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A family of deformations of the Riemann xi-function

by

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1. Introduction. Let $\zeta(s)$ be the Riemann zeta function, and let $\xi(s)$ be the Riemann xi-function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\Gamma(s)$ is the gamma function. We often use the notation

$$\gamma(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2). \tag{1.1}$$

It is well known that $\xi(s)$ is an entire function of order one and maximal type satisfying the functional equation $\xi(s) = \xi(1-s)$. The famous Riemann hypothesis for $\xi(s)$ is the problem whether all zeros of $\xi(s)$ lie on the critical line $\Re(s) = 1/2$.

A possible approach to the Riemann hypothesis is to find an entire function $E(s)$ satisfying

$$\xi(s) = \tfrac{1}{2}(E(s) + E^{\vee}(s)) \tag{1.2}$$

and the Hermite–Biehler condition on the right half-plane $\Re(s) > 1/2$:

$$|E(s)| > |E^{\vee}(s)| \quad \text{for} \quad \Re(s) > 1/2, \tag{1.3}$$

where $E \mapsto E^{\vee}$ is the involution defined by

$$E^{\vee}(s) = \overline{E(1-\bar{s})}, \tag{1.4}$$

because (1.2) and (1.3) imply that all zeros of $\xi(s)$ lie on the line $\Re(s) = 1/2$ by Lemma 5 of de Branges [14] (see also Lemma 2.2 of Lagarias [22]). We mention that this approach differs from those taken by de Branges in [16, 17] that need an essential amendment by Conrey and Li [12], but it is regarded as a variant of Levinson's method [27] (see also Conrey [11]) which starts by writing the functional equation in the form

$$Y(s)X(s)\zeta(s) = X(s)F_1(s) + X(1-s)F_2(1-s),$$

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where $X(s) = \pi^{-s/2}\Gamma(s/2)$ and $Y(s) = Y(1-s)$ is a simple function having only a few zeros. Levinson's method provides good results for the zeros of $\zeta(s)$ on the critical line $\Re(s) = 1/2$ after choosing $F_1(s)$ and $F_2(s)$ carefully.

It is easy to find an entire function $E(s)$ satisfying only (1.2). In fact, we can take $E(s) = \xi(s) + f(s)$ for an arbitrary entire function $f(s)$ satisfying $f^\vee(s) = -f(s)$, since $\xi^\vee(s) = \xi(s)$. Moreover it is not hard to show that there exist infinitely many entire functions $E(s)$ satisfying both (1.2) and (1.3) under the Riemann hypothesis by using the theory of Nevanlinna functions. Therefore a plausible procedure is to find a "nice" entire function $E(s)$ satisfying (1.2) in the sense that (1.3) is expected to be provable without the Riemann hypothesis for $\zeta(s)$. According to Theorem 4 of [24], a reasonable functional equation and a zero-free region of $E(s)$ implies (1.3). Taking notice of this fact, we study entire functions $E(s)$ satisfying (1.2) and a functional equation.

Major results are stated in Section 2. As an example, we take

$$E_{2,0}(s) = \int_0^\infty \Big(2\sum_{n=1}^\infty (2\pi^2 n^4 x^4 - 3\pi n^2 x^2) e^{-\pi n^2 x^2}\Big) \frac{2x^{s+2}}{x^2+1}\,\frac{dx}{x}$$

defined in (2.11) below. By Theorem 2.2, $E_{2,0}(s)$ is an entire function of order at most one satisfying (1.2) and the functional equation $E_{2,0}(s) = E_{2,0}(-1-s)$. In addition, Theorem 2.4 assert that a sufficient condition for (1.3) is that $E_{2,0}(s)$ has no zeros in the right half-plane $\Re(s) \ge 1/2$. Note that the line $\Re(s) = 1/2$ has distance one from the central line of the functional equation $\Re(s) = -1/2$ for $E_{2,0}(s)$. We do not know whether $E_{2,0}(s)$ has a zero with $\Re(s) \ge 1/2$, but at least it has no zeros with $\Re(s) \ge 10$ (Theorem 2.19).

On the other hand, formula (1.2) makes us define

$$\xi^-(s) = \tfrac{1}{2} i (E_{2,0}(s) - E_{2,0}(1-s))$$

as a companion of $\xi(s)$. (The function $\xi^-(s)$ will be written as $\xi_2^-(s;0)$ in Section 2.) The function $\xi^-(s)$ is an entire function of order at most one satisfying the functional equation with negative sign, $\xi^-(1-s) = -\xi^-(s)$.

We say that the zeros of entire functions $A(s)$ and $B(s)$ are *interlaced* if the functions have only simple zeros on a certain line, and these zeros appear alternatingly, that is, between two successive zeros of one of these functions there is exactly one zero of the other. In this sense, the zeros of $\xi(s)$ and $\xi^-(s)$ are interlaced on the line $\Re(s) = 1/2$, as is also the case for $\xi(s)$ and $\xi'(s)$ if $E_{2,0}(s)$ has no zeros in the right half-plane $\Re(s) \ge 1/2$ (Theorem 2.12). See Figure 1 for the conjectural interlacing property of the zeros of $\xi(s)$ and $\xi^-(s)$ with small height on the line $\Re(s) = 1/2$, and Figure 2 for a comparison between $\xi'(s)$ and $\xi^-(s)$ on the line $\Re(s) = 1/2$.

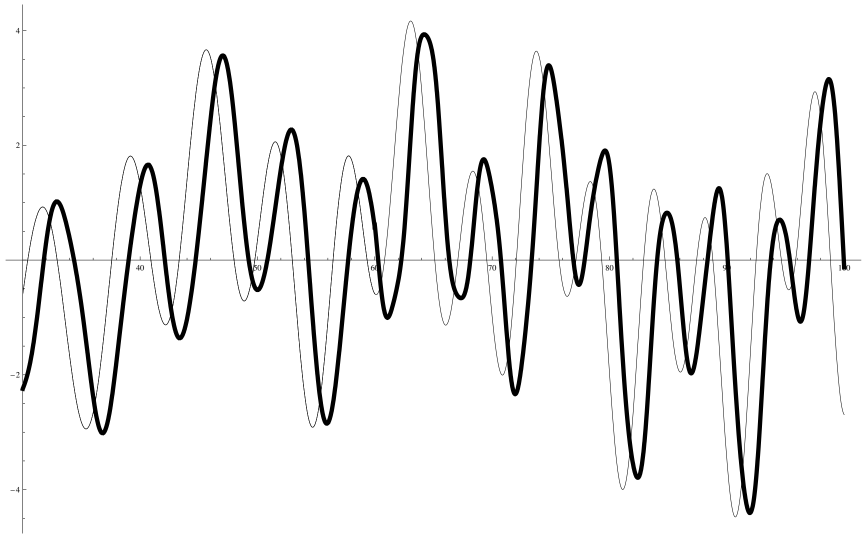

Fig. 1. The thin line is $\xi(1/2+it)|\gamma(1/2+it)|^{-1}$ for $30 \le t \le 100$, and the thick line is $\xi^{-}(1/2+it)|\gamma(1/2+it)|^{-1}$ for the same range.

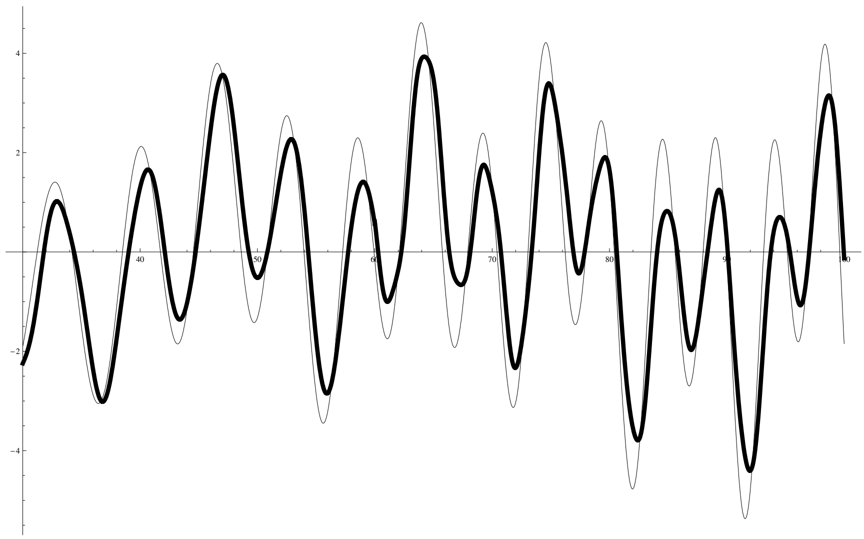

Fig. 2. The thin line is $\xi'(1/2+it)|\gamma(1/2+it)|^{-1}$ for $30 \le t \le 100$, and the thick line is $\xi^{-}(1/2+it)|\gamma(1/2+it)|^{-1}$ for the same range.

They were drawn in Mathematica 6.0 using the first one hundred terms of the series expansion (7.4) for $\xi^{-}(s)$. A relation between $\xi'(s)$ and $\xi^{-}(s)$ is discussed in Section 2.2 after Theorem 2.11.

The primary aim of the paper is to introduce and study a family of entire functions $\xi_\alpha(s;r)$ in order to state a sufficient condition for the Riemann hypothesis to hold for $\xi(s)$ by using $E_{\alpha,r}(s) := \xi_\alpha(s+\alpha/2; r+1)$ as well as $E_{2,0}(s)$.

This paper is organized as follows. In Section 2, we define the main object $\xi_\alpha(s;r)$ which is a one-parameter family of entire functions of two complex variables (s,r), and state our main results, Theorem 2.2 to 2.19. Two basic structures of the family $\xi_\alpha(s;r)$ are also stated in this section (Propositions 2.15 and 2.16). All these results are proved in Sections 3 to 6. In Section 7, we state miscellaneous formulas for $\xi_\alpha(s;r)$ including a series expansion of $\xi_\alpha(s;r)$ for $\alpha = 2$ which is used in the proof of Theorem 2.19.

A more general class of entire functions $E(s)$ satisfying (1.2) and various related topics involving a characterization of the family $\xi_\alpha(s;r)$ are studied in the forthcoming paper [31].

2. Statement of major results

2.1. Definition of a family of deformations. Define the function ϕ on the positive real line by

$$\phi(x) = \frac{1}{2}\frac{d}{dx}\left(x^2 \frac{d}{dx}\theta(x^2)\right) = 2\sum_{n=1}^{\infty}(2\pi^2 n^4 x^4 - 3\pi n^2 x^2)\exp(-\pi n^2 x^2), \tag{2.1}$$

where $\theta(x)$ is the classical theta function defined by $\theta(x) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 x)$. Then $\xi(s)$ is expressed as the Mellin transform of $\phi(x)$ for every $s \in \mathbb{C}$:

$$\xi(s) = \int_0^\infty \phi(x) x^s \,\frac{dx}{x}.$$

It is well-known that $\phi(x)$ satisfies the functional equation

$$\phi(x^{-1}) = x\phi(x), \tag{2.2}$$

and is of rapid decay as $x \to \infty$ and $x \to 0^+$, namely, $\phi(x) = O(\min(x^{-a}, x^a))$ as $x \to \infty$ and $x \to 0^+$ for every positive real number a ([33, §10.1]).

Definition 2.1. Let α be a real number. Define the function $\xi_\alpha(s;r)$ of two complex variables (s,r) by the integral

$$\xi_\alpha(s;r) = \int_0^\infty \phi(x)\left(\frac{x^{\alpha/2} + x^{-\alpha/2}}{2}\right)^{-r} x^s \,\frac{dx}{x}. \tag{2.3}$$

We call the real parameter α the *weight* of $\xi_\alpha(s;r)$, and the (complex) parameter r the *level* of $\xi_\alpha(s;r)$. We call the function $\xi_\alpha(s;r)$ the *deformation* of weight α and level r (for the Riemann xi-function). Note that this terminology does not have a relation to the theory of modular forms.

The integral on the right-hand side of (2.3) converges absolutely and uniformly on every compact subset of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ for an arbitrary fixed real number α, since $\phi(x)$ is of rapid decay at 0 and ∞.

First, we collect the basic analytic properties of $\xi_\alpha(s;r)$.

THEOREM 2.2. *For real α, the function $\xi_\alpha(s;r)$ has the following properties:*

(1) *It is just the Riemann xi-function at $\alpha = 0$ or $r = 0$:*

$$\xi_\alpha(s;0) = \xi_0(s;r) = \xi(s). \tag{2.4}$$

(2) *For fixed α it is an entire function of two complex variables (s,r).*

(3) *For every $(s,r) \in \mathbb{C}^2$, it satisfies the functional equation*

$$\xi_\alpha(s;r) = \xi_\alpha(1-s;r). \tag{2.5}$$

(4) *For every $(s,r) \in \mathbb{C} \times \mathbb{R}$, it satisfies the real symmetry*

$$\xi_\alpha(s;r) = \overline{\xi_\alpha(\bar{s};r)}. \tag{2.6}$$

(5) *For every $(s,r) \in \mathbb{C}^2$, it satisfies the symmetry*

$$\xi_\alpha(s;r) = \xi_{-\alpha}(s;r). \tag{2.7}$$

(6) *For every $(s,r) \in \mathbb{C}^2$, it satisfies the difference equation*

$$\xi_\alpha(s;r) = \tfrac{1}{2}(\xi_\alpha(s+\alpha/2;r+1) + \xi_\alpha(s-\alpha/2;r+1)). \tag{2.8}$$

(7) *For fixed α and r, it is an entire function of order at most one. More precisely, there exists a constant $C > 0$ such that*

$$\xi_\alpha(s;r) \le \exp(CR\log R)$$

for all $(s,r) \in \mathbb{C}^2$ with $R = |s| + (|\alpha|/2)|r| + 1$. In addition,

$$\xi_\alpha(s;r) \le \exp(C|s|\log|s|)$$

uniformly for $|s| \ge 1$ and $\Re(r) \ge 0$.

REMARK 2.3. For fixed real $r \neq 0$, the two properties (3) and (6) characterize the family $\xi_\alpha(s;r)$. See [31] for details.

Proof of Theorem 2.2. Properties (1) to (5) are proved immediately from definition (2.3) and the functional equation (2.2). Property (6) is a special case of Proposition 2.15 below, which will be proved in Section 4. We prove (7) in Section 3.2. ■

For simplicity, we will concentrate on the case that

α is nonnegative real and r is real

in the rest of this paper. Note that by property (2.7) there is no loss of generality if we restrict α to be a nonnegative real number. The functional equation (2.5) and (2.6) show that the Riemann hypothesis for $\xi_\alpha(s;r)$ should be that all zeros of $\xi_\alpha(s;r)$ lie on the line $\Re(s) = 1/2$.

For an entire function $E(s)$, we define

$$\begin{aligned} A(s) &= A_E(s) = \tfrac{1}{2}(E(s) + E^\vee(s)), \\ B(s) &= B_E(s) = \tfrac{1}{2}i(E(s) - E^\vee(s)). \end{aligned} \tag{2.9}$$

Then (1.3) implies that all zeros of $A(s)$ and $B(s)$ lie on the line $\Re(s) = 1/2$ by Lemma 5 of de Branges [14] (see also Lemma 2.2 of Lagarias [22]). The difference equation (2.8) is nothing but (2.9) for $A(s) = \xi_\alpha(s;r)$ with $E(s) = \xi_\alpha(s+\alpha/2;r+1)$, since

$$\xi_\alpha(s-\alpha/2;r+1) = \xi_\alpha((1-s)+\alpha/2;r+1) = E^\vee(s)$$

by the functional equation (2.5) and (2.6).

If the zero-free region of $\xi_\alpha(s;r)$ is sufficiently wide, it implies the Riemann hypothesis for lower level deformations:

Theorem 2.4. *Let α be a positive real number, and let r be a real number. Suppose that $\xi_\alpha(s;r)$ has no zeros on the closed right half-plane $\Re(s) \geq (1+\alpha)/2$. Then $\xi_\alpha(s;r-n)$ satisfies the Riemann hypothesis for every positive integer n: all zeros of $\xi_\alpha(s;r-n)$ lie on the line $\Re(s) = 1/2$ for every positive integer n.*

The deformation $\xi_\alpha(s;0)$ of level zero was just the Riemann xi-function $\xi(s)$ (Theorem 2.2(1)). This yields the following corollary immediately.

Corollary 2.5. *Suppose that there exists a positive real number α and a positive integer n such that $\xi_\alpha(s;n)$ has no zeros on the closed right half-plane $\Re(s) \geq (1+\alpha)/2$. Then the Riemann hypothesis holds for $\xi(s)$.*

Remark 2.6. A computer calculation shows that $\xi_\alpha(s;1)$ has no zeros in the region $\Re(s) \geq (1+\alpha)/2$, $|\Im(s)| \leq T$ for $0 < \alpha \leq 2$ if $T > 0$ is small. However, we also observed that $\xi_{10}(s;1)$ has a zero with $\Re(s) \geq 11/2$. These observations suggest the existence of the "right" weight α_1 for $\xi(s)$. See also the last paragraph of Section 2.5.

As the contraposition of Corollary 2.5, we obtain

Corollary 2.7. *Suppose that the Riemann hypothesis for $\xi(s)$ is false. Then $\xi_\alpha(s;n)$ has at least one zero on the right-half plane $\Re(s) \geq (1+\alpha)/2$ for every positive integer n.*

Remark 2.8. Needless to say, the existence of a zero of $\xi_\alpha(s;n)$ with $\Re(s) \geq (1+\alpha)/2$ for some positive integer n does not imply the falsity of the Riemann hypothesis for $\xi(s)$.

It is well-known that the Riemann xi-function $\xi(s) = \xi_\alpha(s;0)$ has no zeros in the right half-plane $\Re(s) \geq 1$. This implies the following corollary by applying Theorem 2.4 to $r = 0$.

Corollary 2.9. *Suppose $\alpha \geq 1$. Then $\xi_\alpha(s;-n)$ satisfies the Riemann hypothesis for every positive integer n. Further, assuming the Riemann hypothesis for $\xi(s)$, the deformation $\xi_\alpha(s;-n)$ satisfies the Riemann hypothesis for every nonnegative real number α and every positive integer n.*

Corollary 2.9 provides infinitely many examples of entire functions satisfying the Riemann hypothesis unconditionally. Using equality (2.8), we find that every deformation of a negative integer level is a linear combination of $\xi(s)$ in general (see (7.5) in Section 7.5). For example, we have

(2.10)
$$\xi_\alpha(s;-1) = \tfrac{1}{2}(\xi(s+\alpha/2)+\xi(s-\alpha/2)),$$
$$\xi_\alpha(s;-2) = \tfrac{1}{4}(\xi(s+\alpha)+2\xi(s)+\xi(s-\alpha)),$$
$$\xi_\alpha(s;-3) = \tfrac{1}{8}(\xi(s+3\alpha/2)+3\xi(s+\alpha/2)+3\xi(s-\alpha/2)+\xi(s-3\alpha/2)).$$

The level -1 case was studied by many authors [32, 20, 22, 21, 24], and the cases of $\xi_\alpha(s,-n)$ are also proved by de Bruijn [18, Theorem 9] (see also Adams–Cardon [1]. They construct a sequence of functions having only real zeros starting from a function having only real zeros by using Pólya's results [30, Hilfssatz II]).

Here we mention a connection of $\xi_\alpha(s;r)$ and differential operators of infinite order. This was pointed out by the referee. If r is a negative integer, we obtain
$$\xi_\alpha(s;r) = (\cosh(\alpha D/2))^{-r}\xi(s),$$
where $D = d/ds$ is differentiation with respect to s. By this formula, we find that if all zeros of $\xi(s)$ lie in the strip $|\Re(s)-1/2| \le \Delta$ and $r=-n$ for $n\in\mathbb{N}$, then all zeros of $\xi_\alpha(s;-n) = (\cosh(\alpha D/2))^n\xi(s)$ lie in the strip $|\Re(s)-1/2| \le \sqrt{\max\{\Delta^2-n(\alpha/2)^2,0\}}$ by [18, Theorem 12]. Therefore, it is an interesting problem to study the zeros of $g(D)\xi(s)$ for a family of functions $g(iz)$ in the Laguerre–Pólya class. See Borcea [4], Cardon–de Gaston [8], Craven–Csordas [13], and Levin [25, Chapters 8 and 9] for examples.

2.2. Companion of a family of deformations

Definition 2.10. We call an entire function $E(s)$ a *structure function* if it satisfies (1.3). Moreover, we say that a structure function $E(s)$ is *strict* if it has no zeros on the line $\Re(s)=1/2$, and *symmetric* if $\overline{E(\bar{s})} = E(s)$.

The class of entire functions satisfying a condition similar to (1.3) has a long history. Entire functions $E(z)$ satisfying $|E(z)| > |E(\bar{z})|$ for $\Im(z) > 0$ were studied by de Branges [15] in connection with his theory of Hilbert spaces of entire functions. An entire function satisfying this condition is called a structure function, a de Branges function or a de Branges structure function by Lagarias [22, 23]. The terminology of Definition 2.10 is inspired by those papers.

Entire functions satisfying $|E(z)| < |E(\bar{z})|$ for $\Im(z) > 0$ and not vanishing in the closed lower half-plane $\Im(z) \le 0$ were studied in detail in Lecture 27 of Levin [26], and also in Article 6 of Ahiezer–Krein [2]. An entire function satisfying these conditions is called a *Hermite–Biehler function* in the literature. By the change of variable $z \mapsto i(1/2-s)$, the upper half-plane $\Im(z) > 0$ is transformed into the right half-plane $\Re(s) > 1/2$, and the real axis of the z-plane with the direction from $-\infty$ to ∞ is transformed into the vertical line $\Re(s) = 1/2$ of the s-plane with the direction from $1/2 - i\infty$ to $1/2 + i\infty$. In this way, the above classical concepts and results in the z-variable can be interpreted in the s-variable. Condition (1.3) implies that $E(s)$ has no zeros with $\Re(s) > 1/2$, but the nonvanishing of $E(s)$ in this half-plane is not sufficient for (1.3). Note that de Branges allows zeros on the real axis, but Levin does not (he imposes "strictness").

For a symmetric and strict structure function $E(s)$, we have

$$A(s) = \tfrac{1}{2}(E(s) + E(1-s)), \quad B(s) = \tfrac{1}{2}i(E(s) - E(1-s))$$

such that $E(s) = A(s) - iB(s)$, and $A(s)$ and $B(s)$ take real values on the line $\Re(s) = 1/2$. The theory of de Branges asserts that all zeros of $A(s)$ and $B(s)$ lie on the line $\Re(s) = 1/2$ and are simple, and the zeros of $A(s)$ and $B(s)$ are interlaced. These properties show that the functions $A(s)$ and $B(s)$ are possible analogues of the cosine and sine functions ([14]).

The difference equation (2.8) is nothing but formula (2.9) for $A(s) = \xi_\alpha(s;r)$ and $E(s) = \xi_\alpha(s+\alpha/2; r+1)$, since we have the functional equation (2.5) and the real symmetry (2.6). Here is our central result:

THEOREM 2.11. *Let α be a positive real number, and let r be a real number. Define*

$$E_{\alpha,r}(s) := \xi_\alpha(s + \alpha/2; r+1) \tag{2.11}$$

and

$$A_{\alpha,r}(s) := \tfrac{1}{2}(E_{\alpha,r}(s) + E_{\alpha,r}(1-s)) \ (= \xi_\alpha(s;r)),$$
$$B_{\alpha,r}(s) := \tfrac{1}{2}i(E_{\alpha,r}(s) - E_{\alpha,r}(1-s))$$

so that

$$E_{\alpha,r}(s) = A_{\alpha,r}(s) - iB_{\alpha,r}(s).$$

Suppose that $\xi_\alpha(s;r+1)$ does not vanish on the closed right half-plane $\Re(s) \geq (1+\alpha)/2$. Then $E_{\alpha,r}(s)$ is a symmetric and strict structure function.

In what follows, we often use the notation

$$\xi_\alpha^+(s;r) := A_{\alpha,r}(s) = \tfrac{1}{2}(\xi_\alpha(s+\alpha/2; r+1) + \xi_\alpha(s-\alpha/2; r+1)), \tag{2.12}$$

$$\xi_\alpha^-(s;r) := B_{\alpha,r}(s) = \tfrac{1}{2}i(\xi_\alpha(s+\alpha/2; r+1) - \xi_\alpha(s-\alpha/2; r+1)). \tag{2.13}$$

By definition, $\xi_\alpha^+(s;r)$ and $\xi_\alpha^-(s;r)$ are regarded as analogs of the cosine and sine functions with the "exponential function" $\xi_\alpha(s+\alpha/2; r+1)$. In this sense, $\xi_\alpha^-(s;r)$ is a *companion* of $\xi_\alpha^+(s;r)$. Referring to $\sin z = (-\cos z)'$ or

Laguerre's theorem ([26, p. 28]), a natural companion of $\xi(s)$ is its derivative $\xi'(s)$. We have

$$\lim_{\alpha\to 0^+}\frac{2}{\alpha}\xi_\alpha^-(s;r)=i\xi'(s) \tag{2.14}$$

by the formula

$$\xi_\alpha^-(s;r)=i\int_0^\infty \phi(x)\left(\frac{x^{\alpha/2}-x^{-\alpha/2}}{x^{\alpha/2}+x^{-\alpha/2}}\right)\left(\frac{x^{\alpha/2}+x^{-\alpha/2}}{2}\right)^{-r}x^s\,\frac{dx}{x},$$

which is derived from definition (2.3) and (2.13). Formula (2.14) shows that $\xi_\alpha^-(s;r)$ is a deformation of $\xi'(s)$; also $\xi_\alpha^+(s;r)$ is a deformation of $\xi(s)$ as in (2.4). See Figure 2 for a comparison between $\xi_2^-(s;1)$ and $\xi'(s)$.

As a corollary of Theorem 2.11, we obtain the following result.

Theorem 2.12. *Let α be a positive real number, and let r be a real number. Suppose that $\xi_\alpha(s;r)$ has no zeros on the closed right half-plane $\Re(s)\ge(1+\alpha)/2$. Then for every positive integer n all zeros of $\xi_\alpha^\pm(s;r-n)$ lie on the line $\Re(s)=1/2$ and are simple. Moreover the zeros of $\xi_\alpha^+(s;r-n)$ and $\xi_\alpha^-(s;r-n)$ are interlaced for every positive integer n.*

Needless to say, Theorem 2.12 includes Theorem 2.4, and it yields the following result immediately, as well as Corollary 2.9.

Corollary 2.13. *Suppose $\alpha\ge 1$. Then $\xi_\alpha^\pm(s;-n)$ satisfies the Riemann hypothesis and all its zeros are simple for every positive integer n. Further, the same assertions hold for every nonnegative real number α and every positive integer n if we assume that the Riemann hypothesis holds for $\xi(s)$.*

In addition to examples of Corollary 2.9, Corollary 2.13 provides infinitely many other examples of entire functions satisfying the Riemann hypothesis and the simplicity of zeros unconditionally. (Note that de Bruijn [18, Theorem 9] does not imply the simplicity of zeros.) Using (2.8) and definitions (2.12) and (2.13), we find that $\xi_\alpha^-(s;-n)$ is also a linear combination of $\xi(s)$ in general (see (7.6) in Section 7.5). For example,

(2.15)

$$\xi_\alpha^-(s;-1)=\tfrac{1}{2}i(\xi(s+\alpha/2)-\xi(s-\alpha/2)),$$
$$\xi_\alpha^-(s;-2)=\tfrac{1}{4}i(\xi(s+\alpha)-\xi(s-\alpha)),$$
$$\xi_\alpha^-(s;-3)=\tfrac{1}{8}i(\xi(s+3\alpha/2)+\xi(s+\alpha/2)-\xi(s-\alpha/2)-\xi(s-3\alpha/2)),$$
$$\xi_\alpha^-(s;-4)=\tfrac{1}{16}i(\xi(s+2\alpha)+2\xi(s+\alpha)-2\xi(s-\alpha)-\xi(s-2\alpha)).$$

Here we mention the work of Lagarias [22, 23] (see also Li [28]). For $(h,\theta)\in\mathbb{R}\times[0,2\pi)$, he introduced

$$A_h^\theta(s)=\tfrac{1}{2}(\cos\theta)(\xi(s+h)+\xi(s-h))+\tfrac{1}{2}i(\sin\theta)(\xi(s+h)-\xi(s-h)),$$
$$B_h^\theta(s)=-\tfrac{1}{2}i(\sin\theta)(\xi(s+h)+\xi(s-h))+\tfrac{1}{2}i(\cos\theta)(\xi(s+h)-\xi(s-h))$$

in [22] referring to the theory of de Branges. The function $A_h^\theta(s)$ is a kind of continuous deformation of the Riemann xi-function, since $A_0^0(s) = \xi(s)$. He studied the zeros of $A_h^\theta(s)$ and $B_h^\theta(s)$ in detail, and proved that all their zeros lie on the line $\Re(s) = 1/2$ and are simple, and they interlace if $|h| \geq 1/2$ (Theorem 2.1 of [22]). Moreover, he showed that $A_h^\theta(s)$ and $B_h^\theta(s)$ have a limiting normalized zero spacing distribution of k consecutive zeros which corresponds to a delta measure with equal spacings of size one, for any fixed natural number k (Theorem 4.1 of [22]). The same results hold for every real number h with $0 < |h| < 1/2$ under the Riemann hypothesis for $\xi(s)$. (See Li [28] for unconditional results.) Note that

$$A_h^0(s) = \tfrac{1}{2}(E_h(s) + E_h(1-s)), \quad B_h^0(s) = \tfrac{1}{2}i(E_h(s) - E_h(1-s))$$

for $E_h(s) = \xi(s+h)$. In addition, Lagarias studied the entire function

$$E^L(s) := \xi(s) + \xi'(s)$$

in [23] together with

$$A^L(s) = \tfrac{1}{2}(E^L(s) + E^L(1-s)) = \xi(s),$$
$$B^L(s) = \tfrac{1}{2}i(E^L(s) - E^L(1-s)) = i\,\xi'(s).$$

He proved that $E^L(s)$ is a strict structure function if and only if the Riemann hypothesis holds for $\xi(s)$ and all its zeros are simple. See also Lemma 6.1 of [22].

We have $\xi_\alpha^+(s;-1) = A_{\alpha/2}^0(s)$ and $\xi_\alpha^-(s;-1) = B_{\alpha/2}^0(s)$ by (2.10) and (2.15). In this sense $\xi_\alpha^+(s;r)$ and $\xi_\alpha^-(s;r)$ are generalizations of $A_h^0(s)$ and $B_h^0(s)$, respectively. In particular, the results of [22] imply that all zeros of $\xi_\alpha^\pm(s;-1)$ lie on the line $\Re(s) = 1/2$ and are simple. For a positive integer n, the distribution of the zeros of $\xi_\alpha(s;-n)$ is similar to the one of $A_h^0(s)$. In fact, all zeros of $\xi_\alpha(s;-n)$ are simple for $\alpha \geq 1$ by Corollary 2.13, and the set of their imaginary parts has the limiting normalized spacing distribution similar to that of $A_h^0(s)$ for every $\alpha > 0$ by imitating the method of [22], under the Riemann hypothesis for $\xi(s)$. In addition, we find that

$$(1 + 2/\alpha)\xi_\alpha(s + \alpha/2; r) + (1 - 2/\alpha)\xi_\alpha(s - \alpha/2; r)$$

is a deformation of $E^L(s)$ by (2.4) and (2.14). However $E^L(s)$ is understood more naturally in the context of deformations $E_f(s) = \int_0^\infty \phi(x) f(x) x^{s-1}\,dx$ in the follow-up paper [31], where f is a test function on $(0,\infty)$ satisfying $f(x) + \overline{f(x^{-1})} = 2$ and $f(x) = O(x^A)$ as $x \to \infty$ for some $A \geq 0$.

2.3. Spectral interpretations for the zeros of deformations. In this part, we mention the spectral interpretation of zeros of $\xi_\alpha(s;r)$ via the de Branges Hilbert spaces, relying on Theorem 2.11. For the general theory of the de Brange Hilbert spaces in the variable $z = i(1/2 - s)$, see the book of de Branges [15]; see also Lagarias [22, 23] for a relation to L-functions. We describe the de Branges Hilbert spaces in the s-variable. For other interesting approaches to the spectral interpretation of the zeros of the Riemann zeta function related to de Branges Hilbert spaces, see Burnol [5, 6, 7] and their references.

The *de Branges Hilbert space* $\mathcal{H}(E)$ attached to the structure function E is the space of all entire functions F such that F/E and $F^\vee/E$ belong to the Hardy space $H^2(D)$, where $F^\vee(s) = \overline{F(1-\bar{s})}$ and $D = \{s \in \mathbb{C} \mid \Re(s) > 1/2\}$. The scalar product in $\mathcal{H}(E)$ is defined by

$$\langle F, G\rangle_E = \int_{-\infty}^{\infty} \frac{F(1/2+it)\overline{G(1/2+it)}}{|E(1/2+it)|^2}\, dt.$$

Let $(M_E, \mathcal{D}_E)$ be the multiplication operator $F(s) \mapsto i(1/2-s)F(s)$ defined on the natural domain $\mathcal{D}_E = \{F \in \mathcal{H}(E) \mid i(1/2-s)F(s) \in \mathcal{H}(E)\}$. The operator $(M_E, \mathcal{D}_E)$ is a closed symmetric operator, and has the special self-adjoint extension $(M_E(A), \mathcal{D}_E(A))$ specified by the entire function $A(s)$ of (2.9), if the domain $\mathcal{D}_E$ is dense in $\mathcal{H}(E)$. Suppose that E is strict and $\mathcal{D}_E$ is dense in $\mathcal{H}(E)$. Then the self-adjoint extension $(M_E(A), \mathcal{D}_E(A))$ has pure discrete simple spectrum given by the imaginary parts of the zeros of $A(s)$. The denseness of $\mathcal{D}_E$ can be read off from a property of zeros of E, by Corollary 2 of Baranov [3].

THEOREM 2.14. *Let α be a positive real number, and let r be a real number. Let $E = E_{\alpha,r}$ be the function defined in* (2.11). *Suppose that $\xi_\alpha(s;r)$ has no zeros on the right half-plane $\Re(s) \ge (1+\alpha)/2$. Then the domain $\mathcal{D}_E$ of the multiplication operator M_E is dense in the Hilbert space $\mathcal{H}(E)$.*

Hence the zeros of $\xi_\alpha(s;r)$ admit a spectral interpretation under the assumption that $\xi_\alpha(s;r+1)$ has no zeros with $\Re(s) \ge (1+\alpha)/2$ according to the theory of de Branges Hilbert spaces.

2.4. Structures for the level of deformations. For the level parameter r, we have the following general discrete relation "from high level to low level". We call it the *descent formula.*

PROPOSITION 2.15 (descent formula). *Let α be a nonnegative real number, and let r be a real number. Let n be a positive integer. Then*

$$\xi_\alpha^\pm(s;r) = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}\xi_\alpha^\pm(s-\alpha n/2+\alpha j; r+n). \tag{2.16}$$

Conversely, we have the following relation "from low level to high level", which we call the *ascent formula*:

Proposition 2.16 (ascent formula). *Let α be a nonnegative real number, and let r be a real number. Let λ be a positive real number. Then*

$$\xi_\alpha^\pm(s;r) = \frac{1}{2\pi i}\frac{2^\lambda}{\alpha}\int_{(c)} \xi_\alpha^\pm(s-z;r-\lambda)B\left(\frac{\lambda}{2}+\frac{z}{\alpha},\frac{\lambda}{2}-\frac{z}{\alpha}\right)dz, \tag{2.17}$$

for every $s \in \mathbb{C}$, where (c) is the vertical line $\Re(s) = c$ with the direction from $c - i\infty$ to $c + i\infty$, $|c| < \alpha\lambda/2$ and $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the beta function.

Recall that the deformation $\xi_\alpha(s;-1)$ of level -1 satisfies the Riemann hypothesis for every $\alpha \geq 1$ unconditionally. Varying the level parameter r from $r = -1$ to $r = 0$, we obtain the following sufficient condition for the Riemann hypothesis to hold for $\xi(s)$.

Theorem 2.17. *Let $\alpha \geq 1$ be fixed real. Suppose that $\xi_\alpha((1+\alpha)/2 + i\mathbb{R};r) \neq 0$ for every $r \in [0,1]$. Then the Riemann hypothesis holds for $\xi(s)$.*

More precisely, suppose that there exists $0 < r^* \leq 1$ which may possibly depend on α such that $\xi_\alpha((1+\alpha)/2+i\mathbb{R};r) \neq 0$ for every $0 \leq r \leq r^*$. Then the Riemann hypothesis holds for $\xi_\alpha(s;r)$ for every $-1 \leq r \leq -1+r^*$.

Remark 2.18. The proof of Theorem 2.17 uses the fact that $\xi(s) \neq 0$ when $\Re(s) \geq 1$. The restriction $\alpha \geq 1$ is required by this zero-free region.

2.5. A conjecture for the zeros of deformations. Corollary 2.5 shows that the location of zeros of $\xi_\alpha(s;1)$ in a right half-plane is closely related to the Riemann hypothesis for $\xi(s)$. We have

$$\xi_\alpha(s;1) = \frac{1}{\alpha}\int_{-\infty}^{\infty} \xi(s-iv)\,\frac{dv}{\cosh(\pi v/\alpha)}$$

as a special case of Corollary 7.1 below. That is, $\xi_\alpha(s;1)$ is an average of $\xi(s)$ along a vertical line with respect to the hyperbolic cosine distribution. We know that $\xi(s)$ does not vanish on the right half-plane $\Re(s) \geq 1$. Therefore it is not unnatural to expect that $\xi_\alpha(s;1)$ does not vanish on the right half-plane $\Re(s) \geq (1+\alpha)/2$ if $\alpha \geq 1$. In this sense, Corollary 2.5 is interpreted as follows: *If $\xi_\alpha(s;1)$ inherits the standard zero-free region of $\xi(s)$ coming from the Euler product (for $\alpha \geq 1$), then the Riemann hypothesis holds for $\xi(s)$.*

As the first stage for the study of the zero-free region of $\xi_\alpha(s;1)$, we present the following result for the deformation of weight two and level one.

Theorem 2.19. *The deformation $\xi_2(s;1)$ of weight two and level one does not vanish on the closed right half-plane $\Re(s) \geq 11$.*

The zero-free region obtained is far from the desired zero-free region $\Re(s) \geq 3/2$ for $\xi_2(s;1)$ required in Theorem 2.4, but we do not know whether

$\xi_2(s;1)$ has a zero in $3/2 \le \Re(s) < 11$. The restriction $\alpha = 2$ here is due to a technical reason (see the second remark of Section 6).

Using (2.16) we obtain the formal equality

$$\xi(s) = \lim_{n\to\infty} 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \xi_\alpha(s - \alpha n/2 + \alpha j; n), \tag{2.18}$$

where α may depend on each n. This awakens an interest to study the limit behavior of $\xi_\alpha(s;r)$ as $r \to \infty$ (see Section 7.6 for various limit formulas). Corollary 2.9 and the formal equality (2.18) lead to the following natural conjecture. We call it the *Area Riemann Hypothesis* or ARH for short.

Conjecture 2.20 (Area Riemann Hypothesis (ARH)). There exists a connected subset $\mathfrak{D} \subset \mathbb{R} \times \mathbb{R}$ containing the origin $(0,0)$ such that all zeros of $\xi_\alpha(s;r)$ lie on the line $\Re(s) = 1/2$ for every $(\alpha, r) \in \mathfrak{D}$. Moreover, we expect that every section

$$\mathfrak{D}_n := \{(\alpha, r) \in \mathfrak{D}\,;\, r = n\} \quad (n \in \mathbb{Z})$$

contains a nonempty segment $(-a, a) \times \{n\}$ depending on n.

At present, it is not clear which arithmetic properties $\xi_\alpha(s;r)$ inherits from $\xi(s)$. It is natural to expect that the desired zero-free region for $\xi_\alpha(s;r)$ arising in Theorem 2.12 will be derived from unknown arithmetic properties of $\xi_\alpha(s;r)$ (which should be inherited from $\xi(s)$) with the determination of the "right" weight α_1 of $\xi_\alpha(s;r)$ (see Remark 2.6). Moreover, it is possible to expect that the "seriousness" of the generalized Riemann hypothesis for $\xi(s)$ is measured by the volume of the set

$$\mathfrak{D} = \{(\alpha, r) \in \mathbb{R} \times \mathbb{R} \mid \xi_\alpha(s;r) \text{ satisfies RH}\}$$

with respect to an appropriate measure on $\mathbb{R} \times \mathbb{R}$, and the structure of $\mathfrak{D}$ is determined by some arithmetical properties of $\xi(s)$.

3. Proofs of Theorems 2.2 to 2.14

3.1. Preliminaries. For Theorems 2.4 and 2.12, it suffices to prove Theorem 2.11, since the latter implies Theorem 2.12 by the proposition below, and Theorem 2.4 is a part of Theorem 2.12 as mentioned in Section 2.

Proposition 3.1 (de Branges). *Let $E(s)$ be a strict structure function. Define $A(s)$ and $B(s)$ by (2.9) such that $E(s) = A(s) - iB(s)$, and $A(s)$ and $B(s)$ are real-valued on the line $\Re(s) = 1/2$. In addition, define the continuous real-valued function $\vartheta(t)$ by*

$$E(1/2 + it) = |E(1/2 + it)| \exp(i\vartheta(t))$$

such that $0 \le \vartheta(0) < 2\pi$, and extend it to all $t \in \mathbb{R}$ by continuity. Then

(1) *all zeros of $A(s)$ and $B(s)$ lie on the line $\Re(s) = 1/2$,*

(2) $\vartheta(t) = \Im(\log E(1/2+it))$, *and* $\vartheta(t)$ *is a strictly increasing function of* t,
(3) *all zeros of* $A(s)$ *and* $B(s)$ *are simple,*
(4) *the zeros of* $A(s)$ *and* $B(s)$ *are interlaced.*

Proof. See Lemma 5 of [14], or Lemma 2.2 of [22]. ∎

The most important part of the proof of Theorem 2.11 is the following fact which was proved in [24] as a refinement of §10.23 of Titchmarsh [33], which can be traced back to a result of Pólya [30, Hilfssatz II].

Proposition 3.2 (Theorem 4 of [24]). *Let* $E(s)$ *be an entire function of genus zero or one, satisfying the following conditions:*

(i) *It is real on the real axis, and satisfies the functional equation* $E(s) = \pm E(1-s)$ *for some choice of sign.*
(ii) *All zeros of* $E(s)$ *lie in the strip* $|\Re(s)-1/2| < a$ *for some* $a > 0$.

Then, for every real number $c \geq a$,

$$|E(s+c)| > |E(s-c)| \quad \text{for } \Re(s) > 1/2.$$

Proof. See the proof of Theorem 4 in [24] and also Section 3 of Levin [25, Chapter VII]. ∎

Compare Proposition 3.2 with [18, Theorem 8].

In addition to Proposition 3.2, for the proof of Theorem 2.11 we need the fundamental analytic properties of $\xi_\alpha(s;r)$ stated in Theorem 2.2. However, Theorem 2.2(7) remains to be proved, so we now prove it.

3.2. Proof of Theorem 2.2(7). Set

$$I(s;r) = \int_1^\infty \phi(x) w(x;r) x^s \,\frac{dx}{x} \quad \text{with} \quad w(x;r) = \left(\frac{x^{\alpha/2}+x^{-\alpha/2}}{2}\right)^{-r}.$$

Then

$$\xi_\alpha(s;r) = I(s;r) + I(1-s;r) \tag{3.1}$$

by the functional equation (2.2) and $w(x^{-1};r) = w(x;r)$. By the series expansion (2.1), there exists a constant $C_1 > 0$ such that

$$\phi(x) \leq C_1 x^4 \exp(-\pi x^2)$$

for $1 \leq x < \infty$. On the other hand, for every $r \in \mathbb{C}$,

$$|w(x;r)| \leq x^{|\alpha r|/2}$$

for $1 \leq x < \infty$. Therefore

$$|\phi(x)w(x;r)| \leq C_1 x^4 \exp(-\pi x^2) \cdot x^{|\alpha r|/2}$$

for $1 \le x < \infty$. Hence

$$|I(s;r)| \le \int_1^\infty |\phi(x)w(x;r)|\,|x^s|\,\frac{dx}{x} \le C_1 \int_1^\infty e^{-\pi x^2} x^{|s|+|\alpha r|/2+4}\,\frac{dx}{x}$$
$$\le C_1 \int_0^\infty e^{-\pi x^2} x^{|s|+|\alpha r|/2+4}\,\frac{dx}{x}$$
$$= \tfrac{1}{2} C_1 \pi^{-|s|/2-|\alpha r|/4-2} \Gamma(|s|/2+|\alpha r|/4+2).$$

As $\Gamma(X) \le \exp(X \log X)$ for $X \ge 1$, there exists a constant $C_3 > 0$ such that

$$|I(s;r)| \le \exp(C_3 R \log R) \tag{3.2}$$

for $R = |s| + (|\alpha|/2)|r| + 1$. Estimate (3.2) implies the first half of (7) by (3.1).

If $\Re(r) \ge 0$, we have $|w(x;r)| \le 1$ for $x \ge 0$, since $(x^{\alpha/2} + x^{-\alpha/2})/2 \ge 1$. Using this estimate instead of $|w(x;r)| \le x^{|\alpha r|/2}$, we obtain the other half of (7). ■

3.3. Proof of Theorem 2.11. By Theorem 2.2, $\xi_\alpha(s;r+1)$ is an entire function of order at most one satisfying the functional equation (2.5) and (2.6). Since α and r are real, $\xi_\alpha(s;r+1)$ is real on the real axis. Suppose that $\xi_\alpha(s;r+1)$ has no zeros on the closed right half-plane $\Re(s) \ge (1+\alpha)/2$. Then all zeros of $\xi_\alpha(s;r+1)$ lie in the vertical strip $|\Re(s) - 1/2| < \alpha/2$ by the functional equation (2.5). Hence

$$|\xi_\alpha(s+\alpha/2;r+1)| > |\xi_\alpha(s-\alpha/2;r+1)| \quad \text{for } \Re(s) > 1/2$$

by Theorem 2.2 and Proposition 3.2. This implies that $E_{\alpha,r}(s) = \xi_\alpha(s+\alpha/2; r+1)$ is a de Brange function, since

$$E_{\alpha,r}(1-s) = \xi_\alpha((1-s)+\alpha/2;r+1) = \xi_\alpha(s-\alpha/2;r+1)$$

by (2.5), and $|E_{\alpha,r}(s)| = |E_{\alpha,r}(\bar{s})|$ by (2.6). Note that $E_{\alpha,r}(s)$ has no zeros on the line $\Re(s) = 1/2$ by assumption, and is real on the real axis. Hence $E_{\alpha,r}(s)$ is strict and symmetric. ■

3.4. Proof of Theorem 2.4 and 2.12. Suppose that $\xi_\alpha(s;r)$ does not vanish on the closed right half-plane $\Re(s) \ge (1+\alpha)/2$. Then all zeros of $\xi_\alpha^\pm(s;r-1)$ are simple zeros lying on the line $\Re(s) = 1/2$, and they are interlaced by Theorem 2.11 and Proposition 3.1, since

$$A_{\alpha,r-1}(s) = \tfrac{1}{2}\big(E_{\alpha,r-1}(s) + \overline{E_{\alpha,r-1}(1-\bar{s})}\big) = \xi_\alpha^+(s;r-1),$$
$$B_{\alpha,r-1}(s) = \tfrac{1}{2}i\big(E_{\alpha,r-1}(s) - \overline{E_{\alpha,r-1}(1-\bar{s})}\big) = \xi_\alpha^-(s;r-1)$$

for $E_{\alpha,r-1}(s) = \xi_\alpha(s+\alpha/2;r)$. In particular $\xi_\alpha(s;r-1) = \xi_\alpha^+(s;r-1)$ has no zeros with $\Re(s) \ge (1+\alpha)/2$. This implies that all zeros of $\xi_\alpha^\pm(s;r-2)$

are simple zeros lying on the line $\Re(s) = 1/2$, and they are interlaced by Theorem 2.11 and Proposition 3.1 again. Iterating the above process, we obtain the results of Theorem 2.12, which implies Theorem 2.4. ■

3.5. Proof of Theorem 2.14. Let E be a structure function. By Theorem 29 of [15] and Corollary 2 of [3], the domain $\mathcal{D}_E$ of multiplication by $i(1/2 - s)$ is not dense in $\mathcal{H}(E)$ if and only if

$$\sum_{\rho}(1/2 - \Re(\rho)) < \infty, \tag{3.3}$$

where ρ runs over all zeros of E (be careful with the change of variable $z = i(1/2 - s)$). The condition (3.3) means that the zeros of E tend to the line $\Re(s) = 1/2$ (from the left) sufficiently fast whenever $\mathcal{D}_E$ is not dense in $\mathcal{H}(E)$.

All zeros of $\xi_\alpha(s;r)$ lie in the vertical strip $|\Re(s) - 1/2| < \alpha/2$ by the functional equation (2.5) if $\xi_\alpha(s;r) \neq 0$ for $\Re(s) \geq (1+\alpha)/2$. Hence for Theorem 2.14 it is sufficient to show that $\xi_\alpha(s;r)$ has infinitely many zeros, since (3.3) is impossible for $E_{\alpha,r}(s) = \xi_\alpha(s+\alpha/2;r)$ by (2.5) if $\xi_\alpha(s;r) \neq 0$ for $\Re(s) \geq (1+\alpha)/2$ and $\xi_\alpha(s;r)$ has infinitely many zeros.

Suppose that $\xi_\alpha(s;r)$ has only finitely many zeros, and at least one. Then $\xi_\alpha(s;r)$ should be of the form $P(s)\exp(\lambda s)$ for some polynomial $P \not\equiv 1$ and some complex number λ by the Weierstrass factorization theorem, since $\xi_\alpha(s;r)$ is an entire function of order ≤ 1. In addition, λ must be zero by the functional equation. Therefore $\xi_\alpha(s;r)$ is a nonzero and nonconstant polynomial. In particular it is not bounded in the strip $|\Re(s) - 1/2| < \alpha/2$. On the other hand,

$$|\xi_\alpha(s;r)| \leq \int_0^\infty |\phi(x)| \left(\frac{x^{\alpha/2} + x^{-\alpha/2}}{2}\right)^{-r} x^\sigma \frac{dx}{x} = O(1)$$

for every vertical strip with finite width, where $\sigma = \Re(s)$. This is a contradiction. If $\xi_\alpha(s;r)$ has no zeros, it should be a constant, since it is of order ≤ 1 and satisfies (2.5). This also contradicts the formula for real σ:

$$\xi_\alpha(\sigma;r) = \int_1^\infty \phi(x) \left(\frac{x^{\alpha/2} + x^{-\alpha/2}}{2}\right)^{-r} (x^\sigma + x^{1-\sigma}) \frac{dx}{x},$$

since $\phi(x)$ is a positive strictly decreasing function on $(1,\infty)$. Hence $\xi_\alpha(s;r)$ has infinitely many zeros. ■

3.6. Note on the zeros of deformations. Suppose that $E_{\alpha,r}(s)$ defined in (2.11) does not vanish on the line $\Re(s) = 1/2$. We define the *phase function* $\vartheta_\alpha(t;r)$ by

$$E_{\alpha,r}(1/2+it) = |E_{\alpha,r}(1/2+it)| \exp(i\vartheta_\alpha(t;r))$$

such that $0 \le \vartheta_\alpha(0;r) < 2\pi$, and extend it to all $t \in \mathbb{R}$ by continuity. Then the location of the zeros of $\xi_\alpha(s;r)$ on the critical line $\Re(s) = 1/2$ is determined by the phase function $\vartheta_\alpha(t;r)$, since

$$\xi_\alpha(1/2 + it; r) = |E_{\alpha,r}(1/2 + it)| \cos(\vartheta_\alpha(t;r)).$$

Suppose that $\vartheta_\alpha(t;r)$ decomposes into

$$\vartheta_\alpha(t;r) = \vartheta_\alpha^0(t;r) + \vartheta_\alpha^\infty(t;r),$$

where the "nonarchimedean part" $\vartheta_\alpha^0(t;r)$ comes from a hypothetical "Dirichlet series" and the "archimedean part" $\vartheta_\alpha^\infty(t;r)$ comes from a hypothetical "Γ-factor". Then it is natural to expect that the behavior of $\vartheta_\alpha^\infty(t;r)$ is extremely regular. If the contribution of the nonarchimedean part $\vartheta_\alpha^0(t;r)$ is small, the behavior of $\vartheta_\alpha(t;r)$ is almost determined by $\vartheta_\alpha^\infty(t;r)$ only, and it implies that the normalized zero spacing distribution of $\xi_\alpha(s;r)$ is *trivial*, far from GUE. Hence, at least for $r = 0$ (and for $r > 0$), the phase function $\vartheta_\alpha(t;r)$ behaves in a more complicated way than $\vartheta_\alpha(t;-1)$ and $\vartheta_\alpha(t;-r)$.

4. Proof of structures for the level of deformations

4.1. Proof of Proposition 2.15. This is a simple application of the binomial theorem. Let n be a positive integer. Put

$$\phi^+(x) = \phi(s) \quad \text{and} \quad \phi^-(x) = i\phi(x)\frac{x^{\alpha/2} - x^{-\alpha/2}}{x^{\alpha/2} + x^{-\alpha/2}} \tag{4.1}$$

referring to definitions (2.3), (2.12) and (2.13). Then

$$\begin{aligned}
\xi_\alpha^\pm(s;r) &= \int_0^\infty \phi^\pm(x)\left(\frac{2}{1+x^\alpha}\right)^r x^{s+\alpha r/2}\,\frac{dx}{x} \\
&= \int_0^\infty \phi^\pm(x)\left(\frac{2}{1+x^\alpha}\right)^{r+n}\left(\frac{1+x^\alpha}{2}\right)^n x^{s+\alpha r/2}\,\frac{dx}{x} \\
&= 2^{-n}\sum_{j=0}^n \binom{n}{j}\int_0^\infty \phi^\pm(x)\left(\frac{2}{1+x^\alpha}\right)^{r+n} x^{s-\alpha n/2+\alpha j}\,\frac{dx}{x} \\
&= 2^{-n}\sum_{j=0}^n \binom{n}{j}\,\xi_\alpha^\pm(s - \alpha n/2 + \alpha j; r+n). \ \blacksquare
\end{aligned}$$

4.2. Proof of Proposition 2.16. The beta function

$$B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$$

has the integral formula

$$B(p,q) = \int_0^\infty \frac{x^p}{(1+x)^{p+q}}\,\frac{dx}{x}$$

for $\Re(p) > 0$ and $\Re(q) > 0$ ([19, 3.251-2]). Therefore,

$$\int_0^\infty \left(\frac{x^{\alpha/2}+x^{-\alpha/2}}{2}\right)^{-\lambda} x^z \frac{dx}{x} = 2^\lambda \int_0^\infty x^{\alpha\lambda/2+z}(1+x^\alpha)^{-\lambda} x^z \frac{dx}{x}$$
$$= \frac{2^\lambda}{\alpha} \int_0^\infty x^{\lambda/2+z/\alpha}(1+x)^{-\lambda} \frac{dx}{x} = \frac{2^\lambda}{\alpha} B\left(\frac{\lambda}{2}+\frac{z}{\alpha}, \frac{\lambda}{2}-\frac{z}{\alpha}\right)$$

for $-\alpha\lambda/2 < \Re(z) < \alpha\lambda/2$. Put $\phi^\pm(x)$ as in (4.1). Then

$$\xi_\alpha^\pm(s;r) = \int_0^\infty \phi^\pm(x)\left(\frac{x^{\alpha}+x^{-\alpha/2}}{2}\right)^{-(r-\lambda)}\left(\frac{x^{\alpha/2}+x^{-\alpha/2}}{2}\right)^{-\lambda} x^s \frac{dx}{x}$$

for a positive real number λ. Thus, by using Lemma 4.1 below, we have

$$\xi_\alpha^\pm(s;r) = \frac{1}{2\pi i}\left(\frac{2^\lambda}{\alpha}\right)\int_{c-\infty}^{c+i\infty} \xi_\alpha^\pm(s-z;r-\lambda) B\left(\frac{\lambda}{2}+\frac{z}{\alpha}, \frac{\lambda}{2}-\frac{z}{\alpha}\right) dz$$

for $-\alpha\lambda/2 < c < \alpha\lambda/2$. Hence we obtain (2.17) of Proposition 2.16. ∎

LEMMA 4.1. *Let $f(x)$ and $g(x)$ be continuous functions on $(0,\infty)$, and let $F(z)$ and $G(z)$ be their respective Mellin transforms. Suppose that $f(x)$ is of rapid decay as $x \to 0^+$ and $x \to \infty$, and $g(x) \ll \min(x^A, x^{-A})$ for some positive real number A. In addition, suppose that $G(c+it)$ belongs to $L^1(-\infty,\infty)$ for any $-A < c < A$ as a function of t. Then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s-z)G(z)\,dz = \int_0^\infty f(x)g(x)x^s \frac{dx}{x} \tag{4.2}$$

for every $s \in \mathbb{C}$, where c is chosen with $-A < c < A$.

Proof. The Mellin transform $G(z)$ of $g(x)$ is defined by

$$G(z) = \int_0^\infty g(x)x^z \frac{dx}{x}.$$

Under our assumption, the integral converges absolutely for $-A<\Re(z)<A$. Hence it follows from the Mellin inversion formula that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)x^{-z}\,dz = g(x)$$

for $-A < c < A$. Multiplying by $f(x)x^{s-1}$ and integrating over $(0,\infty)$, we have formally

$$\int_0^\infty \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)x^{-z}\,dz\right) f(x)x^s \frac{dx}{x} = \int_0^\infty f(x)g(x)x^s \frac{dx}{x}. \tag{4.3}$$

This equality is valid for every $s \in \mathbb{C}$ if $-A < c < A$. In fact the integral on the right-hand side converges absolutely for every $s \in \mathbb{C}$, because $f(x)g(x)$ is of rapid decay as $x \to 0^+$ and $x \to \infty$. On the left-hand side, we have formally

$$\begin{aligned}(4.4)\qquad &\int_0^\infty \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)x^{-z}\,dz\right) f(x)x^s\,\frac{dx}{x}\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)\left(\int_0^\infty f(x)x^{s-z}\,\frac{dx}{x}\right)dz = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s-z)G(z)\,dz.\end{aligned}$$

This is justified by Fubini's theorem if the integral

$$(4.5)\qquad \int_0^\infty \int_{-\infty}^\infty G(c+it)\,f(x)x^{s-c-it-1}\,dt\,dx$$

converges absolutely. By the assumptions on $G(z)$ and $f(x)$, the double integral (4.5) converges absolutely if $-A < c < A$. Hence we obtain (4.2) for every $s \in \mathbb{C}$ and $-A < c < A$, by (4.3) and (4.4). ∎

5. Proof of Theorem 2.17. Recall that $\xi_\alpha(s;r)$ is an entire function of two complex variables (s,r) by Theorem 2.2. In particular its zero-divisor is locally analytic.

It suffices to prove the latter half of Theorem 2.17, since the first half is a special case with $r^* = 1$.

We argue by contradiction. Suppose that the Riemann hypothesis for $\xi_\alpha(s;r)$ is false for some $-1 < -r_0 \le -1 + r^*$ (≤ 0), that is, there exists a zero of $\xi_\alpha(s;-r_0)$ outside the line $\Re(s) = 1/2$. Then $\xi_\alpha(s;1-r_0)$ has at least one zero ρ_0 such that $\Re(\rho_0) \ge (1+\alpha)/2$ by Corollary 2.7. Since $0 < 1 - r_0 \le r^*$, we vary the level r of $\xi_\alpha(s;r)$ from -1 to $1-r_0$. Then there exists a locus $\{(\rho(r), r)\}$ in $\mathbb{C} \times [-1, 1-r_0]$ such that

$$\xi_\alpha(\rho(r);r) = 0\ (-1 \le r \le r^*) \quad \text{and} \quad \rho(1-r_0) = \rho_0.$$

Note that $\Re(\rho(-1)) = 1/2$, since all zeros of $\xi_\alpha(s;-1)$ lie on the line $\Re(s) = 1/2$ by the assumption $\alpha \ge 1$ and Corollary 2.9. Moreover,

$$\Re(\rho(r)) < (1+\alpha)/2$$

for every $-\delta < r \le 0$ if $\delta > 0$ is sufficiently small, since $\xi(s) = \xi_\alpha(s;0)$ does not vanish on $\Re(s) \ge (1+\alpha)/2$ by $\alpha \ge 1$. Hence $\Re(\rho(r_1)) = (1+\alpha)/2$ for some $0 < r_1 \le 1 - r_0$, since

$$\Re(\rho(1-r_0)) = \Re(\rho_0) \ge (1+\alpha)/2.$$

This contradicts the assumption of Theorem 2.17, since $1 - r_0 \le r^*$. Hence $\xi_\alpha(s;r)$ must satisfy the Riemann hypothesis for every $-1 \le r \le -1 + r^*$ if $\xi_\alpha((1+\alpha)/2 + i\mathbb{R}; r) \ne 0$ for every $0 < r < r^*$. ∎

6. Proof of Theorem 2.19. We prove Theorem 2.19 by using the series expansion

$$(6.1)\qquad \xi_2(s;1) = \Gamma\Big(\frac{s+5}{2}\Big)\sum_{n=1}^{\infty} 4\pi^2 n^4 \Gamma\Big(-\frac{s+3}{2};\pi n^2\Big)e^{\pi n^2} - \Gamma\Big(\frac{s+3}{2}\Big)\sum_{n=1}^{\infty} 6\pi n^2 \Gamma\Big(-\frac{s+1}{2};\pi n^2\Big)e^{\pi n^2},$$

which is the case $r=1$ of the series expansion (7.3) in Section 7.3. For a positive real number λ, we have

$$(6.2)\qquad \Gamma(\lambda s,\lambda) = \lambda^{\lambda s}e^{-\lambda}\int_0^\infty e^{-\lambda(1-s)t-\lambda\rho(t)}\,dt = \frac{\lambda^{\lambda s}e^{-\lambda}}{\lambda(1-s)}(1-Q_\lambda(s))$$

by integration by parts, where $\rho(t)=e^t-(1+t)$ and

$$Q_\lambda(s) = \lambda\int_0^\infty \rho'(t)e^{-\lambda(1-s)t-\lambda\rho(t)}\,dt \quad (\Re(s)\le 1).$$

We need the following properties of $Q_\lambda(s)$ for the proof of Theorem 2.19.

LEMMA 6.1 (Mahler [29]). *Let $s=\sigma+it$. For every $\lambda>0$, we have*

(1) $Q_\lambda(1)=1$ *and* $\Gamma(\lambda,\lambda)\ne 1$,
(2) $|Q_\lambda(s)|<Q_\lambda(\sigma)$ *and* $Q_\lambda(\sigma)>0$ *if* $\sigma\le 1$,
(3) $Q_\lambda(\sigma_2)<Q_\lambda(\sigma_1)$ *if* $\sigma_2<\sigma_1<1$.

In particular, $\Gamma(s,\lambda)$ has no zeros on the left half-plane $\Re(s)\le\lambda$.

Using (6.1) and (6.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big[4\pi^2 n^4\frac{s+3}{2}\Gamma\Big(-\frac{s+3}{2};\pi n^2\Big) - 6\pi n^2\Gamma\Big(-\frac{s+1}{2};\pi n^2\Big)\Big]e^{\pi n^2}\\
&\quad = \pi^{-(s-1)/2}\frac{s+3}{2}\sum_{n=1}^{\infty} n^{-s+1}\frac{8}{2\pi n^2+s+3}\Big[1-Q_{\pi n^2}\Big(-\frac{s+3}{2\pi n^2}\Big)\Big]\\
&\qquad - \pi^{-(s-1)/2}\sum_{n=1}^{\infty} n^{-s+1}\frac{12}{2\pi n^2+s+1}\Big[1-Q_{\pi n^2}\Big(-\frac{s+1}{2\pi n^2}\Big)\Big]\\
&\quad = 4\pi^{-(s-1)/2}\frac{s+3}{2\pi+s+3}\cdot\Big(1-Q_\pi\Big(-\frac{s+3}{2\pi}\Big)\Big)\\
&\qquad \times\Big[1+\sum_{n=2}^{\infty} n^{-s+1}\frac{2\pi+s+3}{2\pi n^2+s+3}\frac{1-Q_{\pi n^2}\big(-\frac{s+3}{2\pi n^2}\big)}{1-Q_\pi\big(-\frac{s+3}{2\pi}\big)}\\
&\qquad\qquad + \sum_{n=1}^{\infty} n^{-s+1}\frac{3(2\pi+s+3)}{(s+3)(2\pi n^2+s+1)}\frac{1-Q_{\pi n^2}\big(-\frac{s+1}{2\pi n^2}\big)}{1-Q_\pi\big(-\frac{s+3}{2\pi}\big)}\Big]
\end{aligned}$$

$$= 4\pi^{-(s-1)/2}\frac{s+3}{2\pi+s+3}\cdot\left(1-Q_\pi\left(-\frac{s+3}{2\pi}\right)\right)\cdot[1+\Sigma_1(s)+\Sigma_2(s)],$$

say. Here we easily find that

$$\left|\frac{2\pi+s+3}{2\pi n^2+s+3}\right| \le 1+\left|\frac{2\pi n^2-2\pi}{2\pi n^2+s+3}\right| \le 1+\left|1-\frac{2\pi+3}{2\pi n^2+3}\right| \le 2$$

for $\Re(s)>0$ and every integer $n\ge 2$, and

$$\left|\frac{3(2\pi+s+3)}{(s+3)(2\pi n^2+s+1)}\right| \le \left|\frac{3}{s+3}\right|\left(1+\left|1-\frac{2\pi+3}{2\pi n^2+1}\right|\right) \le \frac{6}{|s+3|}$$

for $\Re(s)>0$ and every integer $n\ge 1$. Using the properties of $Q_\lambda(s)$ in Lemma 6.1,

$$\left|1-Q_\pi\left(-\frac{s+3}{2\pi}\right)\right| \ge 1-\left|Q_\pi\left(-\frac{s+3}{2\pi}\right)\right| \ge 1-\left|Q_\pi\left(-\frac{3}{2\pi}\right)\right| =: \kappa,$$

say, for $\Re(s)>0$, and

$$\left|1-Q_{\pi n^2}\left(-\frac{s+q}{2\pi n^2}\right)\right| \le 1+\left|Q_{\pi n^2}\left(-\frac{\Re(s)+q}{2\pi n^2}\right)\right| \le 1+Q_{\pi n^2}(0) \le 2$$

for $\Re(s)>0$, every integer $n\ge 1$ and $q=1,3$. Hence we obtain

$$|\Sigma_1(s)| \le \frac{4}{\kappa}\sum_{n=2}^{\infty} n^{-\sigma+1} = \frac{4}{\kappa}(\zeta(\sigma-1)-1)$$

and

$$|\Sigma_2(s)| \le \frac{12}{\kappa|s+3|}\sum_{n=1}^{\infty} n^{-\sigma+1} \le \frac{12}{\kappa(\sigma+3)}\zeta(\sigma-1).$$

Thus

$$1+\Sigma_1(s)+\Sigma_2(s) = 1+O(\sigma^{-1}) \quad \text{as } \sigma\to\infty.$$

This implies that $\xi_2(s;1)\ne 0$ on some right half-plane. Using the value $\kappa \simeq 0.8801687\ldots$, we obtain

$$|\Sigma_1(s)|+|\Sigma_2(s)| \le \frac{4}{\kappa}(\zeta(\sigma-1)-1)+\frac{12}{\kappa(\sigma+3)}\zeta(\sigma-1) < 1$$

for $\Re(s)>11$. This completes the proof. ■

Remark 6.2. It is impossible to prove that $\xi_2(s;1)\ne 0$ for $\Re(s)\ge 3/2$ if we attempt to prove it via the estimate of $|\Sigma_1(s)+\Sigma_2(s)|$, since we can find a point s_0 close to the line $\Re(s)=3/2$ such that $|\Sigma_1(s_0)+\Sigma_2(s_0)|$ is larger than one.

Remark 6.3. We proved Theorem 2.19 by using the series expansion of $\xi_2(s;1)$ coming from Proposition 7.2 which is limited to $\alpha=2$ due to

Lemma 7.3. It is a formula for the integral

$$\int_0^\infty e^{\lambda x} \frac{x^s}{(1+x^\beta)^p} \frac{dx}{x}$$

for $\beta = 1$; however, the author does not know a simple formula for the integral for general real $\beta > 0$ which would suffice to prove Theorem 2.19 for such β, although it is possible to find a somewhat complicated formula which is similar to Lemma 7.3 for a rational $\beta > 0$ by combining Lemma 7.3 with the binomial theorem and factorizations of $x^n \pm 1$. This is why Theorem 2.19 was restricted to $\alpha = 2$.

7. Miscellaneous formulas for deformations

7.1. Corollary of Proposition 2.16. Let α and r be real numbers, and let λ be a positive real number. By equality (2.17), we have

$$\xi_\alpha(s;r) = \frac{1}{\alpha}\frac{2^\lambda}{\Gamma(\lambda)} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_\alpha(s-iv; r-\lambda)\left|\Gamma\left(\frac{\lambda}{2}+\frac{iv}{\alpha}\right)\right|^2 dv \tag{7.1}$$

for every $s \in \mathbb{C}$. In particular,

$$\xi(s) = \frac{1}{\alpha}\frac{2^r}{\Gamma(r)} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(s-iv; -r)\left|\Gamma\left(\frac{r}{2}+\frac{iv}{\alpha}\right)\right|^2 dv$$

and

$$\xi_\alpha(s;r) = \frac{1}{\alpha}\frac{2^r}{\Gamma(r)} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(s-iv)\left|\Gamma\left(\frac{r}{2}+\frac{iv}{\alpha}\right)\right|^2 dv.$$

Using $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ (hence $|\Gamma(1/2+iv/\alpha)|^2 = 2\pi/(e^{\pi v/\alpha} + e^{-\pi v/\alpha})$) and taking $\lambda = 1$ in (7.1), we obtain a formula for $\xi_\alpha(s;r)$ involving an iterated integral.

Corollary 7.1. *For real numbers $\alpha > 0$, r and a positive integer n, we have*

$$\xi_\alpha(s;r) = \left(\frac{2}{\alpha}\right)^n \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \xi_\alpha(s-iv_1-\cdots-iv_n; r-n)\,\frac{dv_1\ldots dv_n}{\prod_{j=1}^n (e^{\pi v_j/\alpha}+e^{-\pi v_j/\alpha})}.$$

In particular,

$$\xi(s) = \left(\frac{2}{\alpha}\right)^n \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \xi(s-iv_1-\cdots-iv_n; -n)\,\frac{dv_1\ldots dv_n}{\prod_{j=1}^n (e^{\pi v_j/\alpha}+e^{-\pi v_j/\alpha})}$$

$$
= \begin{cases} \dfrac{2^{2k}}{\alpha} \dfrac{\Gamma(k)^2}{\Gamma(2k)} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv;-n) \prod_{j=1}^{k-1}\left(1+\frac{v^2}{\alpha^2 j^2}\right) \frac{v}{e^{\pi v/\alpha}-e^{-\pi v/\alpha}}\, dv, & \\ \hfill n=2k, & \\ 2^{2k+1} \dfrac{\Gamma(k)\Gamma(k+1)}{k\Gamma(2k+1)} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv;-n) \prod_{j=1}^{k-1}\left(1-\frac{\dot{v}^2}{j^2}\right) \frac{\dot{v}(k-\dot{v})}{e^{\pi v/\alpha}+e^{-\pi v/\alpha}}\, dv, & \\ \hfill n=2k+1, & \\ \dfrac{2}{\alpha} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv;-1) \frac{dv}{e^{\pi v/\alpha}+e^{-\pi v/\alpha}}, \qquad n=1, & \end{cases}
$$

and

$$
\xi_\alpha(s;n) = \left(\frac{2}{\alpha}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \xi(s-iv_1-\cdots-iv_n) \frac{dv_1 \dots dv_n}{\prod_{j=1}^n (e^{\pi v_j/\alpha}+e^{-\pi v_j/\alpha})}
$$

$$
= \begin{cases} \dfrac{2^{2k}}{\alpha} \dfrac{\Gamma(k)^2}{\Gamma(2k)} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv) \prod_{j=1}^{k-1}\left(1+\frac{v^2}{\alpha^2 j^2}\right) \frac{v}{e^{\pi v/\alpha}-e^{-\pi v/\alpha}}\, dv, \qquad n=2k, & \\ 2^{2k+1} \dfrac{\Gamma(k)\Gamma(k+1)}{k\Gamma(2k+1)} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv) \prod_{j=1}^{k-1}\left(1-\frac{\dot{v}^2}{j^2}\right) \frac{\dot{v}(k-\dot{v})}{e^{\pi v/\alpha}+e^{-\pi v/\alpha}}\, dv, & \\ \hfill n=2k+1, & \\ \dfrac{2}{\alpha} \displaystyle\int_{-\infty}^{\infty} \xi(s-iv) \frac{dv}{e^{\pi v/\alpha}+e^{-\pi v/\alpha}}, \qquad n=1, & \end{cases}
$$

where $\dot{v} = 1/2 + iv/\alpha$.

7.2. Note on the ascent formula. By Corollary 7.1, we have

$$
\xi(s) = \alpha^{-1}[\xi_\alpha(\;;-1)\star J_\alpha](s) = \alpha^{-1}\int_{-\infty}^{\infty} \xi_\alpha(s-iv;-1)\frac{dv}{\cosh((\pi/\alpha)v)}, \tag{7.2}
$$

where the convolution product $\star$ is defined by

$$
(F\star G)(s) = \int_{-\infty}^{\infty} F(s-iv)G(v)\, dv
$$

and $J_\alpha(s) = 1/\cosh((\pi/\alpha)s)$. Recall that all zeros of $\xi_\alpha(s;-1)$ lie on the line $\Re(s) = 1/2$ if $\alpha \geq 1$ unconditionally, and it also holds for every real number α under the Riemann hypothesis. On the other hand, $J_\alpha(s)$ has no zeros, and all poles of $J_\alpha(s)$ lie on the imaginary axis. Equation (7.2) shows that $\xi(s)$ is a convolution product of two functions satisfying the Riemann hypothesis. Compare this fact with the results of Cardon [9] and Cardon–Nielsen [10] (be careful with the direction of the integral).

7.3. Series expansion of deformations for weight two

Proposition 7.2. *Let r, m be nonnegative integers with $r+m>0$. Then*

$$(7.3)\quad \xi_2(s;r)$$

$$= \frac{2\cdot 2^r}{(r+m-1)!}\sum_{j=0}^{m}\binom{m}{j}\Gamma\Big(\frac{s+r}{2}+2+j\Big)\sum_{\nu=0}^{r+m-1}\binom{r+m-1}{\nu}(-1)^{r+m-\nu-1} \times\sum_{n=1}^{\infty}e^{\pi n^2}(\pi n^2)^{r+m-\nu+1}\Gamma\Big(-\frac{s+r}{2}-1-j+\nu;\pi n^2\Big)$$

$$-\frac{3\cdot 2^r}{(r+m-1)!}\sum_{j=0}^{m}\binom{m}{j}\Gamma\Big(\frac{s+r}{2}+1+j\Big)\sum_{\nu=0}^{r+m-1}\binom{r+m-1}{\nu}(-1)^{r+m-\nu-1} \times\sum_{n=1}^{\infty}e^{\pi n^2}(\pi n^2)^{r+m-\nu}\Gamma\Big(-\frac{s+r}{2}-j+\nu;\pi n^2\Big)$$

for $\Re(s)>r+2m-1$, where $\Gamma(s,\lambda)$ is the incomplete gamma function of the second kind:

$$\Gamma(s,\lambda)=\int_{\lambda}^{\infty}e^{-t}t^{s-1}\,dt\quad(\Re(s)>0).$$

The series on the right-hand side of (7.3) converges absolutely and uniformly on every compact subset of the right half-plane $\Re(s)>r+2m-1$.

Proof. Put

$$\psi(x)=2(2\pi^2x^4-3\pi x^2)\exp(-\pi x^2)$$

so that the Mellin transform of $\psi(x)$ is $\gamma(s)$ of (1.1) and $\phi(x)=\sum_{n=1}^{\infty}\psi(nx)$. Then

$$\begin{aligned}\xi_2(s;r) &= 2^r\int_0^{\infty}\phi(x)\Big(\frac{1+x^2}{x}\Big)^m\Big(\frac{x}{1+x^2}\Big)^{r+m}x^s\,\frac{dx}{x}\\ &= 2^r\sum_{j=0}^{m}\binom{m}{j}\sum_{n=1}^{\infty}\int_0^{\infty}\psi(nx)\frac{x^{s+r+2j}}{(1+x^2)^{r+m}}\,\frac{dx}{x}\\ &= 2^r\sum_{j=0}^{m}\binom{m}{j}\sum_{n=1}^{\infty}\int_0^{\infty}(2\pi^2n^4x^2-3\pi n^2x)e^{-\pi n^2x}\frac{x^{(s+r)/2+j}}{(1+x)^{r+m}}\,\frac{dx}{x}\\ &= 2\cdot 2^r\sum_{j=0}^{m}\binom{m}{j}\sum_{n=1}^{\infty}\pi^2n^4\int_0^{\infty}e^{-\pi n^2x}\frac{x^{(s+r)/2+2+j}}{(1+x)^{r+m}}\,\frac{dx}{x}\\ &\quad -3\cdot 2^r\sum_{j=0}^{m}\binom{m}{j}\sum_{n=1}^{\infty}\pi n^2\int_0^{\infty}e^{-\pi n^2x}\frac{x^{(s+r)/2+1+j}}{(1+x)^{r+m}}\,\frac{dx}{x}\end{aligned}$$

if $\Re(s)$ is sufficiently large. Applying Lemma 7.3 below to the final series, we obtain the series expansion (7.3) if $\Re(s)$ is sufficiently large. Each term on the right-hand side of (7.3) is bounded by

$$n^{r+2m-2-\Re(s)}(1+O(n^{-2}))$$

on every compact subset K of $\mathbb{C}$, since

$$\Gamma(s,\lambda) = \lambda^{s-1}e^{-\lambda}(1+O(\lambda^{-1}))$$

for $\lambda \geq 1$ and $s \in K$, where the implied constant depends on K ([19, 8.357-1]). Hence we obtain the convergence of (7.3) as in Proposition 7.2. ■

LEMMA 7.3. *Let λ be a positive real number, and let p be a positive integer. Then*

$$\int_0^\infty e^{-\lambda x}\frac{x^s}{(1+x)^p}\,\frac{dx}{x} = e^{\lambda}\frac{\Gamma(s)}{(p-1)!}\sum_{\nu=0}^{p-1}\binom{p-1}{\nu}(-\lambda)^{p-\nu-1}\Gamma(1-s+\nu;\lambda).$$

Proof. Using

$$\frac{1}{(1+x)^p} = \frac{1}{(p-1)!}\int_0^\infty e^{-(1+x)y}y^{p-1}\,dy \quad (p\in\mathbb{Z}_{>0}),$$

we have

$$\begin{aligned}\int_0^\infty e^{-\lambda x}\frac{x^s}{(1+x)^p}\,\frac{dx}{x} &= \frac{1}{(p-1)!}\int_0^\infty e^{-y}y^{p-1}\int_0^\infty e^{-(y+\lambda)x}x^s\,\frac{dx}{x}\,dy\\ &= \frac{\Gamma(s)}{(p-1)!}\int_0^\infty e^{-y}\frac{y^{p-1}}{(y+\lambda)^s}\,dy\\ &= \frac{\Gamma(s)}{(p-1)!}\sum_{\nu=0}^{p-1}\binom{p-1}{\nu}(-\lambda)^{p-\nu-1}\int_0^\infty e^{-y}\frac{(y+\lambda)^\nu}{(y+\lambda)^s}\,dy\\ &= e^{\lambda}\frac{\Gamma(s)}{(p-1)!}\sum_{\nu=0}^{p-1}\binom{p-1}{\nu}(-\lambda)^{p-\nu-1}\Gamma(1-s+\nu;\lambda).\end{aligned}$$

Hence we obtain the desired formula. ■

Applying Proposition 7.2 to $r=0$, we have

$$\begin{aligned}\xi_2(s;0) = \frac{2}{(m-1)!}\sum_{j=0}^{m}\binom{m}{j}\Gamma\Big(\frac{s}{2}+2+j\Big)\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(-1)^{m-\nu-1}\\ \times\sum_{n=1}^{\infty}e^{\pi n^2}(\pi n^2)^{m-\nu+1}\Gamma\Big(-\frac{s}{2}-1-j+\nu;\pi n^2\Big)\end{aligned}$$

$$- \frac{3}{(m-1)!} \sum_{j=0}^{m} \binom{m}{j} \Gamma\Big(\frac{s}{2}+1+j\Big) \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (-1)^{m-\nu-1} \times \sum_{n=1}^{\infty} e^{\pi n^2} (\pi n^2)^{m-\nu} \Gamma\Big(-\frac{s}{2}-j+\nu; \pi n^2\Big).$$

On the other hand, we have

$$\xi_2(s-r;r) = \frac{2\cdot 2^r}{(m-1)!} \sum_{j=0}^{m-r} \binom{m-r}{j} \Gamma\Big(\frac{s}{2}+2+j\Big) \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (-1)^{m-\nu-1} \times \sum_{n=1}^{\infty} e^{\pi n^2} (\pi n^2)^{m-\nu+1} \Gamma\Big(-\frac{s}{2}-1-j+\nu; \pi n^2\Big) - \frac{3\cdot 2^r}{(m-1)!} \sum_{j=0}^{m-r} \binom{m-r}{j} \Gamma\Big(\frac{s}{2}+1+j\Big) \sum_{\nu=0}^{m-1} \binom{m-1}{\nu} (-1)^{m-\nu-1} \times \sum_{n=1}^{\infty} e^{\pi n^2} (\pi n^2)^{m-\nu} \Gamma\Big(-\frac{s}{2}-j+\nu; \pi n^2\Big)$$

by applying Proposition 7.2 to a positive integer r, and then replacing m by $m-r$. Comparing the above two equalities, we notice that $\xi_2(s;r)$ is not far from a shift of $\xi(s)$, that is,

$$\xi(s) = \xi_2(s;0) \approx \xi_2(s-r;r) \quad (r \in \mathbb{Z}_{>0})$$

for a suitable meaning of the approximate relation $\approx$.

7.4. Series expansion of a companion for weight two. From the series expansion (7.3) and definition (2.13), we obtain a series expansion of $\xi_2^-(s;r)$. In particular, we obtain the series expansion

$$\xi_2^-(s;0) = i\Big[\Gamma\Big(\frac{s+6}{2}\Big) \sum_{n=1}^{\infty} 2\pi^2 n^4 \Gamma\Big(-\frac{s+4}{2}; \pi n^2\Big) e^{\pi n^2} - \Gamma\Big(\frac{s+4}{2}\Big) \sum_{n=1}^{\infty} \pi n^2 (2\pi n^2+3) \Gamma\Big(-\frac{s+2}{2}; \pi n^2\Big) e^{\pi n^2} + \Gamma\Big(\frac{s+2}{2}\Big) \sum_{n=1}^{\infty} 3\pi n^2 \Gamma\Big(-\frac{s}{2}; \pi n^2\Big) e^{\pi n^2}\Big] \tag{7.4}$$

by definition (2.13) with the series expansion (7.3) for $r=1$.

7.5. Formulas for deformations of negative integer levels

PROPOSITION 7.4. *Let α be a positive real number, and let n be a positive integer. Then*

$$\xi_\alpha^+(s;-n) = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}\xi(s+(n/2-j)\alpha), \tag{7.5}$$

$$\begin{aligned}\xi_\alpha^-(s;-n)&\\ &= 2^{-n}\, i\sum_{j=0}^{n}\binom{n-1}{j}(\xi(s+(n/2-j)\alpha)-\xi(s+(n/2-j-1)\alpha)).\end{aligned} \tag{7.6}$$

Proof. Formula (7.5) is a special case $r=-n$ of (2.16) by definition (2.12). On the other hand, we have $\xi_\alpha^-(s;-1)=\xi(s+\alpha/2)-\xi(s-\alpha/2)$ by definition (2.13) and $\xi_\alpha(s;0)=\xi(s)$. Hence (7.6) is obtained as a special case $r=-n-1$ of (2.16). ∎

By (7.6), we have the following formula immediately:

$$\xi_\alpha^-(s;-n) = 2^{-n} i(h_\alpha^+(s;n)-h_\alpha^-(s;n)),$$

where

$$h_\alpha^\pm(s;n) = \begin{cases}\xi(s\pm\alpha/2) & \text{if } n=1,\\ \xi(s\pm\alpha) & \text{if } n=2,\\ \xi(s\pm n\alpha/2) & \\ \quad + \displaystyle\sum_{1\le j<n/2}\binom{n-2}{j-1}\frac{(n-1)(n-2j)}{j(n-j)}\xi(s\pm(n/2-j)\alpha) & \text{if } n\ge 3.\end{cases}$$

7.6. Limit formulas for deformations. Let n be a positive integer. By (2.4), $\xi_\alpha(s;r)$ is close to $\xi(s)$ if α is sufficiently small for every $r\in\mathbb{R}$. On the other hand, $\xi_\alpha(s;n)$ for large positive integer n is closely related to the Riemann hypothesis for $\xi(s)$ by Corollary 2.5. Hence we expect that a self-referentiality will appear when $\alpha\to 0^+$ and $r\to\infty$ simultaneously. Define

$$Z_n(s) := \xi_{1/n}(s;n).$$

By (2.16), we have

$$\xi(s) = 2^{-n}\sum_{j=0}^{n}\binom{n}{j}Z_n\left(s-\frac{1}{2}+\frac{j}{n}\right).$$

Consider the limit of the right-hand side:

$$\lim_{n\to\infty} 2^{-n}\sum_{j=0}^{n}\binom{n}{j}Z_n\left(s-\frac{1}{2}+\frac{j}{n}\right).$$

Using $2^n\cdot(x^{1/2n}+x^{-1/2n})^{-n}\to 1$ $(n\to\infty)$, we have

$$2^{-n}\sqrt{\frac{8\pi}{n}}\,\frac{\Gamma(n)}{\Gamma(n/2)^2}\,Z_n(s) \sim Z_n(s)\to\xi(s)\quad (n\to\infty).$$

On the other hand,

$$\sum_{j=0}^{n} \frac{1}{n}\sqrt{\frac{2n}{\pi}}\binom{n}{j}\binom{n}{n/2}^{-1} \to \int_0^1 \delta_{1/2}(x)\,dx \quad (n\to\infty),$$

where $\delta_a(x)$ is the Dirac mass at $x=a$. Using

$$\binom{n}{n/2} = \frac{4}{n}\,\frac{\Gamma(n)}{\Gamma(n/2)^2},$$

we arrive at the formula

$$\lim_{n\to\infty} 2^{-n}\sum_{j=0}^{n}\binom{n}{j} Z_n\left(s-\frac{1}{2}+\frac{j}{n}\right) = \int_0^1 \delta_{1/2}(x)\xi(s-1/2+x)\,dx = \xi(s).$$

Moreover we have

$$\lim_{n\to\infty} 2^{-n}\sum_{j=0}^{n}\binom{n}{j}\xi_{1/n}\left(s-\frac{1}{2}+\frac{j}{n};r+n\right) = \int_0^1 \delta_{1/2}(x)\xi(s-1/2+x)\,dx = \xi(s)$$

for every real number r.

By Theorem 2.4, if $Z_n(s)$ has no zeros in the right half-plane $\Re(s)\geq 1/2+1/2n$ for some positive integer n, then the Riemann hypothesis holds for $\xi(s)$. Conversely, it is expected that $Z_n(x)$ has no zeros outside the line $\Re(s)=1/2$ for large positive integer n if we assume that the Riemann hypothesis holds for $\xi(s)$, since $Z_n(s)\to\xi(s)$ as $n\to\infty$. The above limit formula explains this self-referentiality.

Certainly, other kind of limit formulas are possible. For example, we have

$$\begin{aligned}\xi(s) &= \lim_{n\to\infty} 2^{-n}\sqrt{\frac{8\pi}{n}}\,\frac{\Gamma(n)}{\Gamma(n/2)^2}\xi_{\pm1/n}(s;n)\\ &= \lim_{n\to\infty} 2^{n}\sqrt{\frac{n}{8\pi}}\,\frac{\Gamma(n/2)^2}{\Gamma(n)}\xi_{\pm1/n}(s;-n) = \lim_{n\to\pm\infty} 2^{-1/n}\xi_{\pm2n}(s;n^{-1}).\end{aligned}$$

The first line suggests that the zeros of $\xi_\alpha(s;r)$ are close to the line $\Re(s)=1/2$ for small $\alpha>0$ and large $r>0$ under the Riemann hypothesis for $\xi(s)$.

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