ACTA ARITHMETICA
154.1 (2012)

Intermediate Diophantine exponents and parametric geometry of numbers

by

OLEG N. GERMAN (Moscow)

1. Introduction. Given a matrix

$$\Theta = \begin{pmatrix} \theta_{11} & \cdots & \theta_{1m} \\ \vdots & \ddots & \vdots \\ \theta_{n1} & \cdots & \theta_{nm} \end{pmatrix}, \quad \theta_{ij} \in \mathbb{R}, \quad n+m \geq 3,$$

consider the system of linear equations

$$\Theta \mathbf{x} = \mathbf{y} \tag{1.1}$$

with variables $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$. The classical measure of how well the space of solutions to this system can be approximated by integer points is defined as follows. Let $|\cdot|$ denote the sup-norm in the corresponding space.

DEFINITION 1.1. The supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities

$$|\mathbf{x}| \leq t, \quad |\Theta\mathbf{x} - \mathbf{y}| \leq t^{-\gamma} \tag{1.2}$$

has a nonzero solution in $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^m \oplus \mathbb{Z}^n$, is called the *regular* (resp. *uniform*) *Diophantine exponent* of Θ and is denoted by β_1 (resp. α_1).

This paper is the result of an attempt to generalize this concept to the case of approximating the space of solutions to (1.1) by p-dimensional rational subspaces of $\mathbb{R}^{m+n}$. Much work in this direction was done by W. Schmidt in [Sch1]. Later, in [L1], [BL], a corresponding definition was given by M. Laurent and Y. Bugeaud in the case when $m = 1$. Their definition enabled them to split the classical Khintchine transference principle into a chain of inequalities for intermediate exponents. However, the way we de-

2010 *Mathematics Subject Classification*: Primary 11J13, 11J82; Secondary 11H06.
Key words and phrases: Diophantine exponents, transference inequalities, consecutive minima.

DOI: 10.4064/aa154-1-5

fined α_1 and β_1 naturally suggests a generalization, which appears to be different from Laurent's:

Definition 1.2. The supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities

$$|\mathbf{x}| \leq t, \qquad |\Theta\mathbf{x} - \mathbf{y}| \leq t^{-\gamma} \tag{1.3}$$

has p solutions $\mathbf{z}_i = (\mathbf{x}_i, \mathbf{y}_i) \in \mathbb{Z}^m \oplus \mathbb{Z}^n$, $i = 1, \ldots, p$, linearly independent over $\mathbb{Z}$, is called the *pth regular* (resp. *uniform*) *Diophantine exponent of the first type* of Θ and is denoted by β_p (resp. α_p).

In Section 2 we propose a definition of intermediate exponents of the second type, which is consistent with Laurent's. In Sections 3, 4 we formulate our main results on these quantities. Sections 5, 6 are devoted to the exponents naturally emerging in parametric geometry of numbers developed by W. Schmidt and L. Summerer in [SchS]. Those exponents are closely connected to the Diophantine exponents, and in Sections 7, 8 we describe this connection. It allows reformulating our main results in terms of Schmidt–Summerer's exponents, which is accomplished in Section 9. Finally, in Section 10, we use this point of view to prove the theorems given in Section 4.

It should be noticed that all our "splitting" results are obtained for the exponents of the second type. It is an interesting question whether anything of this kind can be done with the exponents of the first type.

2. Laurent's exponents and their generalization. Set $d = m + n$. Denote by $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_d$ the columns of the matrix

$$\begin{pmatrix} E_m & -\Theta^{\mathsf{T}} \\ \Theta & E_n \end{pmatrix},$$

where E_m and E_n are the corresponding unit matrices and Θ^{T} is the transpose of Θ. Clearly, $\mathcal{L} = \operatorname{span}_{\mathbb{R}}(\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m)$ is the space of solutions to the system (1.1), and $\mathcal{L}^{\perp} = \operatorname{span}_{\mathbb{R}}(\boldsymbol{\ell}_{m+1}, \ldots, \boldsymbol{\ell}_d)$. Denote also by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the columns of the $d \times d$ unit matrix E_d.

The following definition is a slight modification of Laurent's.

Definition 2.1. Let $m = 1$. The supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities

$$|\mathbf{Z}| \leq t, \qquad |\boldsymbol{\ell}_1 \wedge \mathbf{Z}| \leq t^{-\gamma} \tag{2.1}$$

has a nonzero solution in $\mathbf{Z} \in \bigwedge^p(\mathbb{Z}^d)$ is called the *pth regular* (resp. *uniform*) *Diophantine exponent of the second type* of Θ and is denoted by $\mathfrak{b}_p$ (resp. $\mathfrak{a}_p$).

Here $\mathbf{Z} \in \bigwedge^p(\mathbb{R}^d)$, $\boldsymbol{\ell}_1 \wedge \mathbf{Z} \in \bigwedge^{p+1}(\mathbb{R}^d)$ and for each q we consider $\bigwedge^q(\mathbb{R}^d)$ as a $\binom{d}{q}$-dimensional Euclidean space with the orthonormal basis consisting of the multivectors

$$\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_q}, \qquad 1 \le i_1 < \cdots < i_q \le d,$$

and denote by $|\cdot|$ the sup-norm with respect to this basis.

Laurent denoted the exponents $\mathfrak{b}_p$, $\mathfrak{a}_p$ as ω_{p-1}, $\hat{\omega}_{p-1}$, respectively, and showed that for $p = 1$ they coincide with β_1, α_1. He also noticed that one does not have to require $\mathbf{Z}$ to be decomposable in Definition 2.1, which essentially simplifies working in $\bigwedge^p(\mathbb{R}^d)$.

In order to generalize Definition 2.1 let us set for each $\sigma = \{i_1, \ldots, i_k\}$, $1 \le i_1 < \cdots < i_k \le d$,

$$\mathbf{L}_\sigma = \boldsymbol{\ell}_{i_1} \wedge \cdots \wedge \boldsymbol{\ell}_{i_k}, \tag{2.2}$$

denote by $\mathcal{J}_k$ the set of all k-element subsets of $\{1, \ldots, m\}$, $k = 0, \ldots, m$, and set $\mathbf{L}_\emptyset = 1$.

Let us also set $k_0 = \max(0, m - p)$.

Definition 2.2. The supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities

$$\max_{\sigma \in \mathcal{J}_k} |\mathbf{L}_\sigma \wedge \mathbf{Z}| \le t^{1-(k-k_0)(1+\gamma)}, \qquad k = 0, \ldots, m, \tag{2.3}$$

has a nonzero solution in $\mathbf{Z} \in \bigwedge^p(\mathbb{Z}^d)$ is called the *pth regular* (resp. *uniform*) *Diophantine exponent of the second type* of Θ and is denoted by $\mathfrak{b}_p$ (resp. $\mathfrak{a}_p$).

We have intended to make Definition 2.2 look as simple as possible. However, it will be more convenient to work with in the multilinear algebra setting after it is slightly reformulated. To give the desired reformulation let us set for each $\sigma = \{i_1, \ldots, i_k\}$, $1 \le i_1 < \cdots < i_k \le d$,

$$\mathbf{E}_\sigma = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}, \tag{2.4}$$

denote by $\mathcal{J}'_k$ the set of all k-element subsets of $\{m+1, \ldots, d\}$, $k = 0, \ldots, n$, and set $\mathbf{E}_\emptyset = 1$.

Set also $k_1 = \min(m, d - p)$.

Proposition 2.3. *The inequalities* (2.3) *can be replaced by*

$$\max_{\substack{\sigma \in \mathcal{J}_k \\ \sigma' \in \mathcal{J}'_{d-p-k}}} |\mathbf{L}_\sigma \wedge \mathbf{E}_{\sigma'} \wedge \mathbf{Z}| \le t^{1-(k-k_0)(1+\gamma)}, \qquad k = k_0, \ldots, k_1. \tag{2.5}$$

Proof. The inequality (2.3) is trivial for $k > k_1$. Suppose that $k \le k_1$ and set $q = k + p$. Consider a $d \times q$ matrix M with columns $\mathbf{m}_1, \ldots, \mathbf{m}_q \in \mathbb{R}^d$ and set $\mathbf{M} = \mathbf{m}_1 \wedge \cdots \wedge \mathbf{m}_q$. If $\sigma' = \{i_1, \ldots, i_{d-q}\} \in \mathcal{J}'_{d-q}$, then

$$|\mathbf{M} \wedge \mathbf{E}_{\sigma'}| = |\det(\mathbf{m}_1, \ldots, \mathbf{m}_q, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{d-q}})|$$

is up to sign the $(q\times q)$-determinant involving the jth rows of M with $j\notin\sigma'$. The components of $\mathbf{M}$ are up to sign exactly these determinants, so

$$|\mathbf{M}| = \max_{\sigma'\in\mathcal{J}'_{d-q}} |\mathbf{M}\wedge\mathbf{E}_{\sigma'}|. \tag{2.6}$$

By linearity, (2.6) remains valid if $\mathbf{M}$ is replaced by any element of $\bigwedge^q(\mathbb{R}^d)$, in particular, by $\mathbf{L}_\sigma\wedge\mathbf{Z}$. Therefore,

$$|\mathbf{L}_\sigma\wedge\mathbf{Z}| = \max_{\sigma'\in\mathcal{J}'_{d-p-k}} |\mathbf{L}_\sigma\wedge\mathbf{Z}\wedge\mathbf{E}_{\sigma'}|,$$

which implies the desired statement. ∎

For $p=1$ Definition 2.2 coincides with Definition 1.1, i.e. $\beta_1=\mathfrak{b}_1$ and $\alpha_1=\mathfrak{a}_1$. This is seen from the following

Proposition 2.4. *The quantity β_1 (resp. α_1) equals the supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities*

$$|\mathbf{z}|\le t,\qquad |\mathbf{L}\wedge\mathbf{z}|\le t^{-\gamma}, \tag{2.7}$$

where $\mathbf{L}=\boldsymbol{\ell}_1\wedge\cdots\wedge\boldsymbol{\ell}_m$, has a nonzero solution in $\mathbf{z}\in\mathbb{Z}^d$.

Proof. The parallelepiped in $\mathbb{R}^d$ defined by (1.2) can be written as

$$M_\gamma(t)=\Big\{\mathbf{z}\in\mathbb{R}^d \,\Big|\, \max_{1\le j\le m}|\langle\mathbf{e}_j,\mathbf{z}\rangle|\le t,\ \max_{1\le i\le n}|\langle\boldsymbol{\ell}_{m+i},\mathbf{z}\rangle|\le t^{-\gamma}\Big\},$$

where $\langle\,\cdot,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^d$.

The vectors $\boldsymbol{\ell}_{m+1},\ldots,\boldsymbol{\ell}_d$ form a basis of the orthogonal complement of $\mathcal{L}$. Therefore, since the Euclidean norm of $\mathbf{L}\wedge\mathbf{z}$ equals the $(m+1)$-dimensional volume of the parallelepiped spanned by $\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m,\mathbf{z}$, we have

$$|\mathbf{L}\wedge\mathbf{z}|\asymp\max_{1\le i\le n}|\langle\boldsymbol{\ell}_{m+i},\mathbf{z}\rangle|,$$

with the implied constant depending only on Θ. Moreover,

$$|\mathbf{z}|\asymp\max\Big(\max_{1\le j\le m}|\langle\mathbf{e}_j,\mathbf{z}\rangle|,\ \max_{1\le i\le n}|\langle\boldsymbol{\ell}_{m+i},\mathbf{z}\rangle|\Big)$$

where the implied constant again depends only on Θ.

Hence there is a positive constant c, depending only on Θ, such that the set $M'_\gamma(t)$ defined by (2.7) satisfies

$$c^{-1}M_\gamma(t)\subseteq M'_\gamma(t)\subseteq cM_\gamma(t),$$

at least for $t\ge1$, $\gamma\ge0$, which immediately implies the desired result. ∎

3. Known transference inequalities. The *transference principle* connects the problem of approximating the space of solutions to (1.1) to the analogous problem for the system

$$\Theta^{\intercal}\mathbf{y}=\mathbf{x}. \tag{3.1}$$

Let us denote by β_p^*, α_p^*, $\mathfrak{b}_p^*$, $\mathfrak{a}_p^*$ the intermediate Diophantine exponents corresponding to $\Theta^\intercal$.

The classical transference inequalities estimating $\mathfrak{b}_1$ in terms of $\mathfrak{b}_1^*$, and $\mathfrak{a}_1$ in terms of $\mathfrak{a}_1^*$, belong to A. Ya. Khintchine, V. Jarník, F. Dyson, and A. Apfelbeck. We recall that $\beta_1 = \mathfrak{b}_1$, $\alpha_1 = \mathfrak{a}_1$, $\beta_1^* = \mathfrak{b}_1^*$, $\alpha_1^* = \mathfrak{a}_1^*$, as shown at the end of the previous section.

3.1. Regular exponents. In [Kh] A. Ya. Khintchine proved for $m = 1$ his famous transference inequalities

$$\mathfrak{b}_1^* \geq n\mathfrak{b}_1 + n - 1, \qquad \mathfrak{b}_1 \geq \frac{\mathfrak{b}_1^*}{(n-1)\mathfrak{b}_1^* + n}, \tag{3.2}$$

which were generalized later by F. Dyson [D], who proved that for arbitrary n, m,

$$\mathfrak{b}_1^* \geq \frac{n\mathfrak{b}_1 + n - 1}{(m-1)\mathfrak{b}_1 + m}. \tag{3.3}$$

While (3.2) cannot be improved (see [J1], [J2]) if only $\mathfrak{b}_1$ and $\mathfrak{b}_1^*$ are considered, stronger inequalities can be obtained if $\mathfrak{a}_1$ and $\mathfrak{a}_1^*$ are also taken into account. The corresponding result for $m = 1$ belongs to M. Laurent and Y. Bugeaud (see [L2], [BL]). They proved that if the system (1.1) has no non-zero integer solutions, then

$$\frac{(\mathfrak{a}_1^* - 1)\mathfrak{b}_1^*}{((n-2)\mathfrak{a}_1^* + 1)\mathfrak{b}_1^* + (n-1)\mathfrak{a}_1^*} \leq \mathfrak{b}_1 \leq \frac{(1-\mathfrak{a}_1)\mathfrak{b}_1^* - n + 2 - \mathfrak{a}_1}{n-1}. \tag{3.4}$$

The above inequalities were generalized by the author in [G], where it was proved for arbitrary n, m that if the space of integer solutions of (1.1) is not a one-dimensional lattice, then along with (3.3) we have

$$\mathfrak{b}_1^* \geq \frac{(n-1)(1+\mathfrak{b}_1) - (1-\mathfrak{a}_1)}{(m-1)(1+\mathfrak{b}_1) + (1-\mathfrak{a}_1)}, \tag{3.5}$$

$$\mathfrak{b}_1^* \geq \frac{(n-1)(1+\mathfrak{b}_1^{-1}) - (\mathfrak{a}_1^{-1} - 1)}{(m-1)(1+\mathfrak{b}_1^{-1}) + (\mathfrak{a}_1^{-1} - 1)}, \tag{3.6}$$

with (3.5) stronger than (3.6) if and only if $\mathfrak{a}_1 < 1$.

3.2. Uniform exponents. V. Jarník and A. Apfelbeck proved literal analogues of (3.2) and (3.3) for the uniform exponents, i.e. with $\mathfrak{b}_1$, $\mathfrak{b}_1^*$ replaced by $\mathfrak{a}_1$, $\mathfrak{a}_1^*$, respectively (see [J3], [A]). They also obtained some stronger inequalities of a more cumbersome appearance. Among them, lonely in its elegance, stands the *equality*

$$\mathfrak{a}_1^{-1} + \mathfrak{a}_1^* = 1 \tag{3.7}$$

proved by Jarník for $n = 1$, $m = 2$. The results of Jarník and Apfelbeck were improved by the author in [G], where it was shown that for arbitrary n, m we have

$$\mathfrak{a}_1^* \geq \begin{cases} \dfrac{n-1}{m-\mathfrak{a}_1} & \text{if } \mathfrak{a}_1 \leq 1, \\ \dfrac{n-\mathfrak{a}_1^{-1}}{m-1} & \text{if } \mathfrak{a}_1 \geq 1. \end{cases} \tag{3.8}$$

3.3. Khintchine's inequalities split. Laurent and Bugeaud used the exponents $\mathfrak{b}_p$ to split (3.2) into a chain of inequalities relating $\mathfrak{b}_p$ to $\mathfrak{b}_{p+1}$. Namely, they proved that for $m = 1$ we have $\mathfrak{b}_1^* = \mathfrak{b}_n$ and

$$\mathfrak{b}_{p+1} \geq \frac{(n-p+1)\mathfrak{b}_p + 1}{n-p}, \qquad \mathfrak{b}_p \geq \frac{p\mathfrak{b}_{p+1}}{\mathfrak{b}_{p+1} + p + 1}, \qquad p = 1, \ldots, n-1. \tag{3.9}$$

Moreover, they proved for $m = 1$ that if the system (1.1) has no non-zero integer solutions, then $\mathfrak{a}_1^* = \mathfrak{a}_n$ and

$$\mathfrak{b}_2 \geq \frac{\mathfrak{b}_1 + \mathfrak{a}_1}{1 - \mathfrak{a}_1}, \qquad \mathfrak{b}_{n-1} \geq \frac{1 - \mathfrak{a}_n^{-1}}{\mathfrak{b}_n^{-1} + \mathfrak{a}_n^{-1}}, \tag{3.10}$$

which, combined with (3.9), gives (3.4).

4. Main results for intermediate Diophantine exponents. In this paper we generalize (3.9) and its analogue for the uniform exponents to the case of arbitrary n, m. We show (see Proposition 8.6 in Section 8) that

$$\mathfrak{b}_p^* = \mathfrak{b}_{d-p}, \qquad \mathfrak{a}_p^* = \mathfrak{a}_{d-p}, \qquad p = 1, \ldots, d-1, \tag{4.1}$$

and prove

THEOREM 4.1. *For each $p = 1, \ldots, d-2$ the following statements hold: If $p \geq m$, then*

$$(d-p-1)(1+\mathfrak{b}_{p+1}) \geq (d-p)(1+\mathfrak{b}_p), \tag{4.2}$$

$$(d-p-1)(1+\mathfrak{a}_{p+1}) \geq (d-p)(1+\mathfrak{a}_p). \tag{4.3}$$

If $p \leq m-1$, then

$$(d-p-1)(1+\mathfrak{b}_p)^{-1} \geq (d-p)(1+\mathfrak{b}_{p+1})^{-1} - n, \tag{4.4}$$

$$(d-p-1)(1+\mathfrak{a}_p)^{-1} \geq (d-p)(1+\mathfrak{a}_{p+1})^{-1} - n. \tag{4.5}$$

The second result of the current paper generalizes (3.10). We prove

THEOREM 4.2. *Suppose that the space of integer solutions of* (1.1) *is not a one-dimensional lattice. Then for $m = 1$ we have*

$$\mathfrak{b}_2 \geq \frac{\mathfrak{b}_1 + \mathfrak{a}_1}{1 - \mathfrak{a}_1}, \tag{4.6}$$

and for $m \geq 2$ we have

$$\mathfrak{b}_2 \geq \begin{cases} \dfrac{\mathfrak{a}_1 - 1}{2 + \mathfrak{b}_1 - \mathfrak{a}_1} & \text{if } \mathfrak{a}_1 \neq \infty, \\ \dfrac{1 - \mathfrak{a}_1^{-1}}{\mathfrak{b}_1^{-1} + \mathfrak{a}_1^{-1}}. \end{cases} \tag{4.7}$$

The inequality (4.6) is exactly the first inequality of (3.10). The second inequality of (4.7), in view of (4.1), gives the second inequality of (3.10).

It follows from Theorem 4.1 that for $m \geq 2$,

$$(d-2)(1+\mathfrak{b}_{d-1})^{-1} \leq (1+\mathfrak{b}_2)^{-1} + m - 2. \tag{4.8}$$

Combining this inequality with (4.7) we get (3.5) and (3.6) in case $m \geq 2$.

The third result of this paper splits the inequalities (3.8):

THEOREM 4.3. *For $m = 1$ we have*

$$\mathfrak{a}_2 \geq (1-\mathfrak{a}_1)^{-1} - \frac{n-2}{n-1}. \tag{4.9}$$

For $m \geq 2$ we have

$$\mathfrak{a}_2 \geq \begin{cases} \dfrac{n-1}{-n-(d-2)(1-\mathfrak{a}_1)^{-1}} & \text{if } \mathfrak{a}_1 \leq 1, \\ \dfrac{m-1}{n+(d-2)(\mathfrak{a}_1-1)^{-1}} & \text{if } \mathfrak{a}_1 \geq 1. \end{cases} \tag{4.10}$$

Let us show that Theorem 4.3 splits (3.8) the very same way Theorem 4.2 splits (3.5) and (3.6). It follows from Theorem 4.1 that for $m = 1$,

$$1 + \mathfrak{a}_n \geq (n-1)(1+\mathfrak{a}_2), \tag{4.11}$$

and for $m \geq 2$,

$$(d-2)(1+\mathfrak{a}_{d-1})^{-1} \leq (1+\mathfrak{a}_2)^{-1} + m - 2. \tag{4.12}$$

Combining (4.12) with (4.10), we get (3.8) for $m \geq 2$. As for $m = 1$, we always have $\mathfrak{a}_1 \leq 1$ in this case, so (4.9) and (4.11) indeed give (3.8) with $m = 1$.

5. Schmidt–Summerer's exponents. Let Λ be a unimodular d-dimensional lattice in $\mathbb{R}^d$. Denote by $\mathcal{B}_\infty^d$ the unit ball in sup-norm, i.e. the cube with vertices at the points $(\pm 1, \ldots, \pm 1)$. For each d-tuple $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d$ denote by $D_{\boldsymbol{\tau}}$ the diagonal $d \times d$ matrix with $e^{\tau_1}, \ldots, e^{\tau_d}$ on the main diagonal. Let also $\lambda_p(M)$ denote the pth successive minimum of a compact symmetric convex body $M \subset \mathbb{R}^d$ (centered at the origin) with respect to the lattice Λ.

Suppose we have a path $\mathfrak{T}$ in $\mathbb{R}^d$ defined as $\boldsymbol{\tau} = \boldsymbol{\tau}(s)$, $s \in \mathbb{R}_+$, such that

$$\tau_1(s) + \cdots + \tau_d(s) = 0 \quad \text{for all } s. \tag{5.1}$$

In our further applications to Diophantine approximation we shall confine ourselves to a path that is a ray with endpoint at the origin and all the functions $\tau_1(s),\dots,\tau_d(s)$ being linear. However, in this section, as well as in the next one, all the definitions and statements are given for arbitrary paths and lattices.

Set $\mathcal{B}(s)=D_{\boldsymbol{\tau}(s)}\mathcal{B}^d_\infty$. For each $p=1,\dots,d$ consider the functions

$$\psi_p(\Lambda,\mathfrak{T},s)=\frac{\ln(\lambda_p(\mathcal{B}(s)))}{s},\qquad \Psi_p(\Lambda,\mathfrak{T},s)=\sum_{i=1}^{p}\psi_i(\Lambda,\mathfrak{T},s).$$

Definition 5.1. We call the quantities

$$\underline{\psi}_p(\Lambda,\mathfrak{T})=\liminf_{s\to+\infty}\psi_p(\Lambda,\mathfrak{T},s),\qquad \overline{\psi}_p(\Lambda,\mathfrak{T})=\limsup_{s\to+\infty}\psi_p(\Lambda,\mathfrak{T},s)$$

the pth lower and *upper Schmidt–Summerer exponents of the first type*, respectively.

Definition 5.2. We call the quantities

$$\underline{\Psi}_p(\Lambda,\mathfrak{T})=\liminf_{s\to+\infty}\Psi_p(\Lambda,\mathfrak{T},s),\qquad \overline{\Psi}_p(\Lambda,\mathfrak{T})=\limsup_{s\to+\infty}\Psi_p(\Lambda,\mathfrak{T},s)$$

the *pth lower* and *upper Schmidt–Summerer exponents of the second type*, respectively.

Sometimes, when it is clear from the context which lattice and which path are under consideration, we shall write simply $\psi_p(s)$, $\Psi_p(s)$, $\underline{\psi}_p$, $\overline{\psi}_p$, $\underline{\Psi}_p$, and $\overline{\Psi}_p$.

The following proposition and its corollaries generalize some of the observations made in [SchS] and [BL].

Proposition 5.3. *For any Λ and $\mathfrak{T}$ we have*

$$0\leq-\Psi_d(s)=O(s^{-1}). \tag{5.2}$$

In particular,

$$\underline{\Psi}_d=\overline{\Psi}_d=0. \tag{5.3}$$

Proof. Due to (5.1) the volumes of all the parallelepipeds $\mathcal{B}(s)$ are 2^d, so by Minkowski's second theorem we have

$$\frac{1}{d!}\leq\prod_{i=1}^{d}\lambda_i(\mathcal{B}(s))\leq 1.$$

Hence

$$-\frac{\ln(d!)}{s}\leq\sum_{i=1}^{d}\psi_i(s)\leq 0,$$

which immediately implies (5.2). ∎

COROLLARY 5.4. *For every p with $1 \le p \le d-2$ and every $s > 0$ we have*

$$\frac{p+1}{p}\Psi_p(s) \le \Psi_{p+1}(s) \le \frac{d-p-1}{d-p}\Psi_p(s). \tag{5.4}$$

Proof. In view of (5.2), it follows from the inequalities $\psi_i(s) \le \psi_{i+1}(s)$, $i = 1, \dots, d-1$, that

$$\frac{1}{p}\sum_{i=1}^{p}\psi_i(s) \le \psi_{p+1}(s) \le \frac{-1}{d-p}\sum_{i=1}^{p}\psi_i(s),$$

which immediately implies (5.4). ■

Taking the lim inf and the lim sup of all terms in (5.4), we get

COROLLARY 5.5. *For any Λ and $\mathfrak{T}$ and any p with $1 \le p \le d-2$ we have*

$$\frac{p+1}{p}\underline{\Psi}_p \le \underline{\Psi}_{p+1} \le \frac{d-p-1}{d-p}\underline{\Psi}_p \quad \textit{and} \quad \frac{p+1}{p}\overline{\Psi}_p \le \overline{\Psi}_{p+1} \le \frac{d-p-1}{d-p}\overline{\Psi}_p. \tag{5.5}$$

Next, applying (5.5) we get

COROLLARY 5.6. *For any Λ and $\mathfrak{T}$ we have*

$$(d-1)\underline{\Psi}_1 \le \underline{\Psi}_{d-1} \le \frac{\underline{\Psi}_1}{d-1} \quad \textit{and} \quad (d-1)\overline{\Psi}_1 \le \overline{\Psi}_{d-1} \le \frac{\overline{\Psi}_1}{d-1}. \tag{5.6}$$

Another simple corollary to Proposition 5.3 is the following statement:

COROLLARY 5.7. *For any Λ and $\mathfrak{T}$ we have*

$$\underline{\Psi}_{d-1} = -\overline{\psi}_d \quad \textit{and} \quad \overline{\Psi}_{d-1} = -\underline{\psi}_d. \tag{5.7}$$

As we shall see later, the first inequalities of (5.6) generalize Khintchine's and Dyson's transference inequalities.

6. Schmidt–Summerer's exponents of the second type from the point of view of multilinear algebra. As before, let us consider $\bigwedge^p(\mathbb{R}^d)$ as the $\binom{d}{p}$-dimensional Euclidean space with the orthonormal basis consisting of the multivectors

$$\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p}, \qquad 1 \le i_1 < \dots < i_p \le d.$$

Let us order the set of p-element subsets of $\{1, \dots, d\}$ lexicographically and denote the jth subset by σ_j. To each d-tuple $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ let us associate the r-tuple

$$\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \dots, \widehat{\tau}_r), \qquad \widehat{\tau}_j = \sum_{i \in \sigma_j} \tau_i, \qquad r = \binom{d}{p}. \tag{6.1}$$

Thus, a path $\mathfrak{T} : s \to \boldsymbol{\tau}(s)$ leads (6.1) to the path $\widehat{\mathfrak{T}} : s \to \widehat{\boldsymbol{\tau}}(s)$ satisfying the condition

$$\widehat{\tau}_1(s) + \ldots + \widehat{\tau}_r(s) = 0 \quad \text{for all } s.$$

Finally, to a given lattice $\Lambda \subset \mathbb{R}^d$, we associate the lattice $\widehat{\Lambda} = \bigwedge^p(\Lambda)$ consisting of all linear combinations with integer coefficients of multivectors $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_p$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \Lambda$.

Proposition 6.1. *For any Λ and $\mathfrak{T}$ we have*

$$\underline{\Psi}_p(\Lambda, \mathfrak{T}) = \underline{\Psi}_1(\widehat{\Lambda}, \widehat{\mathfrak{T}}) = \underline{\psi}_1(\widehat{\Lambda}, \widehat{\mathfrak{T}}) \quad \text{and} \quad \overline{\Psi}_p(\Lambda, \mathfrak{T}) = \overline{\Psi}_1(\widehat{\Lambda}, \widehat{\mathfrak{T}}) = \overline{\psi}_1(\widehat{\Lambda}, \widehat{\mathfrak{T}}).$$

Proof. Let us denote by $\lambda_i(M)$ the ith successive minimum of a body M with respect to Λ if $M \subset \mathbb{R}^d$ and with respect to $\widehat{\Lambda}$ if $M \subset \bigwedge^p(\mathbb{R}^d)$.

The matrix $D_{\widehat{\boldsymbol{\tau}}}$ is the pth compound of $D_{\boldsymbol{\tau}}$:

$$D_{\widehat{\boldsymbol{\tau}}} = D_{\boldsymbol{\tau}}^{(p)}.$$

This means that $D_{\widehat{\boldsymbol{\tau}}}\mathcal{B}_\infty^r$ is comparable to Mahler's pth compound convex body of $D_{\boldsymbol{\tau}}\mathcal{B}_\infty^d$ (see [M]), i.e. there is a positive constant c, depending only on d, such that

$$c^{-1} D_{\widehat{\boldsymbol{\tau}}}\mathcal{B}_\infty^r \subset [D_{\boldsymbol{\tau}}\mathcal{B}_\infty^d]^{(p)} \subset c D_{\widehat{\boldsymbol{\tau}}}\mathcal{B}_\infty^r. \tag{6.2}$$

In [Sch2] the set $D_{\widehat{\boldsymbol{\tau}}}\mathcal{B}_\infty^r$ is called the pth *pseudo-compound parallelepiped* for $D_{\boldsymbol{\tau}}\mathcal{B}_\infty^d$.

It follows from Mahler's theory of compound bodies that

$$\lambda_1([D_{\boldsymbol{\tau}}\mathcal{B}_\infty^d]^{(p)}) \asymp \prod_{i=1}^{p} \lambda_i(D_{\boldsymbol{\tau}}\mathcal{B}_\infty^d) \tag{6.3}$$

with the implied constants depending only on d. Combining (6.2) and (6.3) we get

$$\ln(\lambda_1(D_{\widehat{\boldsymbol{\tau}}(s)}\mathcal{B}_\infty^r)) = \sum_{i=1}^{p} \ln(\lambda_i(D_{\boldsymbol{\tau}(s)}\mathcal{B}_\infty^d)) + O(1),$$

whence

$$\psi_1(\widehat{\Lambda}, \widehat{\mathfrak{T}}, s) = \sum_{i=1}^{p} \psi_i(\Lambda, \mathfrak{T}, s) + o(1).$$

It remains to take the lim inf and the lim sup of both sides as $s \to \infty$. ∎

7. Diophantine exponents in terms of Schmidt–Summerer's exponents. Let $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_d$, $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be as in Section 2. Set

$$T = \begin{pmatrix} E_m & 0 \\ \Theta & E_n \end{pmatrix}.$$

Then

$$(T^{-1})^{\intercal} = \begin{pmatrix} E_m & -\Theta^{\intercal} \\ 0 & E_n \end{pmatrix},$$

so the bases $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_m, \mathbf{e}_{m+1}, \ldots, \mathbf{e}_d$ and $\mathbf{e}_1, \ldots, \mathbf{e}_m, \boldsymbol{\ell}_{m+1}, \ldots, \boldsymbol{\ell}_d$ are dual to each other.

Let us specify a lattice Λ and a path $\mathfrak{T}$ as follows. Set

(7.1)
$$\Lambda = T^{-1}\mathbb{Z}^d = \{(\langle \mathbf{e}_1, \mathbf{z}\rangle, \ldots, \langle \mathbf{e}_m, \mathbf{z}\rangle, \langle \boldsymbol{\ell}_{m+1}, \mathbf{z}\rangle, \ldots, \langle \boldsymbol{\ell}_d, \mathbf{z}\rangle\Big)^{\intercal} \in \mathbb{R}^d \mid \mathbf{z} \in \mathbb{Z}^d\}$$

and define $\mathfrak{T} : s \mapsto \boldsymbol{\tau}(s)$ by

$$\tau_1(s) = \cdots = \tau_m(s) = s, \quad \tau_{m+1}(s) = \cdots = \tau_d(s) = -ms/n. \tag{7.2}$$

Schmidt–Summerer's exponents $\underline{\psi}_p$, $\overline{\psi}_p$ corresponding to such Λ and $\mathfrak{T}$ and the exponents β_p, α_p are but two different points of view at the same phenomenon. The same can be said about $\underline{\Psi}_p$, $\overline{\Psi}_p$ and $\mathfrak{b}_p$, $\mathfrak{a}_p$. This manifests itself in the following two propositions.

Proposition 7.1. *We have*

$$(1 + \beta_p)(1 + \underline{\psi}_p) = (1 + \alpha_p)(1 + \overline{\psi}_p) = d/n. \tag{7.3}$$

Proof. The parallelepiped in $\mathbb{R}^d$ defined by (1.2) can be written as

$$M_\gamma(t) = \Big\{\mathbf{z} \in \mathbb{R}^d \;\Big|\; \max_{1\le j\le m} |\langle \mathbf{e}_j, \mathbf{z}\rangle| \le t,\ \max_{1\le i\le n} |\langle \boldsymbol{\ell}_{m+i}, \mathbf{z}\rangle| \le t^{-\gamma}\Big\},$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the inner product in $\mathbb{R}^d$.

Therefore, β_p (resp. α_p) equals the supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the parallelepiped $M_\gamma(t)$ contains p linearly independent integer points.

Hence, considering the parallelepipeds

(7.4)
$$P_\gamma(t) = T^{-1}M_\gamma(t) = \Big\{\mathbf{z} \in \mathbb{R}^d \;\Big|\; \max_{1\le j\le m} |\langle \mathbf{e}_j, \mathbf{z}\rangle| \le t,\ \max_{1\le i\le n} |\langle \mathbf{e}_{m+i}, \mathbf{z}\rangle| \le t^{-\gamma}\Big\},$$

we see that

(7.5)
$$\beta_p = \limsup_{t\to+\infty}\{\gamma \in \mathbb{R} \mid \lambda_p(P_\gamma(t)) = 1\}, \quad \alpha_p = \liminf_{t\to+\infty}\{\gamma \in \mathbb{R} \mid \lambda_p(P_\gamma(t)) = 1\},$$

where $\lambda_p(P_\gamma(t))$ is the pth minimum of $P_\gamma(t)$ with respect to Λ.

But $P_{m/n}(t) = D_{\boldsymbol{\tau}(\ln t)}\mathcal{B}_\infty^d$, so

(7.6)
$$\underline{\psi}_p(\Lambda, \mathfrak{T}) = \liminf_{t\to+\infty} \frac{\ln(\lambda_p(P_{m/n}(t)))}{\ln t}, \quad \overline{\psi}_p(\Lambda, \mathfrak{T}) = \limsup_{t\to+\infty} \frac{\ln(\lambda_p(P_{m/n}(t)))}{\ln t}.$$

A simple calculation shows that

$$P_\gamma(t) = t^{\frac{m-n\gamma}{d}} P_{m/n}(t^{\frac{n+n\gamma}{d}}),$$

i.e.

$$\lambda_p(P_\gamma(t)) = (t')^{\frac{-m+n\gamma}{n+n\gamma}} \lambda_p(P_{m/n}(t'))$$

with $t' = t^{(n+n\gamma)/d}$. Therefore, the equality

$$\lambda_p(P_\gamma(t)) = 1$$

holds if and only if

$$1 - \frac{d}{n+n\gamma} + \frac{\ln(\lambda_p(P_{m/n}(t')))}{\ln t'} = 0.$$

Hence, in view of (7.5), (7.6), we get

$$\beta_p = \limsup_{t\to+\infty} \left\{ \frac{d}{n}\left(1 + \frac{\ln(\lambda_p(P_{m/n}(t)))}{\ln t}\right)^{-1} - 1 \right\} = \frac{d}{n}(1+\underline{\psi}_p)^{-1} - 1,$$

$$\alpha_p = \liminf_{t\to+\infty} \left\{ \frac{d}{n}\left(1 + \frac{\ln(\lambda_p(P_{m/n}(t)))}{\ln t}\right)^{-1} - 1 \right\} = \frac{d}{n}(1+\overline{\psi}_p)^{-1} - 1,$$

which immediately implies (7.3). ∎

Proposition 7.2. *Set* $\varkappa_p = \min\bigl(p, \frac{m}{n}(d-p)\bigr)$. *Then*

$$(1+\mathfrak{b}_p)(\varkappa_p + \underline{\Psi}_p) = (1+\mathfrak{a}_p)(\varkappa_p + \overline{\Psi}_p) = d/n. \tag{7.7}$$

Proof. Let $\mathbf{L}_\sigma$, $\mathbf{E}_\sigma$, $\mathcal{J}_k$, $\mathcal{J}'_k$ be as in Section 2.

Since $T^{-1}\boldsymbol{\ell}_i = \mathbf{e}_i$ and $T^{-1}\mathbf{e}_j = \mathbf{e}_j$ for $1 \le i \le m$ and $m+1 \le j \le d$, we have

$$(T^{-1})^{(k+k')}(\mathbf{L}_\sigma \wedge \mathbf{E}_{\sigma'}) = \mathbf{E}_\sigma \wedge \mathbf{E}_{\sigma'} \quad \text{for each } \sigma \in \mathcal{J}_k,\ \sigma' \in \mathcal{J}'_{k'}, \tag{7.8}$$

where $(T^{-1})^{(k+k')}$ is the $(k+k')$th compound of T^{-1}. Furthermore, since $\Lambda = T^{-1}\mathbb{Z}^d$, we have

$$\widehat{\Lambda} = \textstyle\bigwedge^p(\Lambda) = (T^{-1})^{(p)}(\bigwedge^p(\mathbb{Z}^d)). \tag{7.9}$$

Hence for each $\mathbf{Z} \in \bigwedge^p(\mathbb{Z}^d)$ and each $\sigma \in \mathcal{J}_k$, $\sigma' \in \mathcal{J}'_{d-p-k}$ (with $k_0 \le k \le k_1$) we get

$$|\mathbf{L}_\sigma \wedge \mathbf{E}_{\sigma'} \wedge \mathbf{Z}| = |(T^{-1})^{(d-p)}(\mathbf{L}_\sigma \wedge \mathbf{E}_{\sigma'}) \wedge (T^{-1})^{(p)}\mathbf{Z}| = |\mathbf{E}_\sigma \wedge \mathbf{E}_{\sigma'} \wedge \mathbf{Z}'|, \tag{7.10}$$

where $\mathbf{Z}' \in \widehat{\Lambda}$. Here, besides (7.8), (7.9), we have made use of the fact that for every $\mathbf{V} \in \bigwedge^p(\mathbb{R}^d)$, $\mathbf{W} \in \bigwedge^{d-p}(\mathbb{R}^d)$ the wedge product $\mathbf{V} \wedge \mathbf{W}$ is a real number and

$$|\mathbf{V} \wedge \mathbf{W}| = |T^{(p)}\mathbf{V} \wedge T^{(d-p)}\mathbf{W}|,$$

provided $\det T = 1$.

We conclude from (7.10) and Proposition 2.3 that $\mathfrak{b}_p$ (resp. $\mathfrak{a}_p$) equals the supremum of the real numbers γ such that there are arbitrarily large values of t for which (resp. such that for every t large enough) the system of inequalities

$$\max_{\substack{\sigma\in\mathcal{J}_k\\ \sigma'\in\mathcal{J}'_{d-p-k}}} |\mathbf{E}_\sigma\wedge\mathbf{E}_{\sigma'}\wedge\mathbf{Z}|\leq t^{1-(k-k_0)(1+\gamma)},\qquad k=k_0,\ldots,k_1, \tag{7.11}$$

has a non-zero solution in $\mathbf{Z}\in\widehat{\Lambda}$.

The inequalities (7.11) define the parallelepiped

$$\widehat{P}_\gamma(t)=\Big\{\mathbf{Z}\in\textstyle\bigwedge^p(\mathbb{R}^d)\ \Big|\ \displaystyle\max_{\substack{\sigma\in\mathcal{J}_{m-k}\\ \sigma'\in\mathcal{J}'_{p-m+k}}} |\langle\mathbf{E}_\sigma\wedge\mathbf{E}_{\sigma'},\mathbf{Z}\rangle|\leq t^{1-(k-k_0)(1+\gamma)},\quad k=k_0,\ldots,k_1\Big\}, \tag{7.12}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the inner product in $\bigwedge^p(\mathbb{R}^d)$. By analogy with (7.5) we can write

$$\begin{aligned}\mathfrak{b}_p&=\limsup_{t\to+\infty}\{\gamma\in\mathbb{R}\mid\lambda_1(\widehat{P}_\gamma(t))=1\},\\ \mathfrak{a}_p&=\liminf_{t\to+\infty}\{\gamma\in\mathbb{R}\mid\lambda_1(\widehat{P}_\gamma(t))=1\},\end{aligned} \tag{7.13}$$

where $\lambda_1\big(\widehat{P}_\gamma(t)\big)$ is the first minimum of $\widehat{P}_\gamma(t)$ with respect to $\widehat{\Lambda}$.

Consider the path $\widehat{\mathfrak{T}}$ defined by (6.1) for $\mathfrak{T}$. Then

$$\widehat{\tau}_j(s)=\sum_{i\in\sigma_j}\tau_i(s),$$

and if $\sigma_j\cap\{1,\ldots,m\}\in\mathcal{J}_{m-k}$, we have

$$\begin{aligned}\widehat{\tau}_j(s)&=(m-k)s-\frac{(p-(m-k))m}{n}s=\left(\frac{d}{n}(k_0-k)+\varkappa_p\right)s\\ &=(1-(k-k_0)(1+\gamma_0))\ln t,\end{aligned}$$

where

$$t=e^{\varkappa_p s},\qquad \gamma_0=\frac{d}{n\varkappa_p}-1.$$

Hence

$$\widehat{P}_{\gamma_0}(t)=D_{\widehat{\boldsymbol{\tau}}(s)}\mathcal{B}^r_\infty, \tag{7.14}$$

where, as before, $r=\binom{d}{p}$.

Thus, similar to (7.6), we get

$$\begin{aligned}\underline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}})&=\liminf_{t\to+\infty}\frac{\varkappa_p\ln(\lambda_1(\widehat{P}_{\gamma_0}(t)))}{\ln t},\\ \overline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}})&=\limsup_{t\to+\infty}\frac{\varkappa_p\ln(\lambda_1(\widehat{P}_{\gamma_0}(t)))}{\ln t}.\end{aligned} \tag{7.15}$$

The rest of the argument is very much the same as the corresponding part of the proof of Proposition 7.1. Let us observe that

$$\widehat{P}_{\gamma}(t)=t^{1-\frac{1+\gamma}{1+\gamma_0}}\widehat{P}_{\gamma_0}(t^{\frac{1+\gamma}{1+\gamma_0}}).$$

This implies that

$$\lambda_1(\widehat{P}_{\gamma}(t))=(t')^{1-\frac{1+\gamma_0}{1+\gamma}}\lambda_1(\widehat{P}_{\gamma_0}(t'))$$

with $t'=t^{\frac{1+\gamma}{1+\gamma_0}}$. Therefore, the equality

$$\lambda_1(\widehat{P}_{\gamma}(t))=1$$

holds if and only if

$$1-\frac{1+\gamma_0}{1+\gamma}+\frac{\ln(\lambda_1(\widehat{P}_{\gamma_0}(t')))}{\ln t'}=0.$$

Hence, in view of (7.13), (7.15), we get

$$\begin{aligned}\mathfrak{b}_p&=\limsup_{t\to+\infty}\left\{(1+\gamma_0)\left(1+\frac{\ln(\lambda_1(\widehat{P}_{\gamma_0}(t)))}{\ln t}\right)^{-1}-1\right\}\\&=(1+\gamma_0)(1+\varkappa_p^{-1}\underline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}}))^{-1}-1\end{aligned}$$

and

$$\begin{aligned}\mathfrak{a}_p&=\liminf_{t\to+\infty}\left\{(1+\gamma_0)\left(1+\frac{\ln(\lambda_1(\widehat{P}_{\gamma_0}(t)))}{\ln t}\right)^{-1}-1\right\}\\&=(1+\gamma_0)(1+\varkappa_p^{-1}\overline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}}))^{-1}-1.\end{aligned}$$

Thus,

$$(1+\mathfrak{b}_p)(\varkappa_p+\underline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}}))=(1+\mathfrak{a}_p)(\varkappa_p+\overline{\psi}_1(\widehat{\Lambda},\widehat{\mathfrak{T}}))=d/n.$$

It remains to apply Proposition 6.1. ∎

Remark 7.3. It follows from (7.14) that the volume of $\widehat{P}_{\gamma_0}(t)$ is equal to 2^r. Hence, by Minkowski's convex body theorem, $\widehat{P}_{\gamma_0}(t)$ contains a nonzero point of $\widehat{\Lambda}$. Thus, taking into account (7.13), we get

$$\mathfrak{b}_p\geq\mathfrak{a}_p\geq\gamma_0=\frac{d}{n\varkappa_p}-1,$$

or in terms of Schmidt–Summerer's exponents,

$$-\varkappa_p\leq\underline{\Psi}_p\leq\overline{\Psi}_p\leq 0.$$

8. Transposed system. The subspace spanned by $\boldsymbol{\ell}_{m+1},\dots,\boldsymbol{\ell}_d$ is the space of solutions to the system

$$-\Theta^{\intercal}\mathbf{y}=\mathbf{x}.$$

As we noticed in Section 2, it coincides with the orthogonal complement $\mathcal{L}^\perp$. Denote by β_p^*, α_p^*, $\mathfrak{b}_p^*$, $\mathfrak{a}_p^*$ the corresponding pth regular and uniform Diophantine exponents of the first and of the second types for $\Theta^\intercal$. Obviously, they coincide with the ones corresponding to $-\Theta^\intercal$. The lattice constructed for $-\Theta^\intercal$ the very same way Λ was constructed for Θ, would be

$$\begin{pmatrix} E_n & 0 \\ \Theta^\intercal & E_m \end{pmatrix} \mathbb{Z}^d.$$

But transposing the first n and the last m coordinates turns this lattice into

$$\begin{pmatrix} E_m & \Theta^\intercal \\ 0 & E_n \end{pmatrix} \mathbb{Z}^d = T^\intercal \mathbb{Z}^d = \Lambda^*,$$

which is the lattice dual for Λ. For this reason with $\Theta^\intercal$ we shall associate Λ^*. Now, the most natural way to specify the path determining Schmidt–Summerer's exponents associated to $\Theta^\intercal$ is to take into account the coordinate permutation just mentioned and consider the path $\mathfrak{T}^* : s \to \boldsymbol{\tau}^*(s)$ defined by

$$\tau_1^*(s) = \cdots = \tau_m^*(s) = -ns/m, \quad \tau_{m+1}^*(s) = \cdots = \tau_d^*(s) = s. \tag{8.1}$$

Denoting

$$\underline{\psi}_p^* = \underline{\psi}_p(\Lambda^*, \mathfrak{T}^*), \qquad \overline{\psi}_p^* = \overline{\psi}_p(\Lambda^*, \mathfrak{T}^*),$$
$$\underline{\Psi}_p^* = \underline{\Psi}_p(\Lambda^*, \mathfrak{T}^*), \qquad \overline{\Psi}_p^* = \overline{\Psi}_p(\Lambda^*, \mathfrak{T}^*),$$

we see that any statement proved for an arbitrary Θ concerning the quantities β_p, α_p, $\mathfrak{b}_p$, $\mathfrak{a}_p$, $\underline{\psi}_p$, $\overline{\psi}_p$, $\underline{\Psi}_p$, $\overline{\Psi}_p$ remains valid if Θ is replaced by $\Theta^\intercal$, and the quantities n, m, β_p, α_p, $\mathfrak{b}_p$, $\mathfrak{a}_p$, $\underline{\psi}_p$, $\overline{\psi}_p$, $\underline{\Psi}_p$, $\overline{\Psi}_p$ are replaced by m, n, β_p^*, α_p^*, $\mathfrak{b}_p^*$, $\mathfrak{a}_p^*$, $\underline{\psi}_p^*$, $\overline{\psi}_p^*$, $\underline{\Psi}_p^*$, $\overline{\Psi}_p^*$, respectively. In particular, the analogues of Propositions 7.1, 7.2 hold:

Proposition 8.1. *We have*

$$(1 + \beta_p^*)(1 + \underline{\psi}_p^*) = (1 + \alpha_p^*)(1 + \overline{\psi}_p^*) = d/m. \tag{8.2}$$

Proposition 8.2. *Set* $\varkappa_p^* = \min\bigl(p, \frac{n}{m}(d-p)\bigr)$. *Then*

$$(1 + \mathfrak{b}_p^*)(\varkappa_p^* + \underline{\Psi}_p^*) = (1 + \mathfrak{a}_p^*)(\varkappa_p^* + \overline{\Psi}_p^*) = d/m. \tag{8.3}$$

Further, same as (7.6), we get

$$\underline{\psi}_p^* = \liminf_{t \to +\infty} \frac{\ln(\lambda_p^*(P_{m/n}(t^{-n/m})))}{\ln t}, \qquad \overline{\psi}_p^* = \limsup_{t \to +\infty} \frac{\ln(\lambda_p^*(P_{m/n}(t^{-n/m})))}{\ln t}, \tag{8.4}$$

where λ_p^* denotes the pth minimum with respect to Λ^*.

Let us show that $\underline{\psi}_p^*$, $\overline{\psi}_p^*$ are closely connected with $\underline{\psi}_{d-p}$, $\overline{\psi}_{d-p}$ (which, as before, are related to Λ and the path $\mathfrak{T}$ defined by (7.2)). It follows from the definition of $P_\gamma(t)$ that there is a positive constant c, depending only on Θ, such that

$$c^{-1}P_\gamma(t^{-1}) \subseteq P_\gamma(t)^* \subseteq cP_\gamma(t^{-1}),$$

where $P_\gamma(t)^*$ is the polar reciprocal body for $P_\gamma(t)$. Furthermore, it follows from Mahler's theory that

$$\lambda_p^*(P_\gamma(t)^*)\lambda_{d+1-p}(P_\gamma(t)) \asymp 1$$

with the implied constants depending only on d. Hence

$$\lambda_p^*(P_\gamma(t^{-1}))\lambda_{d+1-p}(P_\gamma(t)) \asymp 1. \tag{8.5}$$

Combining (8.4), (8.5) and (7.6) with p replaced by $d+1-p$ we get

PROPOSITION 8.3. *We have*

$$\underline{\psi}_p^* = -\frac{n}{m}\overline{\psi}_{d+1-p} \quad \textit{and} \quad \overline{\psi}_p^* = -\frac{n}{m}\underline{\psi}_{d+1-p}.$$

COROLLARY 8.4. *We have*

$$(1+\beta_p^*)(m-n\overline{\psi}_{d+1-p}) = (1+\alpha_p^*)(m-n\underline{\psi}_{d+1-p}) = d.$$

Proof. Follows from Propositions 8.1 and 8.3. ∎

COROLLARY 8.5. *We have*

$$\alpha_{d+1-p}\beta_p^* = 1 \quad \textit{and} \quad \alpha_{d+1-p}^*\beta_p = 1.$$

Proof. Follows from Proposition 7.1 and Corollary 8.4. ∎

In order to obtain the corresponding relations between the exponents of the second type, let us go in the opposite direction and prove

PROPOSITION 8.6. *We have*

$$\mathfrak{b}_p = \mathfrak{b}_{d-p}^* \quad \textit{and} \quad \mathfrak{a}_p = \mathfrak{a}_{d-p}^*.$$

Proof. Let $\mathbf{L}_\sigma$, $\mathbf{E}_\sigma$, $\mathcal{J}_k$, $\mathcal{J}_k'$ be as in Section 2.

We recall that the bases $\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_m,\mathbf{e}_{m+1},\ldots,\mathbf{e}_d$ and $\mathbf{e}_1,\ldots,\mathbf{e}_m,\boldsymbol{\ell}_{m+1},\ldots,\boldsymbol{\ell}_d$ are dual to each other. So, if $\sigma\in\mathcal{J}_k$, $\sigma'\in\mathcal{J}_{k'}'$, then

$$*(\mathbf{L}_\sigma\wedge\mathbf{E}_{\sigma'}) = \pm\mathbf{E}_{\overline{\sigma}}\wedge\mathbf{L}_{\overline{\sigma}'},$$

where $*$ denotes the Hodge star operator,

$$\overline{\sigma} = \{1,\ldots,m\}\setminus\sigma, \qquad \overline{\sigma}' = \{m+1,\ldots,d\}\setminus\sigma',$$

and the sign depends on the parity of the corresponding permutation. Hence for any $\sigma\in\mathcal{J}_k$, $\sigma'\in\mathcal{J}_{d-p-k}'$, and any $\mathbf{Z}\in\bigwedge^p(\mathbb{Z}^d)$ we have

$$|\mathbf{L}_\sigma\wedge\mathbf{E}_{\sigma'}\wedge\mathbf{Z}| = |\mathbf{E}_{\overline{\sigma}}\wedge\mathbf{L}_{\overline{\sigma}'}\wedge *\mathbf{Z}|.$$

Thus,

$$\max_{\substack{\sigma\in\mathcal{J}_k\\ \sigma'\in\mathcal{J}_{d-p-k}'}} |\mathbf{L}_\sigma\wedge\mathbf{E}_{\sigma'}\wedge\mathbf{Z}| = \max_{\substack{\sigma'\in\mathcal{J}_{p-m+k}'\\ \sigma\in\mathcal{J}_{m-k}}} |\mathbf{L}_{\sigma'}\wedge\mathbf{E}_\sigma\wedge *\mathbf{Z}| \tag{8.6}$$

for each $\mathbf{Z}\in\bigwedge^p(\mathbb{Z}^d)$.

Set $k_0^* = \max(0, n-(d-p))$, $k_1^* = \min(n,p)$. Then $k_0^* = k_0 + p - m$, $k_1^* = k_1 + p - m$, and the inequalities $k_0 \le k \le k_1$ are equivalent to $k_0^* \le p - m + k \le k_1^*$. Therefore, it follows from (8.6) that (2.5) is equivalent to

$$\max_{\substack{\sigma' \in \mathcal{J}'_k \\ \sigma \in \mathcal{J}_{p-k}}} |\mathbf{L}_{\sigma'} \wedge \mathbf{E}_\sigma \wedge *\mathbf{Z}| \le t^{1-(k-k_0^*)(1+\gamma)}, \qquad k = k_0^*, \ldots, k_1^*. \tag{8.7}$$

It remains to apply Proposition 2.3 and the fact that $*(\bigwedge^p(\mathbb{Z}^d)) = \bigwedge^{d-p}(\mathbb{Z}^d)$. ∎

Corollary 8.7. *Set* $\varkappa_p^{**} = \min\big(d-p, \frac{m}{n}p\big) = \frac{m}{n}\varkappa_p^*$. *Then*

$$(1+\mathfrak{b}_p^*)(\varkappa_p^{**} + \underline{\Psi}_{d-p}) = (1+\mathfrak{a}_p^*)(\varkappa_p^{**} + \overline{\Psi}_{d-p}) = d/n.$$

Proof. Follows from Propositions 7.2 and 8.6. ∎

Corollary 8.8. *We have*

$$\underline{\Psi}_p^* = \frac{n}{m}\underline{\Psi}_{d-p} \quad \textit{and} \quad \overline{\Psi}_p^* = \frac{n}{m}\overline{\Psi}_{d-p}.$$

Proof. Follows from Proposition 8.2 and Corollary 8.7. ∎

9. Main results in terms of Schmidt–Summerer's exponents. It is interesting to rewrite (3.3) in terms of Schmidt–Summerer's exponents. By Propositions 8.6 and 7.2 it becomes simply

$$\underline{\Psi}_{d-1} \le \frac{\underline{\Psi}_1}{d-1}, \tag{9.1}$$

which is one of the statements of Corollary 5.6. But we already have an intermediate variant of this inequality! It is

$$\frac{\underline{\Psi}_{p+1}}{d-p-1} \le \frac{\underline{\Psi}_p}{d-p}, \tag{9.2}$$

one of the statements of Corollary 5.5. Rewriting the corresponding statements of Corollary 5.5 with Λ and $\mathfrak{T}$ defined by (7.1), (7.2) in terms of the intermediate Diophantine exponents gives Theorem 4.1.

As we see, describing the splitting of Dyson's and Apfelbeck's inequalities in terms of Schmidt–Summerer's exponents given by Corollary 5.5 is much more elegant than in terms of Diophantine exponents. Another attraction is its universality for all values of n, m whose sum is equal to d. Moreover, Corollary 5.5 holds actually for arbitrary lattices and paths, while Theorem 4.1 is bound to the specific choice of those.

Let us now translate Theorems 4.2, 4.3 into the language of Schmidt–Summerer's exponents. We recall that, as noticed in Remark 7.3,

$$-1 \le \underline{\Psi}_1 \le \overline{\Psi}_1 \le 0.$$

Theorem 4.2 turns into

THEOREM 9.1. *Suppose that the space of integer solutions of* (1.1) *is not a one-dimensional lattice. Then*

$$\underline{\Psi}_2 \leq \begin{cases} 2\underline{\Psi}_1 + d\dfrac{\overline{\Psi}_1 - \underline{\Psi}_1}{n + n\overline{\Psi}_1} & \text{if } \overline{\Psi}_1 \neq -1, \\ 2\underline{\Psi}_1 + d\dfrac{\overline{\Psi}_1 - \underline{\Psi}_1}{m - n\overline{\Psi}_1}. \end{cases} \tag{9.3}$$

Similarly, Theorem 4.3 turns into

THEOREM 9.2. *We have*

$$\overline{\Psi}_2 \leq \begin{cases} \dfrac{(d-2)\overline{\Psi}_1}{(n-1) + n\overline{\Psi}_1} & \text{if } \overline{\Psi}_1 \geq \dfrac{m-n}{2n}, \\ \dfrac{(d-2)\overline{\Psi}_1}{(m-1) - n\overline{\Psi}_1} & \text{if } \overline{\Psi}_1 \leq \dfrac{m-n}{2n}. \end{cases} \tag{9.4}$$

As we see, this point of view relieves us of singling out the case $m = 1$. In the next section we prove Theorems 9.1 and 9.2.

10. Proofs of Theorems 9.1 and 9.2. Let Λ and $\mathfrak{T}$ be as in (7.1) and (7.2). The following observation is crucial to proving Theorems 9.1 and 9.2.

LEMMA 10.1. *Suppose that* $s, s' \in \mathbb{R}_+$ *satisfy the conditions*

$$\lambda_1(\mathcal{B}(s))\mathcal{B}(s) \subseteq \lambda_1(\mathcal{B}(s'))\mathcal{B}(s'), \tag{10.1}$$

$$\lambda_1(\mathcal{B}(s')) = \lambda_2(\mathcal{B}(s')). \tag{10.2}$$

Then

$$\psi_2(s) \leq \begin{cases} \psi_1(s) + d \cdot \dfrac{\psi_1(s') - \psi_1(s)}{n + n\psi_1(s')} & \text{if } s' \leq s \text{ and } \psi_1(s') \neq -1, \\ \psi_1(s) + d \cdot \dfrac{\psi_1(s') - \psi_1(s)}{m - n\psi_1(s')} & \text{if } s' \geq s. \end{cases} \tag{10.3}$$

Proof. If $s = s'$, then by (10.2) we have $\psi_1(s) = \psi_2(s)$, which implies (10.3). So, we may suppose that $s \neq s'$.

We recall that

$$\mathcal{B}(s) = \left\{ \mathbf{z} = (z_1, \ldots, z_d)^\intercal \in \mathbb{R}^d \;\middle|\; \max_{1 \leq j \leq m} |z_j| \leq e^s, \ \max_{1 \leq i \leq n} |z_i| \leq e^{-ms/n} \right\}.$$

Thus, (10.1) means that

$$\lambda_1(\mathcal{B}(s))e^s \leq \lambda_1(\mathcal{B}(s'))e^{s'}, \quad \lambda_1(\mathcal{B}(s))e^{-ms/n} \leq \lambda_1(\mathcal{B}(s'))e^{-ms'/n}. \tag{10.4}$$

Applying the first inequality of (10.4) for $s' < s$ and the second for $s' > s$, we get in each case

$$\lambda_1(\mathcal{B}(s)) < \lambda_1(\mathcal{B}(s')). \tag{10.5}$$

The inequalities (10.4) cannot both be strict, for this would conflict with the definition of the first minimum. Thus, in view of (10.5), it follows from (10.4) that for $s' < s$ we have

$$\lambda_1(\mathcal{B}(s))e^s = \lambda_1(\mathcal{B}(s'))e^{s'}, \quad \lambda_1(\mathcal{B}(s))e^{-ms/n} < \lambda_1(\mathcal{B}(s'))e^{-ms'/n}, \tag{10.6}$$

and for $s' > s$ we have

$$\lambda_1(\mathcal{B}(s))e^s < \lambda_1(\mathcal{B}(s'))e^{s'}, \quad \lambda_1(\mathcal{B}(s))e^{-ms/n} = \lambda_1(\mathcal{B}(s'))e^{-ms'/n}. \tag{10.7}$$

There are some non-zero lattice points on the boundaries of both $\lambda_1(\mathcal{B}(s))\mathcal{B}(s)$ and $\lambda_1(\mathcal{B}(s'))\mathcal{B}(s')$, while there are no such points in their interiors. But the first m components of each point of Λ are integers (see (7.1)), so both sides of the equality in (10.6) should be equal to a positive integer. As for the equality in (10.7), its sides should both be less than 1, since $\lambda_1(\mathcal{B}(s)) \le 1$ by Minkowski's convex body theorem. Thus,

$$\begin{aligned} \lambda_1(\mathcal{B}(s))e^s = \lambda_1(\mathcal{B}(s'))e^{s'} &\ge 1 \qquad \text{if } s' < s, \\ \lambda_1(\mathcal{B}(s))e^{-ms/n} = \lambda_1(\mathcal{B}(s'))e^{-ms'/n} &< 1 \quad \text{if } s' > s, \end{aligned}$$

or equivalently,

$$\begin{aligned} s(1 + \psi_1(s)) = s'(1 + \psi_1(s')) &\ge 0 \qquad \text{if } s' < s, \\ s(\psi_1(s) - m/n) = s'(\psi_1(s') - m/n) &< 0 \quad \text{if } s' > s. \end{aligned} \tag{10.8}$$

Furthermore, it is clear that

$$\mathcal{B}(s') \subset \begin{cases} e^{-m(s'-s)/n}\mathcal{B}(s) & \text{if } s' < s, \\ e^{s'-s}\mathcal{B}(s) & \text{if } s' > s. \end{cases}$$

Hence, in view of (10.2), it follows that

$$\begin{aligned} \lambda_2(\mathcal{B}(s))e^{-ms/n} \le \lambda_2(\mathcal{B}(s'))e^{-ms'/n} &= \lambda_1(\mathcal{B}(s'))e^{-ms'/n} \qquad \text{if } s' < s, \\ \lambda_2(\mathcal{B}(s))e^{s} \le \lambda_2(\mathcal{B}(s'))e^{s'} &= \lambda_1(\mathcal{B}(s'))e^{s'} \qquad \text{if } s' > s, \end{aligned}$$

or in other words,

$$\begin{aligned} s(\psi_2(s) - m/n) &\le s'(\psi_1(s') - m/n) \quad \text{if } s' < s, \\ s(1 + \psi_2(s)) &\le s'(1 + \psi_1(s')) \qquad \text{if } s' > s. \end{aligned} \tag{10.9}$$

Now, dividing the corresponding inequality in (10.9) by the corresponding equality in (10.8) (naturally excluding the case $\psi_1(s') = -1$), we get (10.3). ∎

For each $\mathbf{z} = (z_1, \dots, z_d)^\intercal \in \mathbb{R}^d$ and each $s > 0$ let us set

$$\mu_s(\mathbf{z}) = e^{-s} \max_{1 \le i \le m} |z_i| \quad \text{and} \quad \nu_s(\mathbf{z}) = e^{ms/n} \max_{m < i \le d} |z_i|.$$

Then

$$\mathcal{B}(s) = \{\mathbf{z} \in \mathbb{R}^d \mid \mu_s(\mathbf{z}) \le 1, \, \nu_s(\mathbf{z}) \le 1\}.$$

The parallelepiped $\lambda_1(\mathcal{B}(s))\mathcal{B}(s)$ contains no non-zero points of Λ in its interior and contains at least one pair of such points on its boundary. Choose any of these points and denote it by $\mathbf{v}_s$. Obviously, the maximum of the quantities $\mu_s(\mathbf{v}_s)$, $\nu_s(\mathbf{v}_s)$ equals $\lambda_1(\mathcal{B}(s))$.

Corollary 10.2. *For each $s > 0$ such that*

$$\mu_s(\mathbf{v}_s) = \nu_s(\mathbf{v}_s) = \lambda_1(\mathcal{B}(s)), \tag{10.10}$$

there are $s', s'' > 0$ such that

$$s(1 + \psi_1(s)) \le s' \le s \le s'' \le s(1 - (n/m)\psi_1(s))$$

and

$$\Psi_2(s) \le \begin{cases} 2\psi_1(s) + d\dfrac{\psi_1(s') - \psi_1(s)}{n + n\psi_1(s')} & \textit{if } \psi_1(s') \ne -1, \\ 2\psi_1(s) + d\dfrac{\psi_1(s'') - \psi_1(s)}{m - n\psi_1(s'')}. \end{cases} \tag{10.11}$$

Proof. Let us show that the relation $\mu_s(\mathbf{v}_s) = \lambda_1(\mathcal{B}(s))$ implies the existence of an $s' \le s$ satisfying the conditions of Lemma 10.1. Denote $\lambda = \lambda_1(\mathcal{B}(s))$. Let

$$\mathcal{P}_\nu = \{\mathbf{z} \in \mathbb{R}^d \mid \mu_s(\mathbf{z}) \le \lambda,\ \nu_s(\mathbf{z}) \le \nu\lambda\}$$

be the minimal (with respect to inclusion) parallelepiped containing no non-zero points of Λ in its interior. The existence of such a parallelepiped follows from Minkowski's convex body theorem. It also implies that $1 \le \nu \le \lambda^{-d/n}$. Then

$$\lambda\mathcal{B}(s) \subseteq \mathcal{P}_\nu = \lambda'\mathcal{B}(s'),$$

where $\lambda' = \lambda\nu^{n/d}$, $s' = s - (n/d)\ln\nu$. For λ', s' we have

$$\lambda' \ge \lambda, \qquad s + \ln\lambda \le s' \le s.$$

On the other hand, $\mathcal{P}_\nu$ contains non-collinear points of Λ on its boundary, so $\lambda_1(\mathcal{B}(s')) = \lambda_2(\mathcal{B}(s')) = \lambda'$. Thus, s, s' satisfy (10.1) and (10.2).

Now let us consider the relation $\nu_s(\mathbf{v}_s) = \lambda_1(\mathcal{B}(s))$. By Minkowski's convex body theorem there is a μ with $1 \le \mu \le \lambda^{-d/m}$ such that the parallelepiped

$$\mathcal{Q}_\mu = \{\mathbf{z} \in \mathbb{R}^d \mid \mu_s(\mathbf{z}) \le \mu\lambda,\ \nu_s(\mathbf{z}) \le \lambda\}$$

contains no non-zero points of Λ in its interior, but contains non-collinear points of Λ on its boundary. Then

$$\lambda\mathcal{B}(s) \subseteq \mathcal{Q}_\mu = \lambda''\mathcal{B}(s''),$$

where $\lambda'' = \lambda\mu^{m/d}$, $s'' = s + (n/d)\ln\mu$. For λ'', s'' we have

$$\lambda'' \ge \lambda, \qquad s \le s'' \le s - (n/m)\ln\lambda.$$

Moreover, s, s'' also satisfy (10.1), (10.2), since $\lambda_1(\mathcal{B}(s'')) = \lambda_2(\mathcal{B}(s'')) = \lambda''$.

It remains to apply Lemma 10.1. ∎

Having Corollary 10.2, it is now easy to prove Theorem 9.1.

First, let us notice that if the system (1.1) has a non-zero integer solution, then it has two linearly independent integer solutions, so in this case $\overline{\Psi}_1 = \underline{\Psi}_1 = -1$, $\underline{\Psi}_2 = -2$, which implies (9.3).

Next, suppose that the system (1.1) has no non-zero integer solutions. Then there are infinitely many local minima of $\psi_1(s)$, each of them satisfies (10.10), and the sequence of these local minima tends to ∞. Moreover, s' and s'' from Corollary 10.2 tend to ∞ as s tends to ∞. Indeed, since (1.1) has no non-zero integer solutions, we have

$$e^{s(1+\psi_1(s))} = e^s\lambda_1(\mathcal{B}(s)) = \lambda_1(e^{-s}\mathcal{B}(s)) \to \infty \quad \text{as } s \to \infty,$$

so

$$s(1+\psi_1(s)) \to \infty \quad \text{as } s \to \infty. \tag{10.12}$$

In particular, it follows from (10.12) that $\psi_1(s)$ is eventually greater than -1 (it can actually be shown that $\psi_1(s) > -1$ starting from the second local minimum point of $\psi_1(s)$). Therefore,

$$\underline{\Psi}_2 \le \liminf \Psi_2(s) \le \begin{cases} 2\liminf \psi_1(s) + d\limsup \dfrac{\psi_1(s') - \psi_1(s)}{n + n\psi_1(s')}, \\ 2\liminf \psi_1(s) + d\limsup \dfrac{\psi_1(s'') - \psi_1(s)}{m - n\psi_1(s'')}, \end{cases} \tag{10.13}$$

where the lim inf and the lim sup are taken over the set of local minima of $\psi_1(s)$. Since $\psi_1(s)$ is never positive, both denominators in (10.13) are eventually positive. Therefore, (10.13) implies (9.3).

Corollary 10.3. *Suppose that the system* (1.1) *has no non-zero integer solutions. Then for each $s > 0$ there is an $s' > 0$ such that $s(1+\psi_1(s)) \le s' \le s(1-(n/m)\psi_1(s))$ and*

$$\Psi_2(s) \le \begin{cases} \dfrac{(d-2)\psi_1(s')}{(n-1)+n\psi_1(s')} & \text{if } \psi_1(s') \ge \dfrac{m-n}{2n}, \\ \dfrac{(d-2)\psi_1(s')}{(m-1)-n\psi_1(s')} & \text{if } \psi_1(s') \le \dfrac{m-n}{2n}. \end{cases} \tag{10.14}$$

Proof. Assume that $\mu_s(\mathbf{v}_s) = \lambda_1(\mathcal{B}(s))$. Then the same argument as in the proof of Corollary 10.2 shows that there is an s', such that $s(1+\psi_1(s)) \le s' \le s$ and

$$\Psi_2(s) \le 2\psi_1(s) + d\frac{\psi_1(s') - \psi_1(s)}{n + n\psi_1(s')} \tag{10.15}$$

unless $\psi_1(s') = -1$. By Corollary 5.4 we have

$$\frac{d-1}{d-2}\Psi_2(s) \le \psi_1(s) \le \frac{1}{2}\Psi_2(s). \tag{10.16}$$

If $\psi_1(s') = -1$, then (10.16) implies (10.14). Suppose that $\psi_1(s') \neq -1$. Then, taking into account that

$$2 - \frac{d}{n + n\psi_1(s')} \geq 0 \quad \text{if and only if} \quad \psi_1(s') \geq \frac{m-n}{2n},$$

we conclude from (10.15) and (10.16) that

$$\Psi_2(s) \leq \begin{cases} 2\psi_1(s') & \text{if } \psi_1(s') \geq \dfrac{m-n}{2n}, \\ \dfrac{(d-2)\psi_1(s')}{(m-1) - n\psi_1(s')} & \text{if } \psi_1(s') \leq \dfrac{m-n}{2n}. \end{cases} \tag{10.17}$$

Assume now that $\nu_s(\mathbf{v}_s) = \lambda_1(\mathcal{B}(s))$. Then the same argument as in the proof of Corollary 10.2 shows that there is an s'' such that $s \leq s'' \leq s(1 - (n/m)\psi_1(s))$ and

$$\Psi_2(s) \leq 2\psi_1(s) + d\frac{\psi_1(s'') - \psi_1(s)}{m - n\psi_1(s'')}. \tag{10.18}$$

Taking into account that

$$2 - \frac{d}{m - n\psi_1(s'')} \geq 0 \quad \text{if and only if} \quad \psi_1(s'') \leq \frac{m-n}{2n},$$

we conclude from (10.18) and (10.16) that

$$\Psi_2(s) \leq \begin{cases} \dfrac{(d-2)\psi_1(s'')}{(n-1) + n\psi_1(s'')} & \text{if } \psi_1(s'') \geq \dfrac{m-n}{2n}, \\ 2\psi_1(s'') & \text{if } \psi_1(s'') \leq \dfrac{m-n}{2n}. \end{cases} \tag{10.19}$$

Since $\psi_1(s')$ and $\psi_1(s'')$ are negative, we have

$$2\psi_1(s') \leq \frac{(d-2)\psi_1(s')}{(n-1) + n\psi_1(s')} \qquad \text{if } \psi_1(s') \geq \frac{m-n}{2n},$$

$$2\psi_1(s'') \leq \frac{(d-2)\psi_1(s'')}{(m-1) - n\psi_1(s'')} \qquad \text{if } \psi_1(s'') \leq \frac{m-n}{2n}.$$

Therefore, (10.17) and (10.19) imply the desired statement. ∎

Deriving Theorem 9.2 from Corollary 10.3 is even easier than deriving Theorem 9.1 from Corollary 10.2.

If the system (1.1) has a non-zero integer solution, then $\overline{\Psi}_1 = -1 < \frac{m-n}{2n}$, and (9.4) follows from (5.5). Suppose now that (1.1) has no non-zero integer solutions. Then it follows from (10.12) that s' from Corollary 10.3 tends to ∞ as s tends to ∞. Hence, taking $\limsup$ of both sides in (10.14), we get (9.4).

Acknowledgements. The author would like to thank the Department of Mathematics of the University of York for the warm welcome during his research visit in February–March 2011, when this paper was conceived. The author is also grateful to the unknown referee for simplifying the proof of Proposition 2.3.

This research was supported by RFBR (grant N° 09-01-00371a) and by grant N° MK-1226.2010.1 of the President of the Russian Federation.

References

[A] A. Apfelbeck, *A contribution to Khintchine's principle of transfer*, Czechoslovak Math. J. 1 (1951), 119–147.

[BL] Y. Bugeaud and M. Laurent, *On transfer inequalities in Diophantine approximation, II*, Math. Z. 265 (2010), 249–262.

[D] F. J. Dyson, *On simultaneous Diophantine approximations*, Proc. London Math. Soc. (2) 49 (1947), 409–420.

[G] O. N. German, *On Diophantine exponents and Khintchine's transference principle*, Moscow J. Combin. Number Theory, to appear; arXiv:1004.4933.

[J1] V. Jarník, *Über einen Satz von A. Khintchine*, Prace Mat.-Fiz. 43 (1936), 151–166.

[J2] V. Jarník, *Über einen Satz von A. Khintchine, 2*, Acta Arith. 2 (1936), 1–22.

[J3] V. Jarník, *Zum Khintchineschen "Übertragungssatz"*, Trav. Inst. Math. Tbilissi 3 (1938), 193–212.

[Kh] A. Ya. Khintchine, *Über eine Klasse linearer Diophantischer Approximationen*, Rend. Circ. Mat. Palermo 50 (1926), 170–195.

[L1] M. Laurent, *On transfer inequalities in Diophantine Approximation*, in: Analytic Number Theory, Essays in Honour of Klaus Roth (W. W. L. Chen et al., eds.), Cambridge Univ. Press, 2009, 306–314.

[L2] M. Laurent, *Exponents of Diophantine approximation in dimension two*, Canad. J. Math. 61 (2009), 165–189.

[M] K. Mahler, *On compound convex bodies (I)*, Proc. London Math. Soc. (3) 5 (1955), 358–379.

[Sch1] W. M. Schmidt, *On heights of algebraic subspaces and diophantine approximations*, Ann. of Math. 85 (1967), 430–472.

[Sch2] W. M. Schmidt, *Diophantine Approximation*, Lecture Notes in Math. 785, Springer, 1980.

[SchS] W. M. Schmidt and L. Summerer, *Parametric geometry of numbers and applications*, Acta Arith. 140 (2009), 67–91.

Oleg N. German
Moscow Lomonosov State University
Vorobiovy Gory, GSP-1
119991 Moscow, Russia
E-mail: german@mech.math.msu.su, german.oleg@gmail.com

Received on 8.7.2011
and in revised form on 31.10.2011 (6758)