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L_2 discrepancy of generalized two-dimensional Hammersley point sets scrambled with arbitrary permutations

by

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1. Introduction. For a point set $\mathcal{P} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N\}$ of $N \geq 1$ points in the two-dimensional unit square $[0,1)^2$ the L_2 *discrepancy* is defined by

$$L_2(\mathcal{P}) := \Big(\int_0^1\int_0^1 |E(x,y,\mathcal{P})|^2\,dx\,dy\Big)^{1/2},$$

where the so-called *discrepancy function* is given as $E(x,y,\mathcal{P}) = A([0,x) \times [0,y),\mathcal{P}) - Nxy$, with $A([0,x)\times[0,y),\mathcal{P})$ denoting the number of indices $1 \leq M \leq N$ for which $\boldsymbol{p}_M \in [0,x) \times [0,y)$. The L_2 discrepancy is a quantitative measure for the irregularity of distribution of $\mathcal{P}$, i.e., the deviation from perfect uniform distribution.

It was first shown by Roth [15] that for the L_2 discrepancy of any finite point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ we have

$$L_2(\mathcal{P}) \geq c\sqrt{\log N} \tag{1}$$

with a constant $c > 0$ independent of $\mathcal{P}$ and N. According to [11, Chapter 2, proof of Lemma 2.5] one can choose $c = 1/(2^8\sqrt{\log 2}) = 0.0046918\ldots.$

The first to obtain the best possible order of L_2 discrepancy for finite two-dimensional point sets was Davenport [5] (see also [2] and [6, Theorem 1.75]), with a modification of so-called $(N\alpha)$-sequences (α having a continued fraction expansion with bounded partial quotients), more precisely with the set consisting of the $N = 2K$ points $(\{\pm M\alpha\}, M/K)$ for $1 \leq M \leq K$ where K is a positive integer and $\{x\}$ denotes the fractional part of x.

Next, observing that $\{-M\alpha\} = 1 - \{M\alpha\}$, Proĭnov [14] obtained the same result with the same set where generalized van der Corput sequences take the place of $(N\alpha)$-sequences, and he named this process *symmetrization*

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of a sequence. Later on, the same process was used by Chaix and Faure [1] for infinite van der Corput sequences (improving at the same time the constants of Proĭnov) and by Larcher and Pillichshammer [12] for $(0, m, 2)$-nets and $(0, 1)$-sequences in base 2. All these results using the symmetrization process give the exact order with bounds only for the implied constants.

Recently, Chen and Skriganov [3] gave concrete examples of point sets in *arbitrary* dimensions of minimal order of L_2 discrepancy (see [4] for an improvement of the result and a simplification of the proof and see [16] for a generalization of the result to L_q discrepancy). Of course, their great merit is to have bounds for any dimension with explicit constructions, but we mention that the constant at the leading term in their result is huge. For example, in dimension 2 (as in this paper) they gave for any integer $N > 1$ a point set $\mathcal{P}$ of N points for which $L_2(\mathcal{P}) \leq C\sqrt{\log N}$ where $C \approx 11^4/(2\sqrt{\log 11}) = 4727.43\ldots$; see [4, Theorem 1]. By comparison, the analogous constants we get are less than 0.2; see Corollary 2 and Table 1.

In this paper we consider the L_2 discrepancy of so-called generalized Hammersley point sets in base b with b^n points. These point sets are generalizations of the Hammersley point set in base b (which is also known as *Roth net* for $b = 2$) and can be considered as finite two-dimensional versions of the generalized van der Corput sequences in base b as introduced by Faure [7].

Throughout the paper let $b \geq 2$ be an integer and let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \ldots, b-1\}$. The identity in $\mathfrak{S}_b$ is denoted id.

Definition 1. Let $b \geq 2$ and $n \geq 1$ be integers and let $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \mathfrak{S}_b^n$. For an integer $1 \leq N \leq b^n$, write

$$N - 1 = \sum_{r=0}^{n-1} a_r(N) b^r$$

in the b-adic system and define

$$S_b^{\Sigma}(N) := \sum_{r=0}^{n-1} \frac{\sigma_r(a_r(N))}{b^{r+1}}.$$

Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated to Σ is defined by

$$\mathcal{H}_{b,n}^{\Sigma} := \left\{ \left(S_b^{\Sigma}(N), \frac{N-1}{b^n} \right) : 1 \leq N \leq b^n \right\}.$$

In the case of $\sigma_i = \sigma$ for all $0 \leq i < n$, we also write $\mathcal{H}_{b,n}^{\sigma}$ instead of $\mathcal{H}_{b,n}^{\Sigma}$. If in the above definition $\sigma_i = \mathrm{id}$ for all $i \in \{0, \ldots, n-1\}$, then we obtain the classical Hammersley point set in base b, which we simply denote by $\mathcal{H}_{b,n}$.

Let $\tau \in \mathfrak{S}_b$ be given by $\tau(k) = b-1-k$. Faure and Pillichshammer [8] investigated the (more general) L_p discrepancy of the generalized two-dimensional Hammersley point set in base b with $\Sigma \in \{\mathrm{id}, \tau\}^n$. In particular, for the L_2 discrepancy they showed that, whenever l is the number of components of Σ which are equal to id, then

$$(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = \left(\frac{b^2-1}{12b}\right)^2((n-2l)^2-n) + \frac{b^2-1}{12b}\left(1-\frac{1}{2b^n}\right)(2l-n) + n\,\frac{b^4-1}{90b^2} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.$$

This result generalizes older ones due to Vilenkin [17], Halton and Zaremba [9], Pillichshammer [13] and Kritzer and Pillichshammer [10] in base $b = 2$ and White [18] in arbitrary bases $b \geq 2$.

Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ with $\Sigma \in \{\mathrm{id}, \tau\}^n$ only depends on n, b and the number of permutations in Σ which are equal to id (and not on their distribution). Setting $l = n$ we get the formula for the L_2 discrepancy of the classical Hammersley point set.

The above result shows that generalized Hammersley point sets can achieve the best possible order of L_2 discrepancy in the sense of Roth's lower bound (1). More specifically, we have

$$\lim_{n\to\infty} \min_{\Sigma\in\{\mathrm{id},\tau\}^n} \frac{L_2(\mathcal{H}_{b,n}^{\Sigma})}{\sqrt{\log b^n}} = \frac{1}{b}\sqrt{\frac{(b^2-1)(3b^2+13)}{720\log b}}. \tag{2}$$

This is not the case for the classical Hammersley point set $\mathcal{H}_{b,n}$ where

$$\lim_{n\to\infty} \frac{L_2(\mathcal{H}_{b,n})}{\log b^n} = \frac{b^2-1}{12b\log b}.$$

In this paper we intend to generalize the result mentioned above. Moreover, we aim to minimize the constant in the leading term in the formula for the L_2 discrepancy, i.e., the quantity $\lim_{n\to\infty} L_2(\mathcal{H}_{b,n}^{\Sigma})/\sqrt{\log b^n}$. More precisely, for $\sigma \in \mathfrak{S}_b$ we define $\overline{\sigma} := \tau \circ \sigma$ and consider sequences of permutations $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. We will show that for arbitrary $\sigma \in \mathfrak{S}_b$ one still can achieve the optimal order of L_2 discrepancy in the sense of (1). However, if we want to study the constant in the leading term, then we need some restrictions on σ for technical reasons.

Let $\mathcal{A}(\tau) := \{\sigma \in \mathfrak{S}_b : \sigma \circ \tau = \tau \circ \sigma\}$. For permutations $\sigma \in \mathcal{A}(\tau)$ and $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ we provide an explicit formula for the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$. This also yields an explicit formula for the quantity

$$\lim_{n\to\infty} \min_{\substack{\sigma\in\mathcal{A}(\tau)\\ \Sigma\in\{\sigma,\overline{\sigma}\}^n}} L_2(\mathcal{H}_{b,n}^{\Sigma})/\sqrt{\log b^n}.$$

With this formula we can then search for the permutations in $\mathcal{A}(\tau)$ which yield the best result (see Section 5).

The results are presented in Section 2. In Section 3 we show some auxiliary results and the proofs are finally given in Section 4.

We close this introduction with some definitions and notations that are used throughout this paper.

Basic notations. Throughout the paper let $b \geq 2$ and $n \geq 1$ be integers. Let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \ldots, b-1\}$, let $\tau \in \mathfrak{S}_b$ be given by $\tau(k) = b - 1 - k$ and define $\mathcal{A}(\tau) := \{\sigma \in \mathfrak{S}_b : \sigma \circ \tau = \tau \circ \sigma\}$. The identity in $\mathfrak{S}_b$ is always denoted by id. In all examples and concrete results we will write down the permutations in the usual cycle notation, e.g. for $\sigma = \binom{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7}{0\ 4\ 2\ 6\ 1\ 5\ 3\ 7}$ we will write $\sigma = (4\ 1)(6\ 3)$.

The analysis of the L_2 discrepancy is based on special functions which have been first introduced by Faure in [7] and which are defined as follows. For $\sigma \in \mathfrak{S}_b$ let

$$\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b).$$

For $h \in \{0, 1, \ldots, b-1\}$ and $x \in [(k-1)/b, k/b)$ where $k \in \{1, \ldots, b\}$ we define

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A([0, h/b); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([h/b, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where for a sequence $X = (x_M)_{M \geq 1}$ we denote by $A(I; k; X)$ the number of indices $1 \leq M \leq k$ such that $x_M \in I$. Further, the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$ for any σ and that $\varphi_{b,h}^\sigma(0) = 0$ for any $\sigma \in \mathfrak{S}_b$ and any $h \in \{0, \ldots, b-1\}$.

Let

$$\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^r$$

where for $r = 1$ we omit the superscript, i.e., $\varphi_b^{\sigma,(1)} =: \varphi_b^\sigma$. Note that φ_b^σ

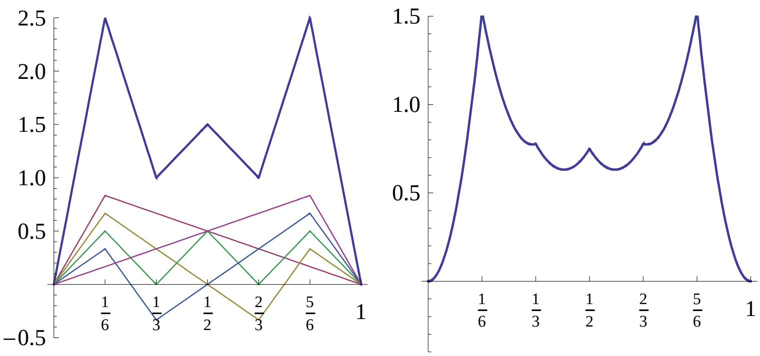

Fig. 1. The functions $\varphi_{b,h}^\sigma$, $0 \leq h < b$, and φ_b^σ (left plot) and $\varphi_b^{\sigma,(2)}$ (right plot) for $b = 6$ and $\sigma = (4, 1)$

is continuous, piecewise linear on the intervals $[k/b,(k+1)/b]$ and $\varphi_b^\sigma(0) = \varphi_b^\sigma(1)$. The function $\varphi_b^{\sigma,(2)}$ is continuous, piecewise quadratic on the intervals $[k/b,(k+1)/b]$ and $\varphi_b^{\sigma,(2)}(0) = \varphi_b^{\sigma,(2)}(1)$. For an example see Fig. 1.

2. The L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$. We start with a general result for the L_2 discrepancy of generalized Hammersley point sets.

THEOREM 1. *Let $\sigma \in \mathfrak{S}_b$ and let $\overline{\sigma} := \tau \circ \sigma$. Let $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and let l denote the number of components of Σ which are equal to σ. Then we have*

$$(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = (\Phi_b^\sigma)^2((n-2l)^2 - n) + O(n),$$

where $\Phi_b^\sigma := (1/b)\int_0^1 \varphi_b^\sigma(x)\,dx$ and where the constant in the O notation only depends on b.

The proof of this result will be given in Section 4.

Theorem 1 shows that one can always obtain $L_2(\mathcal{H}_{b,n}^{\Sigma}) = O(\sqrt{n})$, which is the best possible with respect to Roth's lower bound (1). Either one chooses a permutation $\sigma \in \mathfrak{S}_b$ for which $\Phi_b^\sigma = 0$ or, for arbitrary σ, one chooses l such that the term $(n-2l)^2 = O(n)$.

For permutations σ from the class $\mathcal{A}(\tau)$ we can even give an exact formula for the L_2 discrepancy of generalized two-dimensional Hammersley point sets. This result is a generalization of [8, Theorem 4] which can be obtained by choosing $\sigma = \mathrm{id}$.

THEOREM 2. *Let $\sigma \in \mathcal{A}(\tau)$ and let $\overline{\sigma} := \tau \circ \sigma$. Let $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and let l denote the number of components of Σ which are equal to σ. Then*

$$(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = (\Phi_b^\sigma)^2((n-2l)^2 - n) + \Phi_b^\sigma\left(1 - \frac{1}{2b^n}\right)(2l-n) + n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}},$$

where $\Phi_b^\sigma := (1/b)\int_0^1 \varphi_b^\sigma(x)\,dx$ and $\Phi_b^{\sigma,(2)} := (1/b)\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx$.

The proof of this result will be given in Section 4.

REMARK 1. Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ with $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, $\sigma \in \mathcal{A}(\tau)$, only depends on n, b, σ and the number l of permutations in Σ which are equal to σ. It does *not* depend on the distribution of σ and $\overline{\sigma}$ in Σ.

From Theorem 2 we find that among all sequences of permutations $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, $\sigma \in \mathcal{A}(\tau)$, the one where all components are equal to σ, i.e. $l = n$, gives the worst result for the L_2 discrepancy (except if $\Phi_b^\sigma = 0$, see Corollary 3 below).

Corollary 1. *Let $\sigma \in \mathcal{A}(\tau)$ and let $\overline{\sigma} := \tau \circ \sigma$. Then for any $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ we have $L_2(\mathcal{H}_{b,n}^{\Sigma}) \le L_2(\mathcal{H}_{b,n}^{\sigma})$.*

Still from Theorem 2, if $\Phi_b^\sigma \neq 0$, one can choose l such that $(n-2l)^2 = O(n)$ to obtain the best possible order of L_2 discrepancy in the sense of Roth's lower bound (1). The simplest choice is in general $l = n/2$ if n is even or $l = (n-1)/2$ if n is odd.

Corollary 2. *Let $\sigma \in \mathcal{A}(\tau)$ with $\Phi_b^\sigma \neq 0$ and let $\overline{\sigma} := \tau \circ \sigma$. Then*

$$\min_{\Sigma \in \{\sigma,\overline{\sigma}\}^n} (L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = n(\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2) + O(1).$$

In particular,

$$\lim_{n\to\infty} \min_{\substack{\sigma\in\mathcal{A}(\tau)\\ \Sigma\in\{\sigma,\overline{\sigma}\}^n}} \frac{L_2(\mathcal{H}_{b,n}^{\Sigma})}{\sqrt{\log b^n}} = \min_{\sigma\in\mathcal{A}(\tau)} \sqrt{\frac{\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2}{\log b}}.$$

Proof. The result follows from Theorem 2 together with the fact that the function

$$x \mapsto (\Phi_b^\sigma)^2((n-2x)^2 - n) + \Phi_b^\sigma(1 - 1/2b^n)(2x - n)$$

attains its minimum for $x = n/2 - \frac{1}{4\Phi_b^\sigma}(1 - 1/2b^n)$. ■

If $\Phi_b^\sigma = 0$, the formula from Theorem 2 is *independent* of l (and we can take $l = n$).

Corollary 3. *Let $\sigma \in \mathcal{A}(\tau)$ with $\Phi_b^\sigma = 0$ and let $\overline{\sigma} := \tau \circ \sigma$. Then for any $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ we have*

$$(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = (L_2(\mathcal{H}_{b,n}^{\sigma}))^2 = n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.$$

Remark 2. Concerning Corollary 3, we can give explicit constructions for permutations $\sigma \in \mathcal{A}(\tau)$ in bases $b \equiv 0 \pmod 4$, $b \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$, $b \notin \{3, 7, 11\}$ satisfying $\Phi_b^\sigma = 0$. In bases $b = 3, 7$ and $b \equiv 2 \pmod 4$ there do not exist any permutations $\sigma \in \mathfrak{S}_b$ with $\Phi_b^\sigma = 0$. For $b = 11$ we will give an example in Table 2.

For $b \equiv 0 \pmod 4$ and $b \equiv 1 \pmod 4$, we may choose $\sigma \in \mathcal{A}(\tau)$ defined by

$$\sigma(k) = \begin{cases} k+1 & \text{for even } k \\ b-k & \text{for odd } k \end{cases} \qquad \text{for } 0 \le k < \left\lfloor \frac{b}{2} \right\rfloor,$$

and for $b = 4c + 3$ with $c \ge 3$ we have found

$$\sigma(k)=\begin{cases} 2c-k+1 & \text{for } 1\le k\le c-2,\\ 4c-k+1 & \text{for } c-1\le k\le c+1,\\ 2c+k+1 & \text{for } c+2\le k\le 2c-2,\\ 6c-k & \text{for } 2c-1\le k\le 2c.\end{cases}$$

Note that σ is completely determined since $\sigma\in\mathcal{A}(\tau)$, i.e. the other values are given by symmetry through $\sigma(b-1-k)=b-1-\sigma(k)$. These examples show the existence of permutations with $\Phi_b^\sigma=0$ for arbitrary bases, except $b=3,7$ and $b\equiv 2 \pmod 4$ for which they cannot exist. But on the other hand, numerical experiments suggest that for any $b\not\equiv 2 \pmod 4$, $b\notin\{3,7\}$, there exist many permutations σ with $\Phi_b^\sigma=0$. Getting algorithms to find these σ for arbitrary given b seems a difficult task. We have tabulated those with the minimal L_2 discrepancy for bases $b\le 17$. See Section 5, Table 2, where it appears that the numerical values of $\Phi_b^{\sigma,(2)}$ are not optimal in these cases (except for $b=4$ and 13). One interest of having only one permutation σ instead of two, σ and $\overline{\sigma}$, is that formulas start being valid from $n=1$, i.e. with b points, whereas with two permutations one must start at least from $n=2$, i.e. with b^2 points. Further, more involved calculations would show that σ and $\overline{\sigma}$ produce a permutation ρ in base b^2 for which $\Phi_{b^2}^\rho=0$. Such refinements would lengthen the paper and we think they can be postponed for later investigations. Another interesting observation is that we obtain the optimal order of L_2 discrepancy with only one permutation, which was not evident in the light of former results.

We can also show that the L_2 discrepancy of the two-dimensional generalized Hammersley point set $\mathcal{H}_{b,n}^\Sigma$ with $\Sigma\in\{\sigma,\overline{\sigma}\}^n$ and $\sigma\in\mathcal{A}(\tau)$ satisfies a central limit theorem. More specifically, the following result states that the probability for $L_2(\mathcal{H}_{b,n}^\Sigma)\le c\sqrt{n}$ with randomly chosen $\Sigma\in\{\sigma,\overline{\sigma}\}^n$ can be made arbitrarily close to 1 by choosing the constant c large enough.

Corollary 4. *Let $\sigma\in\mathcal{A}(\tau)$ and let $\overline{\sigma}:=\tau\circ\sigma$. Then for any real $y\ge 0$ we have*

$$\lim_{n\to\infty}\frac{|\{\Sigma\in\{\sigma,\overline{\sigma}\}^n : L_2(\mathcal{H}_{b,n}^\Sigma)\le\sqrt{\Phi_b^{\sigma,(2)}-(\Phi_b^\sigma)^2(1-y^2)}\,\sqrt{n}\}|}{2^n}=2\phi(y)-1,$$

where $\phi(y)=(1/2\pi)\int_{-\infty}^{y}e^{-t^2/2}\,dt$ denotes the normal distribution function.

Proof. We denote the right hand side of the formula in Theorem 2 by $d_b(n,l)$. Then

$$\frac{|\{\Sigma\in\{\sigma,\tau\}^n : L_2(\mathcal{H}_{b,n}^\Sigma)\le x\sqrt{n}\}|}{2^n}=\frac{1}{2^n}\sum_{\substack{l=0\\ \sqrt{d_b(n,l)}\le x\sqrt{n}}}^{n}\binom{n}{l}.$$

We have $\sqrt{d_b(n,l)} \le x\sqrt{n}$ if and only if $a_n^-(x) \le l \le a_n^+(x)$, where

$$a_n^\pm(x) := \frac{n}{2} - \left(1 - \frac{1}{2b^n}\right)\frac{1}{4\Phi_b^\sigma} \pm \frac{\sqrt{4n((\Phi_b^\sigma)^2 - \Phi_b^{\sigma,(2)} + x^2) + O(1)}}{4\Phi_b^\sigma}.$$

Therefore

$$\frac{|\{\Sigma \in \{\sigma, \overline{\sigma}\}^n : L_2(\mathcal{H}_{b,n}^\Sigma) \le x\sqrt{n}\}|}{2^n} = \frac{1}{2^n} \sum_{a_n^-(x) \le l \le a_n^+(x)} \binom{n}{l}.$$

For $x \ge \sqrt{\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2}$ we have

$$\lim_{n\to\infty} \frac{a_n^\pm(x) - n/2}{\sqrt{n/4}} = \pm \frac{\sqrt{(\Phi_b^\sigma)^2 - \Phi_b^{\sigma,(2)} + x^2}}{\Phi_b^\sigma}$$

and the result follows from the central limit theorem together with the substitution $x = \sqrt{\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2(1-y^2)}$. ∎

3. Auxiliary results. In this section we prepare the basic tools which are used for the proof of Theorems 1 and 2. Some of the following results are interesting on their own.

Basic properties of φ_b^σ. We begin with some basic properties of the functions $\varphi_{b,h}^\sigma$ resp. φ_b^σ. It has been shown in [1, Propriété 3.4] that

$$(\varphi_{b,h}^\sigma)'(k/b + 0) = (\varphi_{b,h}^{\mathrm{id}})'(\sigma(k)/b + 0) \tag{3}$$

and from [1, Propriété 3.5] it is known that

$$\varphi_b^\sigma(k/b) = \frac{1}{b}\sum_{j=0}^{k-1} (\varphi_b^\sigma)'(j/b + 0). \tag{4}$$

For $\sigma = \mathrm{id}$ we have

$$\varphi_{b,h}^{\mathrm{id}}(x) = \begin{cases} (b-h)x & \text{if } x \in [0, h/b], \\ h(1-x) & \text{if } x \in [h/b, 1]. \end{cases} \tag{5}$$

A formula for the discrepancy function. The following lemma provides a formula for the discrepancy function of generalized Hammersley point sets. This formula has already been used in [8].

LEMMA 1. *For integers $1 \le \lambda, N \le b^n$ we have*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^\Sigma\right) = \sum_{j=1}^n \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

where the $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ can be given explicitly.

As the exact definition of the ε_j's is not so important here and as this definition is of a very technical nature we omit it here. A proof of the above result together with explicit expressions for the ε_j's can be found in [8, Lemma 1].

REMARK 3. Let $0 \le x, y \le 1$ be arbitrary. Since all points from $\mathcal{H}_{b,n}^{\Sigma}$ have coordinates of the form α/b^n for some $\alpha \in \{0, 1, \dots, b^n - 1\}$, we have

$$E(x, y, \mathcal{H}_{b,n}^{\Sigma}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n) - xy),$$

where for $0 \le x \le 1$ we define $x(n) := \min\{\alpha/b^n \ge x : \alpha \in \{0, \dots, b^n\}\}$.

More involved properties of φ_b^σ. We give a series of lemmas which provide important properties of the functions $\varphi_{b,h}^\sigma$ and φ_b^σ. These results finally lead to the proof of Theorems 1 and 2.

A proof for the subsequent lemma can be found in [8, Lemma 2].

LEMMA 2. *For* $1 \le N \le b^n$, $0 \le j_1 < \dots < j_k < n$ *and* $r_1, \dots, r_k \in \mathbb{N}$ *we have* ($\varepsilon_j = \varepsilon_j(\lambda, n, N)$, *see Lemma* 1)

$$\sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1}}\left(\frac{N}{b^{j_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,\varepsilon_{j_k}}^{\sigma_{j_k}}\left(\frac{N}{b^{j_k}}\right)\right)^{r_k} = b^{n-k} \varphi_b^{\sigma_{j_1},(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{\sigma_{j_k},(r_k)}\left(\frac{N}{b^{j_k}}\right),$$

where $\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^r$.

LEMMA 3. *Let* $\sigma \in \mathfrak{S}_b$ *and let* $\overline{\sigma} = \tau \circ \sigma$. *For any* $h \in \{0, \dots, b-1\}$ *we have* $\varphi_{b,h}^{\overline{\sigma}} = -\varphi_{b,b-h}^\sigma$. *Furthermore*, $\varphi_b^{\overline{\sigma},(r)} = (-1)^r \varphi_b^{\sigma,(r)}$.

Proof. With (3) together with the fact that $\varphi_{b,h}^\tau = -\varphi_{b,b-h}^{\mathrm{id}}$, as shown in [8, Lemma 4], we obtain

$$\begin{aligned}(\varphi_{b,h}^{\overline{\sigma}})'(k/b) &= (\varphi_{b,h}^{\mathrm{id}})'(\overline{\sigma}(k)/b) = (\varphi_{b,h}^{\mathrm{id}})'(\tau(\sigma(k))/b) \\ &= (\varphi_{b,h}^{\tau})'(\sigma(k)/b) = -(\varphi_{b,b-h}^{\mathrm{id}})'(\sigma(k)/b) = -(\varphi_{b,b-h}^{\sigma})'(k/b).\end{aligned}$$

Since for any permutation σ the function $\varphi_{b,h}^\sigma$ is linear on any interval $[k/b, (k+1)/b]$ and since $\varphi_{b,h}^\sigma(0) = 0$ the first result follows. The second result follows easily from the first one. ∎

LEMMA 4. *Let* $\sigma \in \mathfrak{S}_b$. *For* $1 \le i, j \le n$, $i \ne j$, *we have*

$$\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^j}\right) = b^n \int_0^1 \varphi_b^\sigma(x)\,dx, \tag{6}$$

$$\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^i}\right) \varphi_b^\sigma\left(\frac{N}{b^j}\right) = b^n \Big(\int_0^1 \varphi_b^\sigma(x)\,dx\Big)^2, \tag{7}$$

$$\sum_{N=1}^{b^n} \varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) = b^n\left(\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx + \frac{b(b^2-1)}{36b^{2j}}\right). \tag{8}$$

Proof. We first prove (6). Using the periodicity of φ_b^σ we have

$$\sum_{N=1}^{b^n} \varphi_b^{\sigma}\left(\frac{N}{b^j}\right) = \sum_{N=0}^{b^n-1} \varphi_b^{\sigma}\left(\frac{N}{b^j}\right) = b^{n-j}\sum_{N=0}^{b^j-1} \varphi_b^{\sigma}\left(\frac{N}{b^j}\right). \tag{9}$$

Since φ_b^σ is linear on the intervals $[k/b,(k+1)/b]$, from the trapezoidal rule for $0 \le N < b^j$ we obtain

$$\int_{N/b^j}^{(N+1)/b^j} \varphi_b^\sigma(x)\,dx = \frac{\varphi_b^\sigma(N/b^j) + \varphi_b^\sigma((N+1)/b^j)}{2b^j}.$$

Hence

$$\begin{aligned} \int_0^1 \varphi_b^\sigma(x)\,dx &= \sum_{N=0}^{b^j-1} \int_{N/b^j}^{(N+1)/b^j} \varphi_b^\sigma(x)\,dx \\ &= \sum_{N=0}^{b^j-1} \frac{\varphi_b^\sigma\left(\frac{N}{b^j}\right) + \varphi_b^\sigma\left(\frac{N+1}{b^j}\right)}{2b^j} = \frac{1}{b^j}\sum_{N=0}^{b^j-1} \varphi_b^\sigma\left(\frac{N}{b^j}\right), \end{aligned} \tag{10}$$

since $\varphi_b^\sigma(0) = \varphi_b^\sigma(1) = 0$. Inserting (10) into (9) yields (6).

We turn to the proof of (7). Let $i =: i_1$ and $j =: i_2$. We may assume that $i_1 < i_2$. For $0 \le N < b^n$ let $N = N_0 + N_1 b + \cdots + N_{n-1}b^{n-1}$ be its b-adic representation. Then we have

$$\begin{aligned} \sum_{N=1}^{b^n}\prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N}{b^{i_l}}\right) &= \sum_{N=0}^{b^n-1}\prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N}{b^{i_l}}\right) \\ &= \sum_{N_0,\dots,N_{n-1}=0}^{b-1}\prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N_0 + N_1 b + \cdots + N_{n-1}b^{n-1}}{b^{i_l}}\right) \\ &= b^{n-i_2}\sum_{N_0,\dots,N_{i_2-2}=0}^{b-1} \varphi_b^\sigma\left(\frac{N_0 + \cdots + N_{i_1-1}b^{i_1-1}}{b^{i_1}}\right) \\ &\qquad \times \sum_{k=0}^{b-1}\varphi_b^\sigma\left(\frac{k}{b} + \frac{N_0 + \cdots + N_{i_2-2}b^{i_2-2}}{b^{i_2}}\right). \end{aligned}$$

Let $t := (N_0 + \cdots + N_{i_2-2}b^{i_2-2})/b^{i_2} \in [0,1/b)$. From the linearity of $\varphi_b^\sigma(x)$ for $x \in [k/b,(k+1)/b]$ it follows that

$$\varphi_b^\sigma\left(\frac{k}{b}+t\right) = \varphi_b^\sigma\left(\frac{k}{b}\right) + tb\left(\varphi_b^\sigma\left(\frac{k+1}{b}\right) - \varphi_b^\sigma\left(\frac{k}{b}\right)\right).$$

Hence

$$\sum_{k=0}^{b-1}\varphi_b^\sigma\Big(\frac{k}{b}+t\Big)=\sum_{k=0}^{b-1}\varphi_b^\sigma\Big(\frac{k}{b}\Big)+bt\sum_{k=0}^{b-1}\Big(\varphi_b^\sigma\Big(\frac{k+1}{b}\Big)-\varphi_b^\sigma\Big(\frac{k}{b}\Big)\Big)$$
$$=\sum_{k=0}^{b-1}\varphi_b^\sigma\Big(\frac{k}{b}\Big)+bt(\varphi_b^\sigma(1)-\varphi_b^\sigma(0))=b\int_0^1\varphi_b^\sigma(x)\,dx,$$

where we used (10) with $j=1$ and the periodicity of φ_b^σ for the last equality. Therefore we obtain

$$\sum_{N=1}^{b^n}\prod_{l=1}^{2}\varphi_b^\sigma\Big(\frac{N}{b^{i_l}}\Big)$$
$$=b^{n-i_2+1}\Big(\int_0^1\varphi_b^\sigma(x)\,dx\Big)\sum_{N_0,\dots,N_{i_2-2}=0}^{b-1}\varphi_b^\sigma\Big(\frac{N_0+\dots+N_{i_1-1}b^{i_1-1}}{b^{i_1}}\Big)$$
$$=\Big(\int_0^1\varphi_b^\sigma(x)\,dx\Big)\sum_{N=0}^{b^n-1}\varphi_b^\sigma\Big(\frac{N}{b^{i_1}}\Big)=b^n\Big(\int_0^1\varphi_b^\sigma(x)\,dx\Big)^2,$$

where we used (6) for the last equality. This gives (7).

Finally, we prove (8). First let $j\ge 2$. The function $\varphi_b^{\sigma,(2)}(x)$ for $x\in[k/b,(k+1)/b]$ is a quadratic polynomial $a_kx^2+b_kx+c_k$. Hence from Simpson's rule we obtain

$$\int_{N/b^j}^{(N+2)/b^j}\varphi_b^{\sigma,(2)}(x)\,dx=\frac{1}{3b^j}\Big(\varphi_b^{\sigma,(2)}\Big(\frac{N}{b^j}\Big)+4\varphi_b^{\sigma,(2)}\Big(\frac{N+1}{b^j}\Big)+\varphi_b^{\sigma,(2)}\Big(\frac{N+2}{b^j}\Big)\Big),$$

whenever $N/b^j,(N+1)/b^j,(N+2)/b^j\in[k/b,(k+1)/b]$. Hence for $0\le k<b$ we obtain

$$\sum_{N=kb^{j-1}}^{(k+1)b^{j-1}-2}\frac{1}{3b^j}\Big(\varphi_b^{\sigma,(2)}\Big(\frac{N}{b^j}\Big)+4\varphi_b^{\sigma,(2)}\Big(\frac{N+1}{b^j}\Big)+\varphi_b^{\sigma,(2)}\Big(\frac{N+2}{b^j}\Big)\Big)$$
$$=\sum_{N=kb^{j-1}}^{(k+1)b^{j-1}-2}\Big\{\int_{N/b^j}^{(N+1)/b^j}+\int_{(N+1)/b^j}^{(N+2)/b^j}\Big\}\varphi_b^{\sigma,(2)}(x)\,dx$$
$$=\int_{k/b}^{(k+1)/b-1/b^j}\varphi_b^{\sigma,(2)}(x)\,dx+\int_{k/b+1/b^j}^{(k+1)/b}\varphi_b^{\sigma,(2)}(x)\,dx$$
$$=2\int_{k/b}^{(k+1)/b}\varphi_b^{\sigma,(2)}(x)\,dx-\int_{k/b}^{k/b+1/b^j}\varphi_b^{\sigma,(2)}(x)\,dx-\int_{(k+1)/b-1/b^j}^{(k+1)/b}\varphi_b^{\sigma,(2)}(x)\,dx.$$

Summation over all $k = 0, \ldots, b-1$ yields

$$\sum_{N=0}^{b^j-1} \frac{1}{3b^j}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right) + \varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right)$$
$$= 2\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx - \sum_{k=0}^{b-1}\left\{\int_{k/b}^{k/b+1/b^j} + \int_{(k+1)/b-1/b^j}^{(k+1)/b}\right\}\varphi_b^{\sigma,(2)}(x)\,dx$$
$$+ \sum_{k=0}^{b-1} \frac{\varphi_b^{\sigma,(2)}\left(\frac{k+1}{b} - \frac{1}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{k+1}{b}\right) + \varphi_b^{\sigma,(2)}\left(\frac{k+1}{b} + \frac{1}{b^j}\right)}{3b^j}.$$

Now, using again the periodicity of $\varphi_b^{\sigma,(2)}$, we have

$$\sum_{N=0}^{b^j-1}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right) + \varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right)$$
$$= \sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\sum_{N=1}^{b^j}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + \sum_{N=2}^{b^j+1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)$$
$$= 6\sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right).$$

Thus,

$$\sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)$$
$$= b^j\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx - \frac{b^j}{2}\sum_{k=0}^{b-1}\left\{\int_{k/b}^{k/b+1/b^j} + \int_{(k+1)/b-1/b^j}^{(k+1)/b}\right\}\varphi_b^{\sigma,(2)}(x)\,dx$$
$$+ \frac{1}{6}\sum_{k=1}^{b}\left(\varphi_b^{\sigma,(2)}\left(\frac{k}{b} - \frac{1}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right) + \varphi_b^{\sigma,(2)}\left(\frac{k}{b} + \frac{1}{b^j}\right)\right)$$
$$= b^j\left(\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx + \frac{A(j,k,\sigma)}{2}\right),$$

where

$$A(j,k,\sigma) = \sum_{k=1}^{b}\left(\frac{\varphi_b^{\sigma,(2)}\left(\frac{k}{b} - \frac{1}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right) + \varphi_b^{\sigma,(2)}\left(\frac{k}{b} + \frac{1}{b^j}\right)}{3b^j}\right.$$
$$\left. - \int_{k/b-1/b^j}^{k/b+1/b^j} \varphi_b^{\sigma,(2)}(x)\,dx\right).$$

Using the periodicity of $\varphi_b^{\sigma,(2)}$, we get

$$\sum_{N=1}^{b^n} \varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) = b^{n-j} \sum_{N=0}^{b^j} \varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) = b^n \left(\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx + \frac{A(j,k,\sigma)}{2}\right)$$

for all $j \geq 2$. For $j = 1$ this equation can be checked directly.

It remains to evaluate $A(j,k,\sigma)$. For $1 \leq k \leq b$ let $h_k(x) = \alpha_k x^2 + \beta_k x + \gamma_k$ with $h_k\left(\frac{k}{b} - \frac{1}{b^j}\right) = \varphi_b^{\sigma,(2)}\left(\frac{k}{b} - \frac{1}{b^j}\right)$, $h_k\left(\frac{k}{b}\right) = \varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)$ and $h_k\left(\frac{k}{b} + \frac{1}{b^j}\right) = \varphi_b^{\sigma,(2)}\left(\frac{k}{b} + \frac{1}{b^j}\right)$. Then by Simpson's rule we have

$$\int_{k/b-1/b^j}^{k/b+1/b^j} h_k(x)\,dx = \frac{\varphi_b^{\sigma,(2)}\left(\frac{k}{b} - \frac{1}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right) + \varphi_b^{\sigma,(2)}\left(\frac{k}{b} + \frac{1}{b^j}\right)}{3b^j}.$$

By tedious but straightforward algebra it can be shown that

$$\begin{aligned}\int_{k/b-1/b^j}^{k/b+1/b^j} h_k(x)\,dx &- \int_{k/b-1/b^j}^{k/b+1/b^j} \varphi_b^{\sigma,(2)}(x)\,dx \\ &= \frac{1}{6b^{2j}}\left((\varphi_b^{\sigma,(2)})'\left(\frac{k}{b} - 0\right) - (\varphi_b^{\sigma,(2)})'\left(\frac{k}{b} + 0\right)\right).\end{aligned}$$

By definition we have $\varphi_b^{\sigma,(2)} = \sum_{h=0}^{b-1}(\varphi_{b,h}^{\sigma})^2$ and hence

$$\begin{aligned}&(\varphi_b^{\sigma,(2)})'\left(\frac{k}{b} - 0\right) - (\varphi_b^{\sigma,(2)})'\left(\frac{k}{b} + 0\right) \\ &\quad = 2\sum_{h=0}^{b-1} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)\left((\varphi_{b,h}^{\sigma})'\left(\frac{k}{b} - 0\right) - (\varphi_{b,h}^{\sigma})'\left(\frac{k}{b} + 0\right)\right) \\ &\quad = 2\sum_{h=0}^{b-1} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)\left((\varphi_{b,h}^{\sigma})'\left(\frac{k-1}{b} + 0\right) - (\varphi_{b,h}^{\sigma})'\left(\frac{k}{b} + 0\right)\right).\end{aligned}$$

For short we define $f_{h,k} := (\varphi_{b,h}^{\sigma})'(k/b + 0)$. Hence we have

$$A(j,k,\sigma) = \frac{1}{3b^{2j}} \sum_{h=0}^{b-1} \sum_{k=1}^{b} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)(f_{h,k-1} - f_{h,k}). \tag{11}$$

Since $\varphi_{b,h}^{\sigma}$ is linear on every interval $[k/b, (k+1)/b]$ we have $\varphi_{b,h}^{\sigma}(k/b) = \int_0^{k/b}(\varphi_{b,h}^{\sigma})'(x)\,dx = (1/b)\sum_{l=0}^{k-1} f_{h,l}$ and especially $\sum_{l=0}^{b-1} f_{h,l} = 0$. Hence for every fixed h we obtain

$$(12)\qquad \sum_{k=1}^{b} \varphi_{b,h}^{\sigma}\Big(\frac{k}{b}\Big)(f_{h,k-1} - f_{h,k}) \\ = \frac{1}{b}\sum_{l=0}^{b-1} f_{h,l} \sum_{k=l+1}^{b} (f_{h,k-1} - f_{h,k}) = \frac{1}{b}\sum_{l=0}^{b-1} f_{h,l}^2.$$

Inserting (12) into (11) and using (3) gives

$$\begin{aligned} A(j,k,\sigma) &= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1}\Big((\varphi_{b,h}^{\sigma})'\Big(\frac{l}{b}+0\Big)\Big)^2 \\ &= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1}\Big((\varphi_{b,h}^{\mathrm{id}})'\Big(\frac{\sigma(l)}{b}+0\Big)\Big)^2 \\ &= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1}\Big((\varphi_{b,h}^{\mathrm{id}})'\Big(\frac{l}{b}+0\Big)\Big)^2 = A(j,k,\mathrm{id}). \end{aligned}$$

This means that $A(j,k,\sigma)$ does not depend on the choice of the permutation σ. Now we may use known results for the case $\sigma = \mathrm{id}$. It has been shown in [8, Lemma 5] that

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id},(2)}\Big(\frac{N}{b^j}\Big) = b^n\Big(\int_0^1 \varphi_b^{\mathrm{id},(2)}(x)\,dx + \frac{b(b^2-1)}{36b^{2j}}\Big)$$

(we remark that $\int_0^1 \varphi_b^{\mathrm{id},(2)}(x)\,dx = \frac{b^4-1}{90b}$, which follows from [8, Lemma 3]). Hence

$$\frac{A(j,k,\sigma)}{2} = \frac{A(j,k,\mathrm{id})}{2} = \frac{b(b^2-1)}{36b^{2j}}$$

and this finishes the proof. ∎

Lemma 5. *For any $\sigma \in \mathfrak{S}_b$ we have*

$$\int_0^1 \varphi_b^{\sigma}(x)\,dx = \frac{1}{b}\sum_{k=0}^{b-1}\sigma(k)k - \Big(\frac{b-1}{2}\Big)^2.$$

In particular, $\int_0^1 \varphi_b^{\sigma}(x)\,dx = \int_0^1 \varphi_b^{\sigma^{-1}}(x)\,dx$.

Proof. Using integration by parts and (3) we have

$$\int_0^1 \varphi_b^{\sigma}(x)\,dx = \sum_{k=0}^{b-1} \int_{k/b}^{(k+1)/b} \varphi_b^{\sigma}(x)\,dx$$

$$= \sum_{k=0}^{b-1}\Big(x\varphi_b^\sigma(x)\Big|_{k/b}^{(k+1)/b} - \int_{k/b}^{(k+1)/b} x(\varphi_b^\sigma)'(x)\,dx\Big)$$

$$= -\sum_{k=0}^{b-1}\int_{k/b}^{(k+1)/b} x(\varphi_b^\sigma)'\Big(\frac{k}{b}+0\Big)\,dx = -\sum_{k=0}^{b-1}(\varphi_b^{\mathrm{id}})'\Big(\frac{\sigma(k)}{b}+0\Big)\frac{2k+1}{2b^2}.$$

From (5) we obtain

$$(\varphi_{b,h}^{\mathrm{id}})'(x+0) = \begin{cases} b-h & \text{if } x\in[0,h/b],\\ -h & \text{if } x\in[h/b,1],\end{cases}$$

and therefore for any $0 \le l < b$ we have

$$(\varphi_b^{\mathrm{id}})'\Big(\frac{l}{b}+0\Big) = \sum_{h=0}^{b-1}(\varphi_{b,h}^{\mathrm{id}})'\Big(\frac{l}{b}+0\Big) = \sum_{h=0}^{l}(-h) + \sum_{h=l+1}^{b-1}(b-h) = \frac{b(b-1-2l)}{2}. \tag{13}$$

Therefore we have

$$\int_0^1 \varphi_b^\sigma(x)\,dx = -\sum_{k=0}^{b-1}\frac{(b-1-2\sigma(k))(2k+1)}{4b} = \frac{1}{b}\sum_{k=0}^{b-1}\sigma(k)k - \Big(\frac{b-1}{2}\Big)^2. \ \blacksquare$$

Lemma 6. *We have $\sigma\in\mathcal{A}(\tau)$ if and only if $\varphi_b^\sigma(x) = \varphi_b^\sigma(1-x)$ for all $x\in[0,1]$.*

Proof. Since φ_b^σ is continuous, piecewise linear and $\varphi_b^\sigma(0)=\varphi_b^\sigma(1)=0$, we have $\varphi_b^\sigma(x)=\varphi_b^\sigma(1-x)$ if and only if $(\varphi_b^\sigma)'(x) = -(\varphi_b^\sigma)'(1-x)$ for all $x\in[0,1]$. Now if $\sigma\in\mathcal{A}(\tau)$, i.e., $\sigma(k)+\sigma(b-k-1)=b-1$, then by (13) we have

$$(\varphi_b^\sigma)'\Big(\frac{k}{b}+0\Big) = \frac{b(b-1)}{2} - b\sigma(k) = \frac{b(b-1)}{2} - b(b-1-\sigma(b-k-1))$$
$$= -\Big(\frac{b(b-1)}{2} - b\sigma(b-k-1)\Big) = -(\varphi_b^\sigma)'\Big(1-\frac{k+1}{b}+0\Big).$$

This gives the desired property on the interval $[k/b,(k+1)/b]$ for $(\varphi_b^\sigma)'$ and *vice versa.* $\blacksquare$

4. The proof of Theorems 1 and 2. First we give a discrete version of Theorem 2.

Lemma 7. *Let $\sigma\in\mathfrak{S}_b$ and let $\overline{\sigma} := \tau\circ\sigma$. Let $\Sigma\in\{\sigma,\overline{\sigma}\}^n$ and let l denote the number of components of Σ which are equal to σ. Then*

$$\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) = (2l-n)\Phi_b^{\sigma} \tag{14}$$

and

$$\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 = n\Phi_b^{\sigma,(2)} + ((n-2l)^2-n)(\Phi_b^{\sigma})^2 + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right). \tag{15}$$

Here, $\Phi_b^{\sigma} := (1/b)\int_0^1 \varphi_b^{\sigma}(x)\,dx$ *and* $\Phi_b^{\sigma,(2)} := (1/b)\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx$.

Proof. Let $\Sigma = (\sigma_0, \dots, \sigma_{n-1}) \in \{\sigma, \overline{\sigma}\}^n$ and define, for $1 \le i \le n$,

$$s_i := \begin{cases} 1 & \text{if } \sigma_{i-1} = \overline{\sigma}, \\ 0 & \text{if } \sigma_{i-1} = \sigma. \end{cases}$$

For (14) we use Lemmas 1–3 with the definition of the s_i and equation (6) from Lemma 4 (in that order) to obtain (recall that $\varepsilon_j = \varepsilon_j(\lambda, n, N)$)

$$\begin{aligned}
&\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) \\
&\quad= \frac{1}{b^{2n}} \sum_{j=1}^{n} \sum_{N=1}^{b^n} \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = \frac{1}{b^{n+1}} \sum_{j=1}^{n} \sum_{N=1}^{b^n} \varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&\quad= \frac{1}{b^{n+1}} \sum_{j=1}^{n} (-1)^{s_j} \sum_{N=1}^{b^n} \varphi_b^{\sigma}\left(\frac{N}{b^j}\right) = \Phi_b^{\sigma} \sum_{j=1}^{n} (-1)^{s_j} = (2l-n)\Phi_b^{\sigma}.
\end{aligned}$$

Now we prove (15). Using Lemma 1 we have

$$\begin{aligned}
\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 &= \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \sum_{i,j=1}^{n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right) \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&= \frac{1}{b^{2n}} \sum_{i=1}^{n} \sum_{N=1}^{b^n} \sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2 \\
&\quad + \frac{1}{b^{2n}} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{N=1}^{b^n} \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right) \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).
\end{aligned}$$

By Lemma 2 we have

$$\sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2 = b^{n-1} \varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right),$$

and for $i \neq j$,

$$\sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).$$

Therefore we obtain

$$\begin{aligned}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right)\\ &\quad+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n} b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).\end{aligned}$$

From Lemma 3 we find that $\varphi_b^{\sigma,(2)} = \varphi_b^{\overline{\sigma},(2)}$ and $\varphi_b^{\sigma} = -\varphi_b^{\overline{\sigma}}$. Now we obtain

$$\begin{aligned}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^i}\right)\\ &\quad+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}\sum_{N=1}^{b^n} b^{n-2}\varphi_b^{\sigma}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma}\left(\frac{N}{b^j}\right).\end{aligned}$$

Using (8) from Lemma 4 we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^i}\right) &= b^{n-1}\sum_{i=1}^{n} b^n\left(\int_0^1 \varphi_b^{\sigma,(2)}(x)\,dx + \frac{b(b^2-1)}{36b^{2j}}\right)\\ &= b^{2n}n\Phi_b^{\sigma,(2)} + b^{2n}\sum_{i=1}^{n}\frac{b^2-1}{36b^{2j}}\\ &= b^{2n}n\Phi_b^{\sigma,(2)} + \frac{b^{2n}}{36}\left(1-\frac{1}{b^{2n}}\right),\end{aligned}$$

and, by (7) from Lemma 4 for $i \neq j$,

$$\sum_{N=1}^{b^n}\varphi_b^{\sigma}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma}\left(\frac{N}{b^j}\right) = b^n\left(\int_0^1 \varphi_b^{\sigma}(x)\,dx\right)^2 = b^{n+2}(\Phi_b^{\sigma})^2.$$

Hence

$$\begin{aligned}&\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2\\ &\qquad= n\Phi_b^{\sigma,(2)} + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right) + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}(\Phi_b^{\sigma})^2.\end{aligned}$$

Finally we note that $\sum_{i,j=1,\, i\neq j}^{n}(-1)^{s_i+s_j} = (\sum_{i=1}^{n}(-1)^{s_i})^2 - n = (n-2l)^2 - n$, from which the result follows. ∎

Proof of Theorem 2. (The proof of Theorem 1 follows easily, see at the end.) We have

$$\begin{aligned}
(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 &= \int_0^1\int_0^1 (E(x,y,\mathcal{H}_{b,n}^{\Sigma}))^2\,dx\,dy \\
&= \int_0^1\int_0^1 (E(x(n),y(n),\mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n)-xy))^2\,dx\,dy \\
&= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 \\
&\quad + 2b^n\sum_{\lambda,N=1}^{b^n}\int_{(\lambda-1)/b^n}^{\lambda/b^n}\int_{(N-1)/b^n}^{N/b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)dx\,dy \\
&\quad + b^{2n}\sum_{\lambda,N=1}^{b^n}\int_{(\lambda-1)/b^n}^{\lambda/b^n}\int_{(N-1)/b^n}^{N/b^n}\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)^2 dx\,dy \\
&=: \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned}$$

From (15) of Lemma 7 we find that

$$\Sigma_1 = n\Phi_b^{\sigma,(2)} + ((n-2l)^2-n)(\Phi_b^{\sigma})^2 + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right)$$

and straightforward computation shows that $\Sigma_3 = (1+18b^n+25b^{2n})/(72b^{2n})$. So it remains to deal with Σ_2. We have

$$\begin{aligned}
\Sigma_2 &= \frac{2}{b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\lambda N \\
&\quad - \frac{1}{2b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)(2\lambda-1)(2N-1) \\
&= \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}(\lambda+N)E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) - \frac{1}{2b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) \\
&=: \Sigma_4-\Sigma_5.
\end{aligned}$$

From (14) of Lemma 7 we obtain $\Sigma_5 = (2l-n)\Phi_b^{\sigma}/(2b^n)$ and for Σ_4 we have

$$\Sigma_4 = \frac{1}{b^{3n}} \sum_{\lambda,N=1}^{b^n} \lambda E\Big(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\Big) + \frac{1}{b^{3n}} \sum_{\lambda,N=1}^{b^n} N E\Big(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\Big)$$
$$=: \frac{1}{b^{3n}} (\Sigma_{4,1} + \Sigma_{4,2}).$$

Again let $\Sigma = (\sigma_0, \dots, \sigma_{n-1}) \in \{\sigma, \overline{\sigma}\}^n$ and, for $1 \le i \le n$,

$$s_i := \begin{cases} 1 & \text{if } \sigma_{i-1} = \overline{\sigma}, \\ 0 & \text{if } \sigma_{i-1} = \sigma. \end{cases}$$

Then

$$\Sigma_{4,2} = \sum_{i=1}^{n} \sum_{N=1}^{b^n} N \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\Big(\frac{N}{b^i}\Big) = b^{n-1} \sum_{i=1}^{n} (-1)^{s_i} \sum_{N=1}^{b^n} N \varphi_b^{\sigma}\Big(\frac{N}{b^i}\Big),$$

where we used Lemma 2. We have

$$\begin{aligned}\sum_{N=1}^{b^n} N\varphi_b^{\sigma}\Big(\frac{N}{b^j}\Big) = \varphi_b^{\sigma}\Big(\frac{1}{b^j}\Big) &+ \varphi_b^{\sigma}\Big(\frac{2}{b^j}\Big) + \dots + \varphi_b^{\sigma}\Big(\frac{b^n-1}{b^j}\Big) + \varphi_b^{\sigma}\Big(\frac{b^n}{b^j}\Big) \\ &+ \varphi_b^{\sigma}\Big(\frac{2}{b^j}\Big) + \dots + \varphi_b^{\sigma}\Big(\frac{b^n-1}{b^j}\Big) + \varphi_b^{\sigma}\Big(\frac{b^n}{b^j}\Big) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &+ \varphi_b^{\sigma}\Big(\frac{b^n-1}{b^j}\Big) + \varphi_b^{\sigma}\Big(\frac{b^n}{b^j}\Big) \\ &+ \varphi_b^{\sigma}\Big(\frac{b^n}{b^j}\Big).\end{aligned}$$

Since φ_b^{σ} is 1-periodic and since $\sigma \in \mathcal{A}(\tau)$ and hence, by Lemma 6, $\varphi_b^{\sigma}(x) = \varphi_b^{\sigma}(1-x)$ for $x \in [0,1]$, it follows that

$$\sum_{N=1}^{b^n} N\varphi_b^{\sigma}\Big(\frac{N}{b^j}\Big) = \frac{b^n}{2} \sum_{N=1}^{b^n} \varphi_b^{\sigma}\Big(\frac{N}{b^j}\Big) = \frac{b^{2n}}{2} \int_0^1 \varphi_b^{\sigma}(x)\,dx = \frac{b^{2n+1}}{2}\, \Phi_b^{\sigma}.$$

This leads to

$$\Sigma_{4,2} = b^{n-1} \sum_{i=1}^{n} (-1)^{s_i} \frac{b^{2n+1}}{2} \Phi_b^{\sigma} = \frac{b^{3n}}{2} \Phi_b^{\sigma} \sum_{i=1}^{n} (-1)^{s_i} = \frac{b^{3n}}{2} (2l-n) \Phi_b^{\sigma}.$$

It remains to compute $\Sigma_{4,1}$. We have

$$\begin{aligned}\mathcal{H}_{b,n}^{\Sigma} &= \Big\{ \Big(\sum_{i=0}^{n-1} \frac{\sigma_i(a_i)}{b^{i+1}}, \sum_{i=0}^{n-1} \frac{a_{n-1-i}}{b^{i+1}} \Big) : a_0, \dots, a_{n-1} \in \{0, \dots, b-1\} \Big\} \\ &= \Big\{ \Big(\sum_{i=0}^{n-1} \frac{x_i}{b^i}, \sum_{i=0}^{n-1} \frac{\sigma_{n-1-i}^{-1}(x_{n-1-i})}{b^{i+1}} \Big) : x_0, \dots, x_{n-1} \in \{0, \dots, b-1\} \Big\},\end{aligned}$$

with $(\sigma_0, \dots, \sigma_{n-1}) \in \{\sigma, \overline{\sigma}\}^n$. Note that if $\sigma \in \mathcal{A}(\tau)$ then also $\sigma^{-1} \in \mathcal{A}(\tau)$.

Let $g:[0,1]^2\to[0,1]^2$ be the map defined by $g(x,y)=(y,x)$ and for $\Sigma=(\sigma_0,\dots,\sigma_{n-1})$ define $\Sigma^*=(\sigma_{n-1}^{-1},\dots,\sigma_0^{-1})\in\{\sigma^{-1},\overline{\sigma^{-1}}\}^n$. Then we have found that $\mathcal{H}_{b,n}^{\Sigma}=g(\mathcal{H}_{b,n}^{\Sigma^*})$ and therefore

$$\begin{aligned}\Sigma_{4,1}&=\sum_{\lambda,N=1}^{b^n}\lambda E\Big(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\Big)=\sum_{\lambda,N=1}^{b^n}\lambda E\Big(\frac{\lambda}{b^n},\frac{N}{b^n},g(\mathcal{H}_{b,n}^{\Sigma^*})\Big)\\&=\sum_{\lambda,N=1}^{b^n}\lambda E\Big(\frac{N}{b^n},\frac{\lambda}{b^n},\mathcal{H}_{b,n}^{\Sigma^*}\Big)=\frac{b^{3n}}{2}(2l-n)\Phi_b^{\sigma^{-1}},\end{aligned}$$

where for the last equality we used the formula for $\Sigma_{4,2}$ since the number of components of Σ which are equal to σ is the same as the number of components of Σ^* which are equal to σ^{-1}. By Lemma 5 we have $\Phi_b^{\sigma^{-1}}=\Phi_b^{\sigma}$ and hence $\Sigma_{4,1}=(b^{3n}/2)(2l-n)\Phi_b^{\sigma}$. Together we obtain $\Sigma_4=(2l-n)\Phi_b^{\sigma}$.

Now the desired formula follows from $(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2=\Sigma_1+\Sigma_4-\Sigma_5+\Sigma_3$. The evaluation of this sum is a matter of straightforward calculations and hence we omit the details. ∎

For the *proof of Theorem 1* we just remark that the only place in the proof of Theorem 2 where we used that $\sigma\in\mathcal{A}(\tau)$ was in the exact evaluation of Σ_4. However, it is easy to see that for arbitrary permutations $\sigma\in\mathfrak{S}_b$ we always have $\Sigma_4=O(n)$ and hence the result of Theorem 1 follows as well from the proof above.

5. Numerical results. In view of Corollary 2 we search for permutations $\sigma\in\mathcal{A}(\tau)$ giving the minimal L_2 discrepancy for a fixed base b. In fact, we want to minimize the expression $\Phi_b^{\sigma,(2)}-(\Phi_b^{\sigma})^2$. To this end we use an alternative formula for $\Phi_b^{\sigma,(2)}$ that can be derived similarly to the formula for Φ_b^{σ} given in Lemma 5.

LEMMA 8. *For any $\sigma\in\mathfrak{S}_b$ we have*

$$\begin{aligned}\Phi_b^{\sigma,(2)}=\frac{1-6b^2+9b^3-4b^4}{18b^2}+\sum_{k_1,k_2=0}^{b-1}&\frac{\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}}{b^3}\\&\times\Big(b\max\{k_1,k_2\}-\frac{k_1^2+k_1+k_2^2+k_2}{2}\Big).\end{aligned}$$

If in addition $\sigma\in\mathcal{A}(\tau)$ then

$$\begin{aligned}\Phi_b^{\sigma,(2)}=\frac{1}{2b^3}\Big(&2bS_3(\sigma)-S_2(\sigma)-(2b-1)S_1(\sigma)\\&-\frac{b(6-11b+6b^2+3b^3-12b^4+8b^5)}{18}\Big),\end{aligned}$$

where

$$S_1(\sigma) = \sum_{k=0}^{b-1} k\sigma(k),$$

$$S_2(\sigma) = \sum_{k=0}^{b-1} k^2\sigma(k)^2,$$

$$S_3(\sigma) = \sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\max\{\sigma(k_1),\sigma(k_2)\}.$$

Proof. As these results are more of a technical nature and not as essential to the rest, we only give a short sketch of the proof.

The idea is as in Lemma 5 to first separate the integral into intervals where $(\phi_{b,h}^{\sigma})'$ is constant and integrate the occurring quadratic functions explicitly. After some calculation this leads to the expression

$$3b^4\Phi_b^{\sigma,(2)} = \left(\frac{1}{2}\left(3\sum_{k,l=0}^{b-1} F_{l,k}^{\mathrm{id}} - \sum_{k=0}^{b-1} F_{k,k}^{\mathrm{id}}\right) + 3\sum_{l_1,l_2=0}^{b-1} (b-1-\max\{l_1,l_2\})F_{l_1,l_2}^{\sigma}\right),$$

where

$$F_{k,l}^{\sigma} := \sum_{h=0}^{b-1} (\phi_{b,h}^{\sigma})'(k/b+0)(\phi_{b,h}^{\sigma})'(l/b+0)$$

for which by equations (3)–(5) the explicit expression

$$F_{k,l}^{\sigma} = \frac{b}{6}\left(1-3b+2b^2+3\big(\sigma(k)(\sigma(k)+1)+\sigma(l)(\sigma(l)+1)\big)-6b\max\{\sigma(k),\sigma(l)\}\right)$$

can be derived. Since the first two sums in the above expression for $\Phi_b^{\sigma,(2)}$ are thus merely rational polynomials in b (which can be derived by interpolation), the only term depending on σ is the third one. Some more rearranging leads to the result.

For the second formula, the strategy is to use the symmetry

$$\sigma(k) + \sigma(b-1-k) = b-1$$

to reduce the first formula to one using only S_1, S_2, S_3. As an example, $\sum_{k=0}^{b-1} k\sigma(k)^2 = p(b) + (b-1)S_1(\sigma)$, where $p(b)$ is a rational polynomial depending only on b. ■

From the second formula we deduce that σ and σ^{-1} can be interchanged. Therefore for $\sigma \in \mathcal{A}(\tau)$ we can replace σ^{-1} by σ in the first formula for $\Phi_b^{\sigma,(2)}$. Note that, since $\Phi_b^{\sigma,(1)}$ has the same invariance against the inversion of σ, the same applies also to the L_2 discrepancy.

Using the alternative formulas from Lemmas 5 and 8 which are preferable for computation, we have performed a full search over all permutations

Table 1. Numerical results for the full search in $\mathcal{A}(\tau)$

b	$\frac{\Phi_b^{\sigma,(2)}-(\Phi_b^{\sigma})^2}{\log b}$	num. value	σ
2	$\frac{5}{192\log(2)}$	0.037570	id
3	$\frac{4}{81\log(3)}$	0.044950	id
4	$\frac{5}{96\log(4)}$	0.037570	$(2,1)$
5	$\frac{112}{1875\log(5)}$	0.037114	$(3,1)$
6	$\frac{343}{5184\log(6)}$	0.036927	$(4,1)$
7	$\frac{512}{7203\log(7)}$	0.036529	$(2,0)(5,1)(6,4)$
8	$\frac{5}{64\log(8)}$	0.037570	$(4,1)(6,3)$
9	$\frac{512}{6561\log(9)}$	0.035516	$(5,1)(7,3)$
10	$\frac{3391}{40000\log(10)}$	0.036817	$(2,8,4,6,9,7,1,5,3,0)$
11	$\frac{3680}{43923\log(11)}$	0.034940	$(7,1)(4,2)(9,3)(8,6)$
12	$\frac{1759}{20736\log(12)}$	0.034137	$(5,4,10,6,7,1)(8,9,3,2)$
13	$\frac{574}{6591\log(13)}$	0.033953	$(5,12,7,0)(10,11,2,1)(8,9,4,3)$
14	$\frac{41581}{460992\log(14)}$	0.034178	$(2,5,7,3,12,4,0)(9,13,11,8,6,10,1)$
15	$\frac{4714}{50625\log(15)}$	0.034385	$(8,10,12,9,1)(5,13,6,4,2)(11,3)$
16	$\frac{17573}{196608\log(16)}$	0.032237	$(7,6,14,8,9,1)(12,11,5,2)(4,10,13,3)$
17	$\frac{8040}{83521\log(17)}$	0.033977	$(9,1)(4,6,2)(13,3)(11,5)(15,7)(14,12,10)$
18	$\frac{40631}{419904\log(18)}$	0.033478	$(10,15,11,9,5,16,7,2,6,8,12,1)(13,14,4,3)$
19	$\frac{12970}{130321\log(19)}$	0.033800	$(7,12,13,2,14,15,8,1)(10,17,11,6,5,16,4,3)$
20	$\frac{46733}{480000\log(20)}$	0.032500	$(11,1)(7,2)(16,3)(14,5)$ $(9,6)(18,8)(13,10)(17,12)$
21	$\frac{19402}{194481\log(21)}$	0.032768	$(12,1)(7,2)(17,3)(15,5)$ $(9,6)(19,8)(14,11)(18,13)$
22	$\frac{278629}{2811072\log(22)}$	**0.032066**	$(10,5,7,2,15,8,20,11,16,14,19,6,13,1)$ $(4,18,17,3)$
23	$\frac{87112}{839523\log(23)}$	0.033093	$(9,21,13,1)(16,20,6,2)(5,12,18,3)$ $(19,17,10,4)(14,15,8,7)$

$\sigma \in \mathcal{A}(\tau)$ for bases $4 \le b \le 23$. Note that we improved the best results known until now in all of these bases which were obtained for the identical permutation (see (2)—the best value 0.03757 appeared in base 2). In particular, the minimal value occurs in base 22 (see Table 1).

Additionally, we have performed a full search over all permutations $\sigma \in \mathcal{A}(\tau)$ where $\Phi_b^{\sigma} = 0$ for bases $b \le 17$, $b \notin \{2,3,6,7,10,14\}$, and tabulated those with the minimal L_2 discrepancy (see Table 2).

Table 2. Numerical results for the full search in $\mathcal{A}(\tau)$ where $\Phi_b^\sigma = 0$ (see Remark 2)

b	$\frac{\Phi_b^{\sigma,(2)}}{\log b}$	num. value	σ
4	$\frac{5}{96\log(4)}$	0.037570	$(1,3,2,0)$
5	$\frac{26}{375\log(5)}$	0.043079	$(1,4,3,0)$
8	$\frac{5}{64\log(8)}$	0.037570	$(2,4,7,5,3,0)(6,1)$
9	$\frac{20}{243\log(9)}$	0.037458	$(3,8,5,0)(6,7,2,1)$
11	$\frac{38}{363\log(11)}$	0.043656	$(3,1,6,0)(8,2)(10,7,9,4)$
12	$\frac{235}{2592\log(12)}$	0.036486	$(3,1,9,5,4,11,8,10,2,6,7,0)$
13	$\frac{574}{6591\log(13)}$	0.033953	$(5,12,7,0)(10,11,2,1)(8,9,4,3)$
15	$\frac{964}{10125\log(15)}$	0.035158	$(5,3,10,6,14,9,11,4,8,0)(12,13,2,1)$
16	$\frac{37}{384\log(16)}$	0.034752	$(4,15,11,0)(12,14,3,1)(8,13,7,2)(6,10,9,5)$
17	$\frac{28}{289\log(17)}$	0.034196	$(6,16,10,0)(14,15,2,1)(11,13,5,3)(9,12,7,4)$

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