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Pair correlation of lattice points with prime constraint

by

Maosheng Xiong (Kowloon) and
Alexandru Zaharescu (Bucureşti and Urbana, IL)

1. Introduction. Let Ω be a star-shaped region in the plane (i.e., for any $(x,y) \in \Omega$, the line segment joining $(0,0)$ to (x,y) is contained in Ω) and suppose Ω is bounded by a curve $\mathcal{C}$, parametrized by $x = \rho_\Omega(\alpha)\cos\alpha, y = \rho_\Omega(\alpha)\sin\alpha$, where ρ_Ω is continuous and piecewise C^1 on $[0,2\pi]$. For each large integer X, the dilated region

$$\Omega_X = \{(x,y) \in \mathbb{R}^2 : (x/X, y/X) \in \Omega\}$$

has area $X^2\mathrm{Area}(\Omega)$ and is bounded by the curve $\mathcal{C}_X = \{(x,y) \in \mathbb{R}^2 : (x/X, y/X) \in \mathcal{C}\}$. Given a pair of coprime integers a and b, we are interested in the set $\Omega_X^{(a,b)}$ of lattice points in Ω_X defined by

$$\Omega_X^{(a,b)} = \{(x,y) \in \mathbb{Z}^2 \cap \Omega_X : ax+by \text{ is a prime}\}.$$

Here the primes may be positive or negative.

A natural way of studying the distribution of points in $\Omega_X^{(a,b)}$ is to look at the angles of straight lines joining the origin to points of $\Omega_X^{(a,b)}$. This is the set of lines through all the lattice points in $\Omega_X^{(a,b)}$, taken without multiplicities. In this paper we consider the pair correlation of these angles as Ω and a,b are fixed and $X \to \infty$. Our goal is to prove the existence of a limiting pair correlation function and to provide an explicit formula for it.

Without the constraint that $ax+by$ is a prime, the problem has been considered for the set of lattice points in Ω_X which are visible from the origin, i.e., the set

$$\mathcal{F}(\Omega, X) = \{(x,y) \in \mathbb{Z}^2 \cap \Omega_X : \gcd(x,y) = 1\},$$

and it was proved in [1] that as X goes to ∞, the limiting "nearest neighbor spacing distribution" of the angles of straight lines from the origin to the points of $\mathcal{F}(\Omega, X)$ exists, and it was computed explicitly. For Ω a unit square,

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the correlations of directions, i.e., the angles of straight lines from the origin or other base point to the points of the set Ω_X have also been considered in [2]. The results of [1] imply a very strong repulsion between these angles, even stronger than in the GUE model studied by Random Matrix Theory, which is also believed to describe the distribution of the imaginary parts of zeros of primitive L-functions (see the fundamental papers of Montgomery [9], Rudnick and Sarnak [10], and Katz and Sarnak [8]). By contrast, the pair correlation of the angles formed by the points of Ω_X with prime constraint is very close to the Poissonian model, and only depends on the shape of the region. We will see that the pair correlation is Poissonian if the region Ω is a disk.

To formulate the problem precisely, let $0 < \theta_1 < \cdots < \theta_N < 2\pi$ be the angles that correspond to elements from $\Omega_X^{(a,b)}$, where $N = N(\Omega, X)$. We normalize them to $0 < \theta_1/2\pi < \theta_2/2\pi < \cdots < \theta_N/2\pi < 1$. The pair correlation function measures the density of the differences between pairs of elements of a given sequence. Thus for the sequence $(\theta_n/2\pi)_{n\le N}$ in $[0, 1]$, the limiting pair correlation function $R_2(\Omega_{(a,b)}, \lambda)$ is given, if it exists, by

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1 \le n_1 \ne n_2 \le N : \frac{\theta_{n_1}}{2\pi} - \frac{\theta_{n_2}}{2\pi} \in \frac{1}{N}\mathbf{I}\right\} = \int_{\mathbf{I}} R_2(\Omega_{(a,b)}, \lambda)\, d\lambda$$

for any interval $\mathbf{I} \subset \mathbb{R}$. In the Poissonian model the pair correlation function $R_2(\Omega_{(a,b)}, \lambda)$ is identically equal to 1. Our main result is the following.

Theorem 1.1. *As X goes to ∞, the limiting pair correlation function $R_2(\Omega_{(a,b)}, \lambda)$ of the angles of straight lines from the origin to the points of the set $\Omega_X^{(a,b)}$ exists, is independent of a, b and is identically equal to the constant*

$$\frac{\pi}{2A(\Omega)^2}\int_0^{2\pi} \rho_\Omega(\theta)^4\, d\theta,$$

where $A(\Omega)$ is the area of the region Ω.

Since $A(\Omega)^2 = \frac{1}{4}(\int_0^{2\pi} \rho_\Omega(\theta)^2\, d\theta)^2 \le \frac{\pi}{2}\int_0^{2\pi} \rho_\Omega(\theta)^4\, d\theta$, we have $R_2(\Omega_{(a,b)}, \lambda) \ge 1$ for any λ, and equality holds if and only if Ω is a disk. This shows that the pair correlation density we obtained for general Ω corresponds to a Poisson process with non-uniform density.

The angles of straight lines from the origin to the points of $\Omega_X^{(a,b)}$ are related to Farey fractions with prime denominators, whose pair correlation function was obtained in [11]. However, in order to translate from one set to the other by using a deformation lemma ([12, Theorem 2]), it is essential to first obtain a local version, that is, the pair correlation of Farey fractions in short intervals with prime denominators. This is a more subtle problem than the one investigated in [11], and it is treated in Section 2. As in [1], the machinery of Kloosterman sums plays a crucial role. The main difficulty comes, of course, from the constraint that $ax + by$ is a prime. In order to overcome this difficulty, a key role in our proof will be played by the powerful estimates for bilinear forms with Kloosterman fractions developed by Duke, Friedlander and Iwaniec in [4].

It is interesting that present day techniques allow us to solve the pair correlation problem for the angles of lines from the origin to the points of $\Omega_X^{(a,b)}$. By contrast, the local spacing distribution of the sequence of prime numbers is not well understood. The spectacular work by Goldston, Pintz and Yildirim [6] establishes the existence of small gaps between consecutive primes. But the problem of existence of the pair correlation measure, or other local spacing measures, is still wide open. The distribution is conjectured to be Poissonian, and this was proved by Gallagher [5] under the assumption of a uniform version of the k-tuple conjecture for prime numbers.

2. Farey fractions with prime denominators in short intervals. For each positive integer Q, let

$$\mathscr{M}_Q = \{a/p : 1 \le a < p \le Q,\ p \text{ is a positive prime number}\}$$

be the set of *Farey fractions* of order Q with prime denominators. The limiting pair correlation measure of $\mathscr{M}_Q$ on $[0,1]$ as $Q \to \infty$ was established by the authors in [11]. Here we will need a short interval version of this result. We adapt the method of [11] with some necessary modifications. Similar ideas were originated in [3] and have been used in [12].

Let $\mathbf{I}$ be a subinterval of $[0,1]$ and let $|\mathbf{I}|$ denote its length. Define $\mathscr{M}_{\mathbf{I}}(Q) := \mathscr{M}_Q \cap \mathbf{I}$ and denote by $N_{\mathbf{I}}(Q)$ the cardinality of the set $\mathscr{M}_{\mathbf{I}}(Q)$. By the prime number theorem,

$$N_{\mathbf{I}}(Q) = \sum_{p \le Q} \sum_{a/p \in \mathbf{I}} 1 = |\mathbf{I}| \frac{Q^2}{2\log Q}(1 + O(\log^{-1} Q)) + O(Q\log^{-1} Q).$$

Hence

$$N = \frac{N_{\mathbf{I}}(Q)}{|\mathbf{I}|} = \frac{Q^2}{2\log Q}(1 + O(\log^{-1} Q)) + O(|\mathbf{I}|^{-1} Q\log^{-1} Q).$$

Our objective is to estimate, for any positive real number Λ, the quantity

$$S_{Q,\mathbf{I}}(\Lambda) := \#\left\{(x,y) \in \mathscr{M}_{\mathbf{I}}(Q)^2 : x \ne y, x - y \in \frac{(0,\Lambda)}{N} + \mathbb{Z}\right\},$$

as $Q \to \infty$. We prove a more general result. Denote by $\operatorname{Supp} f$ the support of a function f.

Lemma 2.1. *Suppose* $H, G \in C^1(\mathbb{R})$ *with* $\operatorname{Supp} G \subset (0,1)$ *and* $\operatorname{Supp} H \subset (0,\Lambda)$ *for some* $\Lambda > 0$. *Define*

$$h(y) = \sum_{n\in\mathbb{Z}} H(N(y+n)), \quad g(y) = \sum_{n\in\mathbb{Z}} G(y+n),$$

and

$$S_{Q,\mathbf{I},H,G} = \sum_{\gamma,\gamma'\in\mathscr{M}_Q} h(\gamma-\gamma')g(\gamma)g(\gamma').$$

Then

$$S_{Q,\mathbf{I},H,G} = \frac{Q^2}{2\log Q}\Big(\int_0^1 G(z)^2\,dz\Big)\int_0^\Lambda H(x)\,dx + O_{H,G}\Big(\frac{Q^2}{\log^2 Q}\Big).$$

Note that assuming Lemma 2.1 and using the fact that the error term is of order $Q^2\log^{-2} Q$, we have

$$\lim_{Q\to\infty}\frac{S_{Q,\mathbf{I},H,G}}{N_{\mathbf{I}}(Q)} = \frac{\int_0^1 G(z)^2\,dz}{|\mathbf{I}|}\cdot\int_0^\Lambda H(x)\,dx.$$

Letting the smooth function G approach $\chi_{\mathbf{I}}$, the characteristic function of the interval $\mathbf{I}$, we have

$$\frac{\int_0^1 G(z)^2\,dz}{|\mathbf{I}|}\to 1,$$

and letting H approach $\chi_{(0,\Lambda)}$, by a standard approximation argument we obtain

Theorem 2.2. *As* Q *goes to* ∞, *the limiting pair correlation function of* $\mathscr{M}_Q$ *on any subinterval* $\mathbf{I}\subset[0,1]$ *exists, is independent of the location and length of* $\mathbf{I}$, *and is identically equal to* 1.

Remark. Denote by $\widetilde{\mathscr{M}_Q}$ the set of general Farey fractions of order Q with prime denominators, that is,

$$\widetilde{\mathscr{M}_Q} = \{a/p : a\in\mathbb{Z},\ p\le Q,\ p \text{ is a positive prime number}\}.$$

It will be clear that the proof of Lemma 2.1 can be extended to the set $\widetilde{\mathscr{M}_Q}$, and consequently, as Q goes to ∞, the limiting pair correlation function of $\widetilde{\mathscr{M}_Q}$ on any finite interval $\mathbf{I}\subset\mathbb{R}$ exists, is independent of the location and length of $\mathbf{I}$, and is identically equal to 1. This observation will be used in Section 4.

3. Proof of Lemma 2.1. In what follows, p, q stand for (positive) prime numbers, and for simplicity, all the constants in the proof implied by the big "O" and "$\ll$" notations may depend on the functions H and G.

3.1. Fourier series expansion and Poisson summation. Suppose that the Fourier series expansions of the functions h and g are given by

$$h(y)=\sum_{n\in\mathbb{Z}} c_n e(ny) \quad \text{and} \quad g(y)=\sum_{n\in\mathbb{Z}} a_n e(ny),$$

where $e(ny)=\exp(2\pi iny)$ for $y\in\mathbb{R}$. Then we have

$$\begin{aligned} S_{Q,\mathbf{I},H,G} &= \sum_{\gamma,\gamma'\in\mathscr{M}_Q}\sum_m c_m e(m(\gamma-\gamma'))\sum_n a_n e(n\gamma))\sum_r a_r e(r\gamma') \\ &= \sum_{m,n,r} c_m a_n a_r \sum_{\gamma\in\mathscr{M}_Q} e((m+n)\gamma)\sum_{\gamma'\in\mathscr{M}_Q} e((r-m)\gamma'). \end{aligned}$$

Changing the summation index so that $m+n=m'$, $r-m=n'$, $m=r'$, hence $m=r'$, $n=m'-r'$, $r=n'+r'$, and using the formula

$$\sum_{x\in\mathscr{M}_Q} e(rx)=\sum_{p\le Q}\sum_{a=1}^{p-1} e(ra/p)=\sum_{p\le Q,\, p|r} p-\pi(Q), \quad r\in\mathbb{Z},$$

where $\pi(Q)$ is the number of prime numbers in the interval $[1,Q]$, we have

$$\begin{aligned} &S_{Q,\mathbf{I},H,G} \\ &=\sum_{m',n',r'} c_{r'}a_{m'-r'}a_{n'+r'}\Big(\sum_{\substack{p,q\le Q\\ p|m,\, q|n}} pq+\pi(Q)^2-\pi(Q)\Big(\sum_{p\le Q,\, p|m} p+\sum_{q\le Q,\, q|n} q\Big)\Big). \end{aligned}$$

We can rewrite this equality as

$$S_{Q,\mathbf{I},H,G}=S_1+S_2-S_3,$$

where

$$\begin{aligned} S_1 &= \sum_{p,q\le Q} pq\sum_{m,n,r\in\mathbb{Z}} c_r a_{pm-r}a_{qn+r}, \\ S_2 &= \pi(Q)^2\sum_{m,n,r\in\mathbb{Z}} c_r a_{m-r}a_{n+r}, \\ S_3 &= \sum_{p\le Q} p\sum_{m,n,r\in\mathbb{Z}} c_r a_{pm-r}a_{n+r}+\sum_{q\le Q} q\sum_{m,n,r\in\mathbb{Z}} c_r a_{m-r}a_{qn+r}. \end{aligned}$$

Similar to the computation in [12], by considering the Fourier transform of the function

$$G_{r,d}(x)=\frac{1}{d}G\Big(\frac{x}{d}\Big)e\Big(\frac{rx}{d}\Big), \quad x\in\mathbb{R},$$

and applying the Poisson summation formula, we obtain

$$\sum_{n\in\mathbb{Z}} a_{dn+r}=\sum_{n\in\mathbb{Z}}\widehat{G}_{-r,d}(n)=\sum_{n\in\mathbb{Z}} G_{-r,d}(n)=\sum_{n\in\mathbb{Z}}\frac{1}{d}G\Big(\frac{n}{d}\Big)e\Big(\frac{-rn}{d}\Big)$$

for any integers $d > 0$ and r. It follows that

$$\sum_{r\in\mathbb{Z}} c_r \sum_{m\in\mathbb{Z}} a_{pm-r} \sum_{n\in\mathbb{Z}} a_{qn+r} \\ = \frac{1}{pq}\sum_{m,n} G\Big(\frac{m}{p}\Big) G\Big(\frac{n}{q}\Big) \sum_r c_r e\Big(\Big(\frac{m}{p}-\frac{n}{q}\Big) r\Big).$$

From the Fourier expansion of $h(y)$ we derive

$$\sum_r c_r e\Big(\Big(\frac{m}{p}-\frac{n}{q}\Big) r\Big) = h\Big(\frac{m}{p}-\frac{n}{q}\Big) = \sum_r H\Big(N\Big(r+\frac{m}{p}-\frac{n}{q}\Big)\Big).$$

Therefore we may write S_1 as

$$S_1 = \sum_{1\le p,q\le Q} \sum_{m,n\in\mathbb{Z}} G\Big(\frac{m}{p}\Big) G\Big(\frac{n}{q}\Big) \sum_{r\in\mathbb{Z}} H\Big(N\Big(r+\frac{m}{p}-\frac{n}{q}\Big)\Big).$$

Similarly for S_2 and S_3 we have

$$S_2 = \pi(Q)^2 \sum_{m,n\in\mathbb{Z}} G(m)G(n) \sum_{r\in\mathbb{Z}} H(N(r+m-n)),$$

$$S_3 = 2\pi(Q) \sum_{1\le p\le Q} \sum_{m,n\in\mathbb{Z}} G\Big(\frac{m}{p}\Big) G(n) \sum_{r\in\mathbb{Z}} H\Big(N\Big(r+\frac{m}{p}-n\Big)\Big).$$

Since we assume that $\operatorname{Supp} G \subset (0,1)$, $S_2 = S_3 = 0$. It suffices to treat S_1.

3.2. Further reductions. First, using the facts that $\operatorname{Supp} G \subset (0,1)$, $\operatorname{Supp} H \subset (0,\Lambda)$, $p,q \le Q$, and $N \sim Q^2/2\log Q$ when Q is sufficiently large, we obtain $H(N(r+m/p-n/q)) = 0$ if $r \ne 0$ or $p = q$. Now for $r = 0$ and $p \ne q$, since $(p,q) = 1$, there is a unique integer $\bar{q}$ such that $0 < \bar{q} < p, \bar{q}q \equiv 1 \pmod{ptp}$. Take $a = (1-\bar{q}q)/p$, so that $ap+\bar{q}q = 1$. Changing the summation index to

$$m' = qm - pn, \quad n' = am + \bar{q}n,$$

we have

$$m = \bar{q}m' + pn', \quad n = -am' + qn'.$$

Therefore S_1 can be rewritten as

$$S_1 = \sum_{p\ne q\le Q} \sum_{m,n\in\mathbb{Z}} G\Big(\frac{m}{p}\Big) G\Big(\frac{n}{q}\Big) H\Big(N\Big(\frac{m}{p}-\frac{n}{q}\Big)\Big) \\ = \sum_{p\ne q\le Q} \sum_{m,n\in\mathbb{Z}} G\Big(\frac{\bar{q}m}{p}+n\Big) G\Big(\frac{-am}{q}+n\Big) H\Big(\frac{Nm}{pq}\Big).$$

Since $\operatorname{Supp} H \subset (0,\Lambda)$, to get a non-zero contribution from the term $H(Nm/pq)$ we must have

$$0 < \frac{Nm}{pq} < \Lambda.$$

This implies

$$0 < m < \frac{pq\Lambda}{N} \le \frac{Q^2\Lambda}{N} \ll \log Q, \tag{3.1}$$

and

$$p > \frac{Nm}{q\Lambda} > \frac{N}{Q\Lambda} \gg \frac{Q}{\log Q}.$$

On the other hand, since $\operatorname{Supp} G \subset (0,1)$, to get a non-zero contribution from the term $G(\bar{q}m/p + n)$, we need

$$0 < \bar{q}m/p + n < 1. \tag{3.2}$$

There is at most one integer n with this property for each m. Notice that

$$G\left(\frac{-am}{q} + n\right) = G\left(\frac{\bar{q}m}{p} + n - \frac{m}{pq}\right) = G\left(\frac{\bar{q}m}{p} + n\right) + O\left(\frac{m}{pq}\right).$$

Using (3.1) and the remark after (3.2) we obtain

$$S_1 = \sum_{m,n\in\mathbb{Z}} \sum_{p\ne q\le Q} G\left(\frac{\bar{q}m}{p} + n\right)^2 H\left(\frac{Nm}{pq}\right) + O(\log^5 Q).$$

For any fixed integers m, n that satisfy (3.1) and (3.2), define

$$f(x) = f_{m,n}(x) = G(mx+n)^2, \quad h(x,y) = h_m(x,y) = H\left(\frac{Nm}{xy}\right), \quad x, y \in \mathbb{R}.$$

It is enough to estimate

$$S_{m,n} = \sum_{p\ne q\le Q} f(\bar{q}/p) h(p,q).$$

It is clear that

$$|f(x)| \ll 1, \quad |f'(x)| \ll \log Q, \quad x \in \mathbb{R}.$$

Let K be a large positive integer which will be chosen later. We can rewrite $S_{m,n}$ as

$$S_{m,n} = \sum_{i=0}^{k-1} \sum_{\substack{p\ne q\le Q \\ \bar{q}\in \mathbf{I}_{p,i}}} (f(i/K) + O(\log Q/K)) h(p,q),$$

where $\mathbf{I}_{p,i}$ is the closed interval with endpoints pi/k and $p(i+1)/k$. This gives

$$S_{m,n} = \sum_{i=0}^{k-1} f(i/K) \sum_{\substack{p\ne q\le Q \\ \bar{q}\in \mathbf{I}_{p,i}}} h(p,q) + O(Q^2/K). \tag{3.3}$$

3.3. Kloosterman sums with prime entries. For each i, the above inner sum over p, q can be written as

$$I = \sum_{p\le Q} \sum_{\substack{q\le Q \\ \bar{q}\in \mathbf{I}_{p,i}}} h(p,q) = \sum_{p\le Q} \sum_{q\le Q} h(p,q) \sum_{y\in \mathbf{I}_{p,i}} \frac{1}{p} \sum_{|l|<p/2} e(l(\bar{q}-y)/p).$$

The term corresponding to $l=0$ is the main term, which gives

$$I_1 = \sum_{p\le Q} \sum_{q\le Q} h(p,q) \frac{1}{p} (|\mathbf{I}_{p,i}| + O(1)) = \frac{1}{K} \sum_{p,q\le Q} h(p,q) + O(Q).$$

The terms corresponding to $l\neq 0$ give

$$I_2 = \sum_{p\le Q} \frac{1}{p} \sum_{0\neq |l|<p/2} \sum_{y\in \mathbf{I}_{p,i}} e(-ly/p) \sum_{q\le Q} h(p,q) e(l\bar{q}/p).$$

Using the bound

$$\Big| \sum_{y\in \mathbf{I}_{p,i}} e(-ly/p) \Big| \ll \frac{p}{|l|},$$

and defining $\delta_{l,p}$ to be the complex number, depending only on l and p, such that

$$\Big| \sum_{q\le Q} h(p,q) e(l\bar{q}/p) \Big| = \sum_{q\le Q} h(p,q) e(l\bar{q}/p) \delta_{l,p},$$

we obtain

$$\begin{aligned} |I_2| &\ll \sum_{p\le Q} \frac{1}{p} \sum_{0\neq |l|<p/2} \frac{p}{|l|} \sum_{q\le Q} h(p,q) e(l\bar{q}/p) \delta_{l,p} \\ &= \sum_{0\neq |l|\le Q/2} \frac{1}{|l|} \sum_{\substack{2|l|\le p\le Q \\ q\le Q,\, q\neq p}} h(p,q) e(l\bar{q}/p) \delta_{l,p}. \end{aligned}$$

To deal with the inner sum, we apply Theorem 2 of [4] which states:

Consider general bilinear forms of the type

$$\mathscr{B}_F(M,N) = \sum_{\gcd(m,n)=1} \alpha_m \beta_n e\left(a\frac{\bar{m}}{n} \right) F(m,n),$$

where a is a fixed positive integer and α_m, β_n are arbitrary complex numbers for $M < m \le 2M$, $N < n \le 2N$, respectively, and $m\bar{m} \equiv 1 \pmod{ptn}$, and F is a smooth function whose partial derivatives satisfy

$$F^{(j,k)}(m,n) \ll \eta^{j+k} m^{-j} n^{-k} \tag{3.4}$$

for $0 \le j,k \le 2$ and some $\eta \ge 1$. Then

$$\mathscr{B}_F(M,N) \ll \eta^2 \|\alpha\| \, \|\beta\| (a+MN)^{3/8} (M+N)^{11/48+\epsilon},$$

where $\|\cdot\|$ denotes the l_2 norm and the implied constant depends only on ϵ.

Denote

$$\alpha_n^{(l)} = \begin{cases} \delta_{l,m}, & 2|l| \le n \le Q \text{ is a prime,} \\ 0, & \text{otherwise,} \end{cases} \qquad \beta_m^{(l)} = \begin{cases} 1, & m \le Q \text{ is a prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|\alpha^{(l)}\| \, \|\beta^{(l)}\| \le \pi(Q).$$

It is also easy to see that the function h satisfies the property (3.4) with $\eta = 1$. By dividing the interval $[1, Q]$ into dyadic intervals for both p, q and applying Theorem 2 of [4], we obtain

$$\sum_{\substack{2|l| \le p \le Q \\ q \le Q,\, q \ne p}} h(p,q) e(l\bar{q}/p) \delta_{l,p} \ll_\epsilon \pi(Q)(|l| + Q^2)^{3/8} Q^{11/48+\epsilon} \log^2 Q \ll_\epsilon Q^{2-1/48+\epsilon}.$$

Hence

$$I_2 \ll Q^{2-\gamma}$$

for some sufficiently small γ such that $0 < \gamma < 1/50$, which may be different at each occurrence. Combining the results for I_1 and I_2 we obtain the estimate

$$I = \frac{1}{K} \sum_{p,q \le Q} h(p,q) + O(Q^{2-\gamma}).$$

3.4. Completion of the proof of Lemma 2.1. Returning to (3.3) we have

$$S_{m,n} = \sum_{i=0}^{k-1} f(i/K) \left(\frac{1}{K} \sum_{p,q \le Q} h(p,q) + O(Q^{2-\gamma}) \right) + O(Q^2/K).$$

Choosing $K = Q^{\gamma/2}$ and noting that by the simple formula

$$\frac{1}{K} \sum_{i=0}^{k-1} f(i/K) = \int_0^1 f(x)\, dx + O(K^{-1}),$$

we obtain

$$S_{m,n} = \int_0^1 f(x)\, dx \sum_{p,q \le Q} h(p,q) + O(Q^{2-\gamma})$$

for some sufficiently small number $\gamma > 0$. Since $S_2 = S_3 = 0$ for Q sufficiently large, by using the definition of f and h we have

$$\begin{aligned} S_{Q,\mathbf{I},H,G} = S_1 &= \sum_{m,n \in \mathbb{Z}} S_{m,n} + O(\log^5 Q) \\ &= \sum_{m,n \in \mathbb{Z}} \int_0^1 G(mx+n)^2\, dx \sum_{p,q \le Q} H\left(\frac{Nm}{pq}\right) + O(Q^{2-\gamma}). \end{aligned}$$

Observe that

$$\int_0^1 G(mx+n)^2\,dx = \frac{1}{m}\Big(\int_n^{n+1} G(z)^2\,dz + \cdots + \int_{n+m-1}^{n+m} G(z)^2\,dz\Big).$$

Consequently,

$$\sum_{n\in\mathbb{Z}}\int_0^1 G(mx+n)^2\,dx = \frac{1}{m}\sum_{n\in\mathbb{Z}}\Big(\int_n^{n+1} G(z)^2\,dz + \cdots + \int_{n+m-1}^{n+m} G(z)^2\,dz\Big)$$
$$= \int_0^1 G(z)^2\,dz.$$

On the other hand, the term $\sum_{m\in\mathbb{Z}}\sum_{p,q\le Q} H(Nm/pq)$ has been handled in the computation of the pair correlation function of $\mathscr{M}_Q$ over the interval $[0,1]$ ([11, Lemma 4]), which yields the estimate

$$\sum_{m\in\mathbb{Z}}\sum_{p,q\le Q} H\Big(\frac{Nm}{pq}\Big) = \frac{Q^2}{2\log Q}\int_{\mathbb{R}} H(x)\,dx + O\Big(\frac{Q^2}{\log^2 Q}\Big).$$

Therefore we obtain the desired result

$$S_{Q,\mathbf{I},H,G} = \frac{Q^2}{2\log Q}\Big(\int_0^1 G(z)^2\,dz\Big)\int_0^{\Lambda} H(x)\,dx + O\Big(\frac{Q^2}{\log^2 Q}\Big).$$

This completes the proof of Lemma 2.1.

4. Proof of Theorem 1.1. Since $\gcd(a,b)=1$, we can find integers c,d such that $ad-bc=1$. We may assume, via a linear transformation of the xy-plane, that $a=1$, $b=0$. For each large integer X we consider the set

$$\Omega_X^{(1,0)} = \{(x,y)\in\mathbb{Z}^2\cap\Omega_X : x \text{ is a prime number}\}.$$

4.1. The case when Ω is a triangle. For $0\le\alpha<\beta<\pi/4$ and $m>0$, we first consider the case $\Omega = T_{\alpha,\beta,m}$, the triangle bounded by $y=x\tan\alpha$, $y=x\tan\beta$ and $x=m$. For each large integer X, denote

$$\mathscr{M}(T_{\alpha,\beta,m},X) := \{(x,y)\in\mathbb{Z}^2 : (x/X,y/X)\in T_{\alpha,\beta,m} \text{ and } x \text{ is a prime number}\}.$$

The map $(x,y)\mapsto y/x$ identifies $\mathscr{M}(T_{\alpha,\beta,m},X)$ with the set $\mathscr{M}_{\mathbf{I}}(mX)$, the set of Farey fractions of order mX with prime denominators in the interval $\mathbf{I}=[\tan\alpha,\tan\beta]$. By the prime number theorem, the set $\mathscr{M}(T_{\alpha,\beta,m},X)$ has cardinality

$$N(X) = N_{\alpha,\beta,mX} \sim (\tan\beta-\tan\alpha)\frac{(mX)^2}{2\log X} = \frac{X^2}{\log X}A(T_{\alpha,\beta,m}), \tag{4.1}$$

as $X\to\infty$.

For each $\mathbf{p} \in \mathscr{M}(T_{\alpha,\beta,m}, X)$, denote by $\theta(\mathbf{p}) \in [\alpha, \beta]$ the angle between the x-axis and the ray formed by joining $\mathbf{p}$ to the origin $\mathbf{0}$. It is clear that the map

$$f : \mathscr{M}_{\mathbf{I}}(mX) \subset \mathbf{I} \to \{\theta(\mathbf{p}) : \mathbf{p} \in \mathscr{M}(T_{\alpha,\beta,m}, X)\} \subset [\alpha, \beta] \tag{4.2}$$

given by $y/x \mapsto \arctan(y/x)$ is a one-to-one correspondence between the two sets. We consider, for any $\lambda > 0$,

$$\mathcal{R}_2(T_{\alpha,\beta,mX}; \lambda) = \frac{1}{N(X)} \# \left\{ (\mathbf{p}, \mathbf{q}) \in T^2_{\alpha,\beta,mX} : 0 < \frac{\theta(\mathbf{p}) - \theta(\mathbf{q})}{\beta - \alpha} \le \frac{\lambda}{N(X)} \right\},$$

$$\mathcal{R}_2(\mathscr{M}_{\mathbf{I}}(mX); \lambda) = \frac{1}{N(X)} \# \left\{ (\gamma_1, \gamma_2) \in \mathscr{M}_{\mathbf{I}}(mX)^2 : 0 < \frac{\gamma_1 - \gamma_2}{|\mathbf{I}|} \le \frac{\lambda}{N(X)} \right\},$$

which are the pair correlation measures of $\{\theta(\mathbf{p}) : \mathbf{p} \in \mathscr{M}(T_{\alpha,\beta,m}, X)\}$ and $\mathscr{M}_{\mathbf{I}}(mX)$, respectively. As for the latter, by Theorem 2.2, as X goes to ∞, the limiting pair correlation function of $\mathscr{M}(mX)$ on any subinterval $\mathbf{I} \subset [0, 1]$ exists, is independent of the location and length of $\mathbf{I}$ and is identically equal to 1. Hence

$$\lim_{X \to \infty} \mathcal{R}_2(\mathscr{M}_{\mathbf{I}}(mX); \lambda) = \lambda = \int_0^\lambda 1\, dx.$$

Because of the correspondence (4.2), the two functions $\mathcal{R}_2(T_{\alpha,\beta,mX}; \lambda)$ and $\mathcal{R}_2(\mathscr{M}_{\mathbf{I}}(mX); \lambda)$ are naturally related, and the relation between their limiting behavior as X goes to ∞ is revealed by Theorem 2 of [12], which states, roughly speaking, that if one limit exists, so does the other, and there is a simple explicit deformation formula relating the two limiting functions. In our case, this theorem implies that as $X \to \infty$, the limit $\lim_{X \to \infty} \mathcal{R}_2(T_{\alpha,\beta,mX}; \lambda)$ exists and is given by

$$\lim_{X \to \infty} \mathcal{R}_2(T_{\alpha,\beta,mX}; \lambda) = \frac{(\beta - \alpha)\lambda}{(\tan \beta - \tan \alpha)^2} \int_{\tan \alpha}^{\tan \beta} (1 + t^2)\, dt. \tag{4.3}$$

One can easily show that this is consistent with Theorem 1.1 for $\Omega = T_{\alpha,\beta,m}$.

4.2. Proof of Theorem 1.1 for a general region Ω. For any region $\Omega \subset \mathbb{R}^2$, we denote by $\mathscr{M}(\Omega)$ the set of lattice points from Ω such that the x-coordinate is a prime number. As $a = 1$, $b = 0$, we consider the set

$$\mathscr{M}(\Omega_X) = \Omega_X^{(1,0)} = \{(x, y) \in \mathbb{Z}^2 \cap \Omega_X : x \text{ is a prime number}\}.$$

For the number of elements of $\mathscr{M}(\Omega_X)$, the prime number theorem yields

$$N(\Omega, X) = \#\mathscr{M}(\Omega_X) \sim \frac{X^2}{\log X} A(\Omega), \tag{4.4}$$

where $A(\Omega)$ is the area of the region Ω. A sharp estimate for $N(\Omega, X)$ can be obtained by employing the work of Huxley and Nowak [7]. For $\lambda > 0$

denote

$$\mathcal{R}_{\Omega,X}(\lambda) = \frac{1}{N(\Omega,X)} \#\left\{(\mathbf{p},\mathbf{q}) \in \mathscr{M}(\Omega_X)^2 : 0 < \frac{\theta(\mathbf{p})-\theta(\mathbf{q})}{2\pi} \le \frac{\lambda}{N(\Omega,X)}\right\}.$$

To establish Theorem 1.1 for Ω, we need to prove convergenceof $\mathcal{R}_{\Omega,X}(\lambda)$ as $X \to \infty$, for all $\lambda > 0$. For this purpose, we decompose the region Ω into eight regions $\Omega_1, \dots, \Omega_8$, where Ω_k contains the points from Ω with argument between $(k-1)\pi/4$ and $k\pi/4$. For simplicity, we first consider the case where $\Omega\ (=\Omega_1) \subset \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x\}$.

For such a region $\Omega\ (=\Omega_1)$, we fix a large integer $L > 0$ and denote

$$\alpha = \frac{\pi}{4L}, \qquad \alpha_i = i\alpha, \qquad 0 \le i \le L.$$

Assume that $\xi_i, \eta_i \in [\alpha_i, \alpha_{i+1}]$, $0 \le i \le L-1$, are chosen so that $x_\Omega(\xi_i) = m_i = \min_{\alpha\in[\alpha_i,\alpha_{i+1}]} x_\Omega(\alpha)$ and $x_\Omega(\eta_i) = M_i = \max_{\alpha\in[\alpha_i,\alpha_{i+1}]} x_\Omega(\alpha)$. Denote $\triangle_i = T_{\alpha_i,\alpha_{i+1},m_i}$, $\triangle_i' = T_{\alpha_i,\alpha_{i+1},M_i}$ and

$$N_i(X) = \#\mathscr{M}(\triangle_{i,X}), \quad N_i'(X) = \#\mathscr{M}(\triangle_{i,X}').$$

Fix a sufficiently small real $\epsilon > 0$ and assume that $\gamma_i \in [\alpha_i - \epsilon, \alpha_i + \epsilon]$, $1 \le i \le L-1$, are chosen so that $x_\Omega(\gamma_i) = M_i' = \max_{\alpha\in[\alpha_i-\epsilon,\alpha_i+\epsilon]} x_\Omega(\alpha)$. Denote

$$\triangle_i^{(\epsilon)} = T_{\alpha_i-\epsilon,\alpha_i+\epsilon,M_i'} \quad \text{and} \quad N_i^{(\epsilon)}(X) = \#\mathscr{M}(\triangle_{i,X}^{(\epsilon)}).$$

Now for each i and $\lambda > 0$ we define

$$\mathcal{R}_{i,X}(\lambda) = \frac{1}{N_i(X)} \#\left\{(\mathbf{p},\mathbf{q}) \in \mathscr{M}(\triangle_{i,X})^2 : 0 < \frac{\theta(\mathbf{p})-\theta(\mathbf{q})}{2\pi} \le \frac{\lambda}{N(\Omega,X)}\right\},$$

$$\mathcal{R}_{i,X}'(\lambda) = \frac{1}{N_i'(X)} \#\left\{(\mathbf{p},\mathbf{q}) \in \mathscr{M}(\triangle_{i,X}')^2 : 0 < \frac{\theta(\mathbf{p})-\theta(\mathbf{q})}{2\pi} \le \frac{\lambda}{N(\Omega,X)}\right\},$$

$$\mathcal{R}_{i,X}^{(\epsilon)}(\lambda) = \frac{1}{N_i^{(\epsilon)}(X)} \#\left\{(\mathbf{p},\mathbf{q}) \in \mathscr{M}(\triangle_{i,X}^{(\epsilon)})^2 : 0 < \frac{\theta(\mathbf{p})-\theta(\mathbf{q})}{2\pi} \le \frac{\lambda}{N(\Omega,X)}\right\},$$

Since Ω is star-shaped with respect to the origin, the slice of Ω between the lines $y = x\tan\alpha_i$ and $y = x\tan\alpha_{i+1}$ contains $\triangle_i$, hence

$$\sum_{i=0}^{L-1} \frac{N_i(X)}{N(\Omega,X)} \mathcal{R}_{i,X}(\lambda) \le \mathcal{R}_{\Omega,X}(\lambda). \tag{4.5}$$

On the other hand, the slice of Ω between the lines $y = x\tan\alpha_i$ and $y = x\tan\alpha_{i+1}$ is contained in $\triangle_i'$, and the slice of Ω between the lines $y = x\tan(\alpha_i-\epsilon)$ and $y = x\tan(\alpha_i+\epsilon)$ is contained in $\triangle_i^{(\epsilon)}$. Since $\lim_{X\to\infty} N(\Omega,X) = \infty$, if X is sufficiently large, we obtain

$$\mathcal{R}_{\Omega,X}(\lambda) \le \sum_{i=0}^{L-1} \frac{N_i'(X)}{N(\Omega,X)} \mathcal{R}_{i,X}(\lambda) + \sum_{i=1}^{L-1} \frac{N_i^{(\epsilon)}(X)}{N(\Omega,X)} \mathcal{R}_{i,X}^{(\epsilon)}(\lambda). \tag{4.6}$$

By using (4.1) and (4.4) we have

$$\lim_{X\to\infty}\frac{N_i(X)}{N(\Omega,X)}=\frac{A(\triangle_i)}{A(\Omega)}=\frac{m_i^2(\tan\alpha_{i+1}-\tan\alpha_i)}{2A(\Omega)},$$

$$\lim_{X\to\infty}\frac{N_i'(X)}{N(\Omega,X)}=\frac{A(\triangle_i')}{A(\Omega)}=\frac{M_i^2(\tan\alpha_{i+1}-\tan\alpha_i)}{2A(\Omega)},$$

$$\lim_{X\to\infty}\frac{N_i^{(\epsilon)}(X)}{N(\Omega,X)}=\frac{A(\triangle_i^{(\epsilon)})}{A(\Omega)}=\frac{M_i'^2(\tan(\alpha_i+\epsilon)-\tan(\alpha_i-\epsilon))}{2A(\Omega)}.$$

Moreover, applying (4.3) for $\Omega=\triangle_i$ we obtain

$$\lim_{X\to\infty}\mathcal{R}_{i,X}(\lambda)=\frac{(\alpha_{i+1}-\alpha_i)\widetilde{\lambda}}{(\tan\alpha_{i+1}-\tan\alpha_i)^2}\int_{\tan\alpha_i}^{\tan\alpha_{i+1}}(1+t^2)\,dt,$$

where

$$\widetilde{\lambda}=\lim_{X\to\infty}\frac{2\pi\lambda N_i(X)}{(\alpha_{i+1}-\alpha_i)N(\Omega,X)}=\frac{2\pi\lambda A(\triangle_i)}{(\alpha_{i+1}-\alpha_i)A(\Omega)}.$$

By a simple calculation, we conclude that

$$\lim_{X\to\infty}\mathcal{R}_{i,X}(\lambda)=\frac{\pi\lambda m_i^2}{A(\Omega)}\left(1+\frac{\tan^2\alpha_{i+1}+\tan^2\alpha_i+\tan\alpha_{i+1}\tan\alpha_i}{3}\right).$$

Similarly for $\Omega=\triangle_i'$ and $\triangle_i^{(\epsilon)}$ respectively we obtain

$$\lim_{X\to\infty}\mathcal{R}_{i,X}'(\lambda)=\frac{\pi\lambda M_i^2}{A(\Omega)}\left(1+\frac{\tan^2\alpha_{i+1}+\tan^2\alpha_i+\tan\alpha_{i+1}\tan\alpha_i}{3}\right)$$

and

$$\lim_{X\to\infty}\mathcal{R}_{i,X}^{(\epsilon)}(\lambda)=\frac{\pi\lambda M_i'^2}{A(\Omega)}\left(1+\frac{\tan^2(\alpha_i+\epsilon)+\tan^2(\alpha_i-\epsilon)+\tan(\alpha_i+\epsilon)\tan(\alpha_i-\epsilon)}{3}\right).$$

It is clear from the above that

$$\lim_{\epsilon\to0^+}\lim_{X\to\infty}\sum_{i=1}^{L-1}\frac{N_i^{(\epsilon)}(X)}{N(\Omega,X)}\mathcal{R}_{i,X}^{(\epsilon)}(\lambda)=0.$$

Therefore by letting $X\to\infty$ and then $\epsilon\to0^+$ in (4.5) and (4.6) we obtain

$$\sum_{i=0}^{L-1}\frac{\pi\lambda m_i^4}{2A(\Omega)}(\tan^3\alpha_{i+1}-\tan^3\alpha_i)\le\liminf_{X\to\infty}\mathcal{R}_{\Omega,X}(\lambda)\le\limsup_{X\to\infty}\mathcal{R}_{\Omega,X}(\lambda)\le\sum_{i=0}^{L-1}\frac{\pi\lambda M_i^4}{2A(\Omega)}(\tan^3\alpha_{i+1}-\tan^3\alpha_i),$$

where $\alpha_i = i\pi/4L$, $m_i = x_\Omega(\xi_i)$, $M_i = x_\Omega(\eta_i)$, $\xi_i, \eta_i \in [\alpha_i, \alpha_{i+1}]$ for $0 \le i \le L-1$. The above inequalities hold true for any large positive integer L. By letting $L \to \infty$, and using polar coordinates $x_\Omega(\alpha) = \rho_\Omega(\alpha)\cos\alpha$, we conclude that the limit $\lim_{X\to\infty} \mathcal{R}_{\Omega,X}(\lambda)$ exists and is given by

$$\lim_{X\to\infty} \mathcal{R}_{\Omega,X}(\lambda) = \frac{\pi\lambda}{2A(\Omega)^2} \int_0^{\pi/4} \rho_\Omega(\theta)^4 \, d\theta.$$

Taking the derivative with respect to λ, we see that the limiting pair correlation function exists for the region Ω and is given by the desired formula. This completes the proof of Theorem 1.1 for $\Omega = \Omega_1$.

For a general Ω, noting that for $m > 0$ and $0 \le \alpha < \beta < 2\pi$ such that $\pm\pi/2 \notin [\alpha, \beta]$, the set $\mathscr{M}(T_{\alpha,\beta,mX})$ of lattice points from the triangle $T_{\alpha,\beta,mX}$ with x-coordinate a prime number corresponds naturally to the set $\widetilde{\mathscr{M}_{\mathbf{I}}}(mX)$ of general Farey fractions of order mX in the finite subinterval $\mathbf{I} = [\tan\alpha, \tan\beta] \subset \mathbb{R}^2$ with prime denominators, for which Theorem 2.2 holds true (see Remark after Theorem 2.2). In other words the limiting pair correlation function of $\widetilde{\mathscr{M}_{\mathbf{I}}}(mX)$ exists as $X \to \infty$ and is identically equal to 1. By using a similar argument to the case of $\Omega = \Omega_1$, we can see that Theorem 1.1 also holds for a general region Ω.

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Maosheng Xiong
Department of Mathematics
Hong Kong University
of Science and Technology
Clear Water Bay
Kowloon, Hong Kong, P.R. China
E-mail: mamsxiong@ust.hk

Alexandru Zaharescu Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
70700 Bucureşti, Romania
and
Department of Mathematics
University of Illinois at Urbana-Champaign
273 Altgeld Hall, MC-382
1409 W. Green Street
Urbana, IL 61801, U.S.A.
E-mail: zaharesc@math.uiuc.edu

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