEMMA 4. *Let k be a positive integer and $Z_1, Z_2 \ge 2$. If $Z_1Z_2 \le k^{19/10}$ then*

$$E := \sum_{\substack{Z_1\le ab<2Z_1\\ Z_2\le cd<2Z_2\\ ac\equiv\pm bd\,(\mathrm{mod}\,k)\\ ac\ne bd\\ (abcd,k)=1}} \frac{1}{\left|\log\frac{ac}{bd}\right|} \ll \frac{(Z_1Z_2)^{1+\varepsilon}}{k}$$

for any fixed $\varepsilon > 0$, while if $Z_1Z_2 > k^{19/10}$ then

$$E \ll \frac{Z_1Z_2}{k}(\log Z_1Z_2)^3. \tag{2.1}$$

Proof. We note that in each case the contribution of the terms with $|\log ac/bd| > \log 2$ is satisfactory, by the corresponding lemma of Soundararajan [5, Lemma 3]. Thus, by symmetry, it is enough to consider the terms with $bd < ac \le 2bd$. We shall show how to handle the terms in which $ac \equiv bd\,(\mathrm{mod}\ k)$, the alternative case being dealt with similarly. We write $n = bd$ and $ac = kl + bd$ and observe that $kl \le bd$. We deduce that $n \le 2\sqrt{Z_1Z_2}$ and $1 \le l \le 2\sqrt{Z_1Z_2}/k$. Since $\log ac/bd \gg kl/n$ the contribution of these terms to E is

$$\ll \frac{1}{k}\sum_{l\le 2\sqrt{Z_1Z_2}/k} \frac{1}{l}\sum_{\substack{n\le 2\sqrt{Z_1Z_2}\\ (n,k)=1}} nd(n)d(kl+n).$$

We estimate the sum over n using a bound from Heath-Brown's paper [1, (17)]. This shows that the above expression is

$$\ll \frac{Z_1 Z_2 (\log Z_1 Z_2)^2}{k} \sum_{l \le 2\sqrt{Z_1 Z_2}/k} \frac{1}{l} \sum_{d|l} d^{-1} \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3.$$

This suffices to complete the proof. The reader will observe that when $Z_1 Z_2 \le k^{19/10}$ it is only the terms with $|\log ac/bd| > \log 2$ which prevent us from achieving the bound (2.1). ∎

Finally, we shall require the following two lemmas [5, Lemmas 4 and 5].

LEMMA 5. *For $q \ge 2$ we have*

$$\sum_{\substack{n \le x \\ (n,q)=1}} \frac{1}{n} = \frac{\varphi(q)}{q} (\log x + O(1 + \log \omega(q))) + O\left(\frac{2^{\omega(q)} \log x}{x}\right).$$

LEMMA 6. *For $x \ge \sqrt{q}$ we have*

$$\sum_{\substack{n \le x \\ (n,q)=1}} \frac{2^{\omega(n)}}{n} \ll \left(\frac{\varphi(q)}{q}\right)^2 (\log x)^2$$

and

$$\sum_{\substack{n \le x \\ (n,q)=1}} \frac{2^{\omega(n)}}{n} \left(\log \frac{x}{n}\right)^2 = \left(1 + O\left(\frac{1 + \log \omega(q)}{\log q}\right)\right) \frac{(\log x)^4}{12\zeta(2)} \prod_{p|q} \frac{1 - 1/p}{1 + 1/p}.$$

3. The main term. Applying Lemma 3 we have

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_0^T A(t,\chi)^2\, dt = M + E,$$

where

$$M = \frac{\varphi^*(q)}{2} \sum_{\mathfrak{a}=0,1} \sum_{\substack{ab, cd \le Z \\ ac = bd \\ (abcd, q)=1}} \frac{1}{\sqrt{abcd}} \int_0^T W_{\mathfrak{a}}\left(\frac{\pi ab}{q}; t\right) W_{\mathfrak{a}}\left(\frac{\pi cd}{q}; t\right) dt$$

and

$$E = \sum_{k|q} \varphi(k) \mu(q/k) E(k),$$

with

$$E(k)=\sum_{\mathfrak{a}=0,1}\sum_{\substack{ab,cd\le Z\\ ac\equiv\pm bd\,(\mathrm{mod}\,k)\\ ac\ne bd\\ (abcd,q)=1}}\frac{1}{\sqrt{abcd}}\int_0^T\left(\frac{ac}{bd}\right)^{-it}W_{\mathfrak{a}}\Big(\frac{\pi ab}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi cd}{q};t\Big)\,dt.$$

We first estimate the error term E. We integrate by parts, using Lemma 2. This produces

$$E(k)\ll\sum_{\substack{ab,cd\le Z\\ ac\equiv\pm bd\,(\mathrm{mod}\,k)\\ ac\ne bd\\ (abcd,q)=1}}\frac{1}{\sqrt{abcd}\,\left|\log\frac{ac}{bd}\right|}.$$

We divide the terms $ab,cd\le Z$ into dyadic blocks $Z_1\le ab<2Z_1$ and $Z_2\le cd<2Z_2$. From Lemma 4, the contribution of this range to $E(k)$ is

$$\ll\frac{1}{\sqrt{Z_1Z_2}}\,\frac{Z_1Z_2}{k}\,(\log Z_1Z_2)^3=\frac{\sqrt{Z_1Z_2}}{k}\,(\log Z_1Z_2)^3$$

if $Z_1Z_2>k^{19/10}$, and is $O((Z_1Z_2)^{1/2+\varepsilon}k^{-1})$ if $Z_1Z_2\le k^{19/10}$. Summing over all such dyadic blocks we have

$$E(k)\ll\frac{Z}{k}\,(\log Z)^3+k^{-1/20+2\varepsilon}.$$

Thus

$$E\ll Z2^{\omega(q)}(\log Z)^3\ll qT(\log qT)^3. \tag{3.1}$$

We now turn to the main term M. Since $ac=bd$, we can write $a=gr$, $b=gs$, $c=hs$ and $d=hr$, where $(r,s)=1$. We put $n=rs$. Hence

$$M=\frac{\varphi^*(q)}{2}\sum_{\mathfrak{a}=0,1}\sum_{\substack{n\le Z\\ (n,q)=1}}\frac{2^{\omega(n)}}{n}\sum_{\substack{g,h\le\sqrt{Z/n}\\ (gh,q)=1}}\frac{1}{gh}\int_0^T W_{\mathfrak{a}}\Big(\frac{\pi g^2n}{q};t\Big)W_{\mathfrak{a}}\Big(\frac{\pi h^2n}{q};t\Big)\,dt.$$

From Lemma 2 we have $W_{\mathfrak{a}}(\pi g^2n/q;t)=1+O(g^{1/2}(n/qt)^{1/4})$, whence

$$M=\varphi^*(q)T\sum_{\substack{n\le Z\\ (n,q)=1}}\frac{2^{\omega(n)}}{n}\bigg(\sum_{\substack{g\le\sqrt{Z/n}\\ (g,q)=1}}\frac{1}{g}+O(1)\bigg)^2.$$

We split the terms $n\le Z$ into the cases $n\le Z_0$ and $Z_0<n\le Z$, where $Z_0=Z/9^{\omega(q)}$. In the first case, from Lemma 5 the sum over g is

$$\frac{\varphi(q)}{2q}\log\frac{Z_0}{n}+O(1+\log\omega(q)),$$

since the first error term in Lemma 5 dominates the second. Hence the contribution of such values of n to M is

$$\varphi^*(q)T\left(\frac{\varphi(q)}{2q}\right)^2\sum_{\substack{n\le Z_0\\(n,q)=1}}\frac{2^{\omega(n)}}{n}\left(\left(\log\frac{Z_0}{n}\right)^2+O(\omega(q)\log Z)\right).$$

Here we use the fact that $q/\varphi(q)\ll 1+\log\omega(q)$. This estimate will be employed a number of times in what follows, without further comment. In view of Lemma 6 the contribution from terms with $n\le Z_0$ is now seen to be

$$\frac{\varphi^*(q)T}{8\pi^2}\prod_{p\mid q}\frac{(1-1/p)^3}{1+1/p}(\log Z_0)^4\left(1+O\left(\frac{\omega(q)}{\log q}\right)\right). \tag{3.2}$$

For $Z_0\le n\le Z$, we extend the sum over g to all $g\le 3^{\omega(q)}$ that are coprime to q. By Lemma 5, this sum is $\ll\omega(q)\varphi(q)/q$. Hence the contribution of these terms to M is

$$\ll\varphi^*(q)T\left(\omega(q)\,\frac{\varphi(q)}{q}\right)^2\sum_{Z_0\le n\le Z}\frac{2^{\omega(n)}}{n}\ll\varphi^*(q)T\left(\frac{\varphi(q)}{q}\right)^4\omega(q)^2(\log Z)^2.$$

Combining this with (3.1) and (3.2) we obtain

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_0^T A(t,\chi)^2\,dt = \left(1+O\left(\frac{\omega(q)}{\log q}\right)\right)\frac{\varphi^*(q)T}{8\pi^2}\prod_{p\mid q}\frac{(1-1/p)^3}{1+1/p}(\log qT)^4. \tag{3.3}$$

4. The error term. We have

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_0^T B(t,\chi)^2\,dt\le\sum_{\chi\,(\mathrm{mod}\,q)}\int_0^T B(t,\chi)^2\,dt$$
$$=\frac{\varphi(q)}{2}\sum_{\mathfrak{a}=0,1}\sum_{\substack{ab,cd>Z\\ac\equiv\pm bd\,(\mathrm{mod}\,q)\\(abcd,q)=1}}\frac{1}{\sqrt{abcd}}\int_0^T\left(\frac{ac}{bd}\right)^{-it}W_{\mathfrak{a}}\left(\frac{\pi ab}{q};t\right)W_{\mathfrak{a}}\left(\frac{\pi cd}{q};t\right)dt.$$

Using Lemma 2 and integration by parts, the integral over t is

$$\ll\frac{1}{\left|\log\frac{ac}{bd}\right|}\left(1+\frac{ab}{qT}\right)^{-2}\left(1+\frac{cd}{qT}\right)^{-2}$$

if $ac\ne bd$, and is

$$\ll T\left(1+\frac{ab}{qT}\right)^{-2}\left(1+\frac{cd}{qT}\right)^{-2}$$

if $ac = bd$. Hence

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \int_0^T B(t,\chi)^2\,dt = O(R_1 + R_2),$$

where

$$R_1 = \varphi(q)T \sum_{\substack{ab,cd>Z\\ ac=bd\\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}}\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2},$$

$$R_2 = \varphi(q) \sum_{\substack{ab,cd>Z\\ ac\equiv \pm bd \,(\mathrm{mod}\, q)\\ ac\neq bd\\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}\,\big|\log\frac{ac}{bd}\big|}\Big(1+\frac{ab}{qT}\Big)^{-2}\Big(1+\frac{cd}{qT}\Big)^{-2}.$$

To estimate R_2, we again break the terms into dyadic ranges $Z_1 \le ab < 2Z_1$ and $Z_2 \le cd < 2Z_2$, where $Z_1, Z_2 > Z$. By Lemma 4, the contribution of each such block is

$$\ll \frac{\varphi(q)}{\sqrt{Z_1Z_2}}\Big(1+\frac{Z_1}{qT}\Big)^{-2}\Big(1+\frac{Z_2}{qT}\Big)^{-2}\frac{Z_1Z_2}{q}(\log Z_1Z_2)^3.$$

Summing over all the dyadic ranges we obtain

$$R_2 \ll \varphi(q)T(\log qT)^3. \tag{4.1}$$

To handle R_1 we argue as in the previous section. We write $a = gr$, $b = gs$, $c = hs$ and $d = hr$, where $(r,s) = 1$, and we put $n = rs$. Then

$$R_1 \ll \varphi(q)T \sum_{(n,q)=1} \frac{2^{\omega(n)}}{n}\Big(\sum_{\substack{g>\sqrt{Z/n}\\ (g,q)=1}} \frac{1}{g}\Big(1+\frac{g^2n}{qT}\Big)^{-2}\Big)^2. \tag{4.2}$$

We split the sum over n into the ranges $n \le qT$ and $n > qT$. In the first case, the sum over g is

$$\ll 1 + \sum_{\substack{\sqrt{Z/n}\le g\le\sqrt{qT/n}\\ (g,q)=1}} \frac{1}{g}.$$

When $n \le Z_0$ this is

$$\ll \frac{\varphi(q)}{q}\,\omega(q)$$

by Lemma 5. In the alternative case $n > Z_0$ we extend the sum over g to include all $g \le 3^{\omega(q)}$ that are coprime to q. Lemma 5 then gives the same bound as before. Thus the contribution of the terms $n \le qT$ to (4.2), using

Lemma 6, is

$$\ll \varphi(q)T\Big(\frac{\varphi(q)}{q}\,\omega(q)\Big)^2 \sum_{\substack{n\le qT\\(n,q)=1}} \frac{2^{\omega(n)}}{n} \ll qT\Big(\frac{\varphi(q)}{q}\Big)^5 \omega(q)^2(\log qT)^2. \tag{4.3}$$

In the remaining case $n > qT$, the sum over g in (4.2) is $O(q^2T^2/n^2)$. Hence the contribution of such terms is

$$\ll \varphi(q)T \sum_{n>qT} \frac{2^{\omega(n)}}{n}\,\frac{q^4T^4}{n^4} \ll \varphi(q)T\log qT.$$

In view of (4.1) and (4.3) we now have

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T B(t,\chi)^2\,dt \ll qT\Big(\frac{\varphi(q)}{q}\Big)^5 \omega(q)^2(\log qT)^2 + \varphi(q)T(\log qT)^3. \tag{4.4}$$

5. Deduction of Theorem 1. From Lemma 1 we have

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T |L(1/2+it,\chi)|^4\,dt = 4\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T (A(t,\chi)^2 + 2A(t,\chi)B(t,\chi) + B(t,\chi)^2)\,dt.$$

The first and third terms on the right hand side are handled by (3.3) and (4.4). Also, by Cauchy's inequality we have

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T A(t,\chi)B(t,\chi)\,dt \le \Big(\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T A(t,\chi)^2\,dt\Big)^{1/2}\Big(\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \int_0^T B(t,\chi)^2\,dt\Big)^{1/2}.$$

Hence (3.3) and (4.4) also yield an estimate for the cross term. Combining these results leads to the theorem.

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