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On the spinor zeta functions problem: higher power moments of the Riesz mean

by

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1. Introduction and statement of results. Let $S_k(\Gamma_g)$ be the space of cusp forms of weight k and genus g on the full Siegel modular group $\Gamma_g = Sp_g(\mathbb{Z})$. If $F \in S_k(\Gamma_g)$ is a common Hecke eigenform, define

$$(1.1)\qquad Z_{F,p}(X) := (1-\alpha_{0,p}X)\prod_{\nu=1}^{n}\prod_{1\le i_1<\cdots<i_\nu\le n}(1-\alpha_{0,p}\alpha_{i_1,p}\cdots\alpha_{i_\nu,p}X).$$

For $\Re s \gg 0$, the *spinor zeta function* by definition is

$$(1.2)\qquad Z_F(s) = \prod_p [Z_{F,p}(p^{-s})]^{-1} := \sum_{n=1}^{\infty} c_n n^{-s}.$$

We shall only use the spinor zeta function for eigenforms of genus 2. In this case, Andrianov [1] has proved the following properties.

LEMMA 1.1 (Andrianov, 1974). *Suppose that $f|T(n) = \lambda(n)f$ for all $n \ge 1$. Then*

$$Z_{F,p}(X) = 1-\lambda(p)X+(\lambda(p)^2-\lambda(p^2)-p^{2k-4})X^2-\lambda(p)p^{2k-3}X^3+p^{4k-6}X^4,$$

where $\lambda(n)$ denotes the eigenvalue of F with respect to the Hecke operator $T(n)$. The function

$$Z_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$Z_F^*(2k-2-s) = (-1)^k Z_F^*(s).$$

In [2] and [8], the authors proved that $Z_F^*(s)$ is holomorphic if either k is odd or if k is even and F is contained in the orthogonal complement of the Maass subspace $S_k^*(\Gamma_2)$. Since $Z_F^*(s)$ is entire, $Z_F(s)$ is entire and in particular vanishes at the points $s = k-2, k-3, \ldots$.

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Under our assumption on F, by Weissauer's theorem [13] the Satake p-parameters $\alpha_{1,p}$, $\alpha_{2,p}$ are of absolute value 1. As in [9], we can define $\alpha_{1,p} := \chi_1(p)$, $\alpha_{2,p} = \chi_2(p)$ and $\alpha_{0,p} = p^{k-3/2}\sigma(p)$, where χ_1, χ_2 and σ are unramified characters of $\mathbb{Q}_p^\times$. Then combining (1.1) and (1.2) we obtain

$$Z_F(s) = \prod_p ((1-\alpha_{0,p}p^{-s})(1-\alpha_{0,p}\alpha_{1,p}p^{-s})(1-\alpha_{0,p}\alpha_{2,p}p^{-s}) \times (1-\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}p^{-s}))^{-1}$$
$$= \sum_{n=1}^{\infty} \frac{n^{k-3/2}(\sigma^4(n)\chi_1^2(n)\chi_2^2(n))}{n^s}.$$

Hence we have

$$c_n \ll n^{k-3/2+\epsilon}. \tag{1.3}$$

Thus we conclude that the Dirichlet series $Z_F(s)$ converges absolutely for $\Re s > k - 1/2$.

The Riesz mean of the coefficients c_n of the function $Z_F(s)$ is defined by the relation

$$D_\rho(x; Z_F) := \frac{1}{\Gamma(\rho+1)} {\sum_{n\le x}}' (x-n)^\rho c_n,$$

where $\rho \ge 0$, and $\sum'$ means that if $\rho = 0$ and x is an integer, then c_x is replaced by $c_x/2$.

When $Z_F(s)$ is entire, using Hafner's method [4], we gave the truncated Voronoï-type formula for $D_\rho(x, Z_F)$ in [7]:

Lemma 1.2. *Assume that k is odd or k is even and F is contained in the orthogonal complement of the Maass subspace $S_k^*(\Gamma_2)$. Fix $1/2 < \rho \le 1$. Let $N \gg 1$. Then for any $x > 1$,*

$$\begin{aligned} D_\rho(x; Z_F) &= (2\pi)^{-(\rho+1)} x^{3\rho/4+k-9/8} \sum_{n\le N} c_n n^{-\rho/4-k+7/8} \sin\left(8\pi(nx)^{1/4} + \frac{3-2\rho}{4}\pi\right) \\ &\quad + O(x^{3(\rho-1)/4+k+\epsilon} N^{-(1+\rho)/4}) + O(x^{(3\rho-5)/4+k} N^{(1-\rho)/4+\epsilon}) \end{aligned} \tag{1.4}$$

for any $\epsilon > 0$.

Letting $N = x^{3/5}$ in (1.4) one obtains

$$D_\rho(x; Z_F) \ll x^{3\rho/5+k-9/10}. \tag{1.5}$$

In this paper we are concerned with estimates of power moments of $D_\rho(x; Z_F)$. Before stating our results, we introduce some notations. Suppose $f : \mathbb{N} \to \mathbb{R}$ is any function such that $f(n) \ll n^\epsilon$, and $h \ge 2$ is a fixed integer.

Define

$$(1.6)\qquad s_\rho(l,h;f) := \sum_{\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_l}=\sqrt[4]{n_{l+1}}+\cdots+\sqrt[4]{n_h}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \qquad (1\le l<h),$$

and

$$(1.7)\qquad B_\rho(h;f) := \sum_{l=1}^{h-1}\binom{h-1}{l} s_\rho(l,h;f)\cos\left(\frac{(1-2\rho)(h-2l)}{4}\pi\right).$$

We shall use $s_\rho(l,h;f)$ to denote both the series (1.6) and its value.

Let

$$b_\rho(h) = 4^{h-2} + (3-2\rho)h/8 - 1/4.$$

For a real number $A_0 := A_0(\rho) > 3$, let H_0 be the least even integer n such that $n \ge A_0$. Define

$$\sigma_\rho(h,A_0) := \frac{(1+2\rho)(A_0-h)}{4(A_0-2)}, \qquad 3\le h<A_0,$$

$$(1.8)\qquad \delta_\rho(h,A_0) := \sigma_\rho(h,A_0)\min\left(\frac{2\rho-1}{2\rho+1}, \frac{1}{4b_\rho(H_0)}\right),$$

$$(1.9)\qquad \lambda_\rho(h,A_0) := \frac{\sigma_\rho(h,A_0)}{4b_\rho(h)+4\sigma_\rho(h,A_0)}.$$

Our results are as follows:

THEOREM 1.3. *Suppose that the assumption of Lemma* 1.2 *holds. Fix* $1/2<\rho\le 1$. *Then*

$$\int_1^X D_\rho(x;Z_F)^2\,dx = CX^{3\rho/2+2k-5/4} + O(T^{\rho+2k-1}),$$

where

$$C = 2^{-2\rho-3}\pi^{-2\rho-2}(3\rho/2+2k-5/4)^{-1}\sum_{n=1}^{\infty} c_n^2 n^{-\rho/2-2k+7/4}.$$

Using the result of Robert–Sargos [10] and following the proof of Theorem 2 in [6], one can prove the fourth power moments

$$\int_1^T D_1^4(x;Z_F)\,dx \ll T^{4k-1/2+\epsilon} + T^{4k-2+\epsilon}.$$

THEOREM 1.4. *Suppose that the assumption of Lemma* 1.2 *holds. Fix* $1/2<\rho\le 1$. *If there exists a real number* $A_0>3$ *such that*

$$(1.10)\qquad \int_1^T |D_\rho(x;Z_F)|^{A_0}\,dx \ll T^{1+(3\rho/4+k-9/8)A_0+\epsilon},$$

then for any integer $3 \le h < A_0$, *we have the asymptotic formula*

$$\int_1^T D_\rho^h(x; Z_F)\, dx = \frac{B_\rho(h; f)}{(2\pi)^{(\rho+1)h} 2^{h-1}} (1 + (3\rho/4 + k - 9/8)h)^{-1} T^{1+(3\rho/4+k-9/8)h} + O(T^{1+(3\rho/4+k-9/8)h-\delta_\rho(h,A_0)+\epsilon}),$$

where $B_\rho(h; f)$ *and* $\delta_\rho(h, A_0)$ *are given in* (1.7) *and* (1.8).

THEOREM 1.5. *Suppose that the assumption of Lemma* 1.2 *holds and* $\rho = 1$. *Then*

$$\int_1^T |D_1(x; Z_F)|^{16/3}\, dx \ll T^{1+16(k-3/8)/3+\epsilon}.$$

Thus the asymptotic formula

$$\int_1^T D_1^h(x; Z_F)\, dx = \frac{B_1(h; f)}{2^{3h-1}\pi^{2h}} (1 + (k - 3/8)h)^{-1} T^{1+(k-3/8)h} + O(T^{1+(k-3/8)h-\lambda_1(h,16/3)+\epsilon}).$$

holds for $h = 3, 4, 5$, *where* $\lambda_\rho(h, A_0)$ *is defined by* (1.9).

REMARK. From Theorem 1.5 we find that the estimate

$$\int_1^T |D_1(x; Z_F)|^{A_0}\, dx \ll T^{1+(k-3/8)A_0+\epsilon} \tag{1.11}$$

holds for $A_0 = 16/3$. The value of A_0 for which (1.11) holds is closely related to the upper bound of $D_1(x; Z_F)$.

REMARK. To prove Theorem 1.5, we use the method given in Section 4. But this method breaks down for the other ρ $(1/2 < \rho < 1)$.

2. Proof of Theorem 1.3. In the section, we estimate the integral

$$\int_1^X D_\rho(x; Z_F)^2\, dx$$

for $1/2 < \rho \le 1$. We follow the approach of Fomenko [3] and Ivić [5, Theorem 13.5]. It is sufficient to consider the integral over $[T, 2T]$. In this case, $T \le x \le 2T$, and, for $N = T$, Lemma 1.2 yields

$$D_\rho(x; Z_F) = (2\pi)^{-(\rho+1)} x^{3\rho/4+k-9/8} \sum_{n \le N} \frac{f(n)}{n^{(2\rho+5)/8}} \cos\left(8\pi(nx)^{1/4} + \frac{1-2\rho}{4}\pi\right) + O(T^{\rho/2+k-1+\epsilon}),$$

where $f(n) = c_n n^{-k+3/2} \ll n^\epsilon$.

On integrating the latter equality term-by-term, we obtain

$$
\begin{aligned}
(2.1)\qquad & \int_T^{2T} D_\rho(x;Z_F)^2\,dx \\
&= (2\pi)^{-2(\rho+1)} \int_T^{2T} x^{3\rho/2+2k-9/4} \sum_{m,n\le T} f(m)f(n)(mn)^{-(2\rho+5)/8} \\
&\quad \times \cos\left\{8\pi(nx)^{1/4} + \left(\frac{1}{4}-\frac{\rho}{2}\right)\pi\right\} \cos\left\{8\pi(mx)^{1/4} + \frac{1-2\rho}{4}\pi\right\} dx \\
&\quad + O\biggl(T^{\rho/2+k-1+\epsilon} \int_T^{2T} x^{3\rho/4+k-9/8} \\
&\qquad \times \left| \sum_{n\le T} \frac{f(n)}{n^{(2\rho+5)/8}} \cos\left\{8\pi(nx)^{1/4} + \frac{1-2\rho}{4}\pi\right\}\right| dx\biggr) \\
&\quad + O(T^{\rho+2k-1+\epsilon}).
\end{aligned}
$$

In the first term in (2.1), we distinguish the cases $m=n$ and $m\ne n$. The contribution of the terms with $m=n$ is

(2.2)

$$
\begin{aligned}
&(2\pi)^{-2(\rho+1)} \sum_{n\le T} \int_T^{2T} x^{3\rho/2+2k-9/4} f^2(n) n^{-(2\rho+5)/4} \cos^2\left\{8\pi(nx)^{1/4} + \frac{1-2\rho}{4}\pi\right\} dx \\
&= 2^{-2\rho-3}\pi^{-2(\rho+1)} \sum_{n\le T} f^2(n) n^{-(2\rho+5)/4} \int_T^{2T} x^{3\rho/2+2k-9/4} \\
&\qquad \times \left\{1+\cos\left(16\pi(nx)^{1/4} + \left(\frac{1}{2}-\rho\right)\pi\right)\right\} dx \\
&= 2^{-2\rho-3}\pi^{-2(\rho+1)} \left(\frac{3}{2}\rho+2k-\frac{5}{4}\right)^{-1} ((2T)^{3\rho/2+2k-5/4} - T^{3\rho/2+2k-5/4}) \\
&\qquad \times \sum_{n=1}^\infty f^2(n) n^{-(2\rho+5)/4} + O(T^{\rho+2k-3/2+\epsilon}) + O(T^{\rho+2k-1+\epsilon}).
\end{aligned}
$$

In deriving (2.2), we have used the bound

$$
(2.3)\qquad \sum_{n\le x} f(n) \ll \sum_{n\le x} |f(n)| = \sum_{n\le x} \frac{|c_n|}{n^{k-3/2}} \ll x,
$$

partial summation, and the following classical lemma (see [11, Lemma 4.3]).

Lemma 2.1. *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotone and $F'(x)\ge m>0$ or $F'(x)\le -m<0$ for $a\le x\le b$. Then*

$$
(2.4)\qquad \left|\int_a^b G(x)e^{iF(x)}\,dx\right| \le 4Gm^{-1},
$$

where $G(x)$ is a positive monotone function for $a \le x \le b$ such that $|G(x)| \le G$.

The formula $2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta)$ implies that the terms in (2.1) for which $m \ne n$ are a multiple of

$$\sum_{m\ne n\le T} f(m)f(n)(mn)^{-(2\rho+5)/8}\int_T^{2T}\cos(8\pi(mx)^{1/4}-8\pi(nx)^{1/4})\cdot x^{3\rho/2+2k-9/4}\,dx$$
$$+\sum_{m\ne n\le T} f(m)f(n)(mn)^{-(2\rho+5)/8}$$
$$\times\int_T^{2T}\cos(8\pi(mx)^{1/4}+8\pi(nx)^{1/4}+(1/2-\rho)\pi)x^{3\rho/2+2k-9/4}\,dx$$
$$=: S_1+S_2, \quad \text{say}.$$

By estimating the integral in S_2 with the use of (2.4), we obtain

$$S_2 \ll \sum_{m<n\le T} f(m)f(n)(mn)^{-(2\rho+5)/8}T^{3\rho/2+2k-3/2}(\sqrt[4]{m}+\sqrt[4]{n})^{-1}.$$

Using (2.3), we derive

$$S_2 \ll T^{\rho+2k-1}.$$

Analogously, we have

$$S_1 \ll T^{3\rho/2+2k-3/2}\sum_{m,n\le T}|f(m)|\,|f(n)|(mn)^{-(2\rho+5)/8}(\sqrt[4]{m}-\sqrt[4]{n})^{-1}$$
$$= T^{3\rho/2+2k-3/2}\Big(\sum_{n\le m/2}+\sum_{n>m/2}\Big) =: T^{3\rho/2+2k-3/2}(S_1'+S_2'), \quad \text{say}.$$

Further, we also have

$$S_1' \ll \sum_{m\le T,\, n\le m/2}|f(m)|\,|f(n)|(mn)^{-(2\rho+5)/8}m^{-1/4}$$
$$\le \Big(\sum_{m\le T}|f(m)|m^{-(2\rho+7)/8}\Big)\Big(\sum_{n\le T}|f(n)|n^{-(2\rho+5)/8}\Big) \ll T^{(1-\rho)/2},$$

and

$$S_2' \ll \sum_{m\le T}|f(m)|m^{-(2\rho+5)/8}\sum_{m/2<n<m}|f(n)|n^{-(2\rho+5)/8}(\sqrt[4]{m}-\sqrt[4]{n})^{-1}$$
$$\ll T^{(3-2\rho)/8}\sum_{m/2<n<m}|f(n)|n^{(1-2\rho)/8}(m-n)^{-1} \ll T^{(1-\rho)/2}\log T.$$

Combining the above estimates proves Theorem 1.3.

3. Proof of Theorem 1.4

LEMMA 3.1. *Suppose $y > 1$. Define*

$$s_\rho(l,h,y;f) := \sum_{\substack{\sqrt[4]{n_1}+\cdots+\sqrt[4]{n_l}=\sqrt[4]{n_{l+1}}+\cdots+\sqrt[4]{n_h}\\ n_j\le y,\,1\le j\le h}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \quad (1\le l<h).$$

Then

$$|s_\rho(l,h;f) - s_\rho(l,h,y;f)| \ll y^{-(1+2\rho)/4+\epsilon}.$$

Proof. This lemma can be proved by the same argument of Lemma 3.1 of [14], so we omit the details. ∎

LEMMA 3.2. *Suppose $h \ge 3$, $(i_1,\dots,i_{h-1}) \in \{0,1\}^{h-1}$ and*

$$\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{h-1}}\sqrt[4]{n_h} \ne 0.$$

Then

$$|\sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{h-1}}\sqrt[4]{n_h}| \gg \max(n_1,\dots,n_h)^{-(4^{h-2}-4^{-1})}.$$

Proof. See for example Lemma 2.2 in [14] or Lemma 1 in [12]. ∎

Let $T \ge 10$ and $y > T^\varepsilon$ be a parameter to be determined later. For any $T \le x \le 2T$, define

$$\mathcal{R}_1 = \mathcal{R}_1(x;y) := (2\pi)^{-(\rho+1)} x^{3\rho/4+k-9/8} \times \sum_{n\le y} \frac{f(n)}{n^{(2\rho+5)/8}} \cos\left(8\pi\sqrt[4]{nx} + \frac{1-2\rho}{4}\pi\right),$$

$$\mathcal{R}_2 = \mathcal{R}_2(x;y) := D_\rho(x;Z_F) - \mathcal{R}_1.$$

We shall show that the higher-power moment of $\mathcal{R}_2$ is small and hence the integral $\int_T^{2T} D_\rho^h(x;Z_F)\,dx$ can be well approximated by $\int_T^{2T}\mathcal{R}_1^h\,dx$, which is easy to evaluate.

3.1. Evaluation of the integral $\int_T^{2T}\mathcal{R}_1^h\,dx$. For simplicity we set $\mathbb{I} = \{0,1\}$ and

$$\mathbb{N}^h = \{\mathbf{n} = (n_1,\dots,n_h) : n_j \in \mathbb{N},\ 1\le j\le h\}.$$

For each element $\mathbf{i} = (i_1,\dots,i_{h-1}) \in \mathbb{I}^{h-1}$, put $|\mathbf{i}| = i_1 + \cdots + i_{h-1}$. By the elementary formula

$$\cos a_1 \cdots \cos a_h = \frac{1}{2^{h-1}} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \cos(a_1 + (-1)^{i_1}a_2 + (-1)^{i_2}a_3 + \cdots + (-1)^{i_{h-1}}a_h),$$

we have

$$\mathcal{R}_1^h = (2\pi)^{-(\rho+1)h} x^{(3\rho/4+k-9/8)h} \sum_{n_1\le y} \cdots \sum_{n_h\le y} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \prod_{j=1}^{h} \cos\left(8\pi\sqrt[4]{n_j x} + \frac{1-2\rho}{4}\pi\right)$$

$$= \frac{x^{(3\rho/4+k-9/8)h}}{(2\pi)^{(\rho+1)h}2^{h-1}} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{n_1\le y} \cdots \sum_{n_h\le y} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \cos\left(8\pi\sqrt[4]{x}\,\alpha(\mathbf{n};\mathbf{i}) + \frac{1-2\rho}{4}\pi\beta(\mathbf{i})\right),$$

where

$$\alpha(\mathbf{n};\mathbf{i}) := \sqrt[4]{n_1} + (-1)^{i_1}\sqrt[4]{n_2} + (-1)^{i_2}\sqrt[4]{n_3} + \cdots + (-1)^{i_{h-1}}\sqrt[4]{n_h},$$
$$\beta(\mathbf{i}) := 1 + (-1)^{i_1} + (-1)^{i_2} + \cdots + (-1)^{i_{h-1}}.$$

Thus we can write

$$\mathcal{R}_1^h = \frac{1}{(2\pi)^{(\rho+1)h}2^{h-1}}(S_1(x) + S_2(x)), \tag{3.1}$$

where

$$S_1(x) = x^{(3\rho/4+k-9/8)h} \times \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \cos\left(\frac{1-2\rho}{4}\pi\beta(\mathbf{i})\right) \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}},$$

$$S_2(x) = x^{(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \cos\left(8\pi\sqrt[4]{x}\,\alpha(\mathbf{n};\mathbf{i}) + \frac{1-2\rho}{4}\pi\beta(\mathbf{i})\right).$$

For the contribution of $S_1(x)$, we have

$$\int_T^{2T} S_1(x)\,dx = \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \cos\left(\frac{1-2\rho}{4}\pi\beta(\mathbf{i})\right) \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \int_T^{2T} x^{(3\rho/4+k-9/8)h}\,dx.$$

It is easily seen that if $\alpha(\mathbf{n};\mathbf{i}) = 0$, then $1 \le |\mathbf{i}| \le h-1$. Let $l = |\mathbf{i}|$. Then

$$\sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} = s_\rho(l,h,y;f),$$

where $s_\rho(l, h, y; f)$ has been defined in Lemma 3.1. Hence

$$\int_T^{2T} S_1(x)\,dx = B_\rho^*(h; f)\int_T^{2T} x^{(3\rho/4+k-9/8)h}\,dx + O(T^{1+(3\rho/4+k-9/8)h+\epsilon}y^{-(1+2\rho)/4}),$$

where

$$B_\rho^*(h; f) := \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \cos\biggl(\frac{1-2\rho}{4}\pi\beta(\mathbf{i})\biggr) \sum_{\substack{(n_1,\ldots,n_h)\in\mathbb{N}^h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}}.$$

For any $\mathbf{i}\in\mathbb{I}^{h-1}\backslash 0$, let

$$S_\rho(\mathbf{i}, h; f) := \sum_{\substack{(n_1,\ldots,n_h)\in\mathbb{N}^h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}}.$$

It is easily seen that if $|\mathbf{i}| = |\mathbf{i}'|$ or $|\mathbf{i}| + |\mathbf{i}'| = h$, then

$$S_\rho(\mathbf{i}, h; f) = S_\rho(\mathbf{i}', h; f) = s_\rho(|\mathbf{i}|, h; f).$$

From $(-1)^j = 1 - 2j$ $(j = 0, 1)$ we also have $\beta(\mathbf{i}) = h - 2|\mathbf{i}|$. So we get

$$\begin{aligned} B_\rho^*(h; f) &= \sum_{l=1}^{h-1}\sum_{|\mathbf{i}|=l} \cos\biggl(\frac{1-2\rho}{4}\pi\beta(\mathbf{i})\biggr) s_\rho(|\mathbf{i}|, h; f) \\ &= \sum_{l=1}^{h-1} s_\rho(l, h; f)\cos\biggl(\frac{1-2\rho}{4}(h-2l)\pi\biggr)\sum_{|\mathbf{i}|=l} 1 \\ &= \sum_{l=1}^{h-1} \binom{h-1}{l} s_\rho(l, h; f)\cos\biggl(\frac{1-2\rho}{4}(h-2l)\pi\biggr) = B_\rho(h; f). \end{aligned}$$

Now we consider the contribution of $S_2(x)$. By Lemma 3.2 we get

$$\begin{aligned} \int_T^{2T} S_2(x)\,dx &\ll T^{(3\rho/4+k-9/8)h+3/4} \\ &\quad\times \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{\substack{n_j\le y,\,1\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}|\alpha(\mathbf{n};\mathbf{i})|} \\ &\ll T^{(3\rho/4+k-9/8)h+3/4} y^{4^{h-2}+(3-2\rho)h/8-1/4} \\ &\ll T^{(3\rho/4+k-9/8)h+3/4} y^{b_\rho(h)}. \end{aligned}$$

Here we have used the elementary estimate

$$\int_T^{2T} \cos(A\sqrt[4]{t} + B)dt \ll T^{3/4}|A|^{-1}, \quad A \ne 0. \tag{3.2}$$

Combining the above estimates, we obtain the following:

LEMMA 3.3. *For any fixed $h \geq 3$, we have*

$$\int_T^{2T} \mathcal{R}_1^h \, dx = \frac{B_\rho(h;f)}{(2\pi)^{(\rho+1)h} 2^{h-1}} \int_T^{2T} x^{(3\rho/4+k-9/8)h} \, dx + O(T^{1+(3\rho/4+k-9/8)h+\epsilon} y^{-(1+2\rho)/4} + T^{(3\rho/4+k-9/8)h+3/4} y^{b_\rho(h)}).$$

3.2. Higher-power moments of $\mathcal{R}_2$. Taking $N = T$ in Lemma 1.2 and combining with the definition of $\mathcal{R}_2$ we get

(3.3)
$$\mathcal{R}_2 = (2\pi)^{-(\rho+1)} x^{3\rho/4+k-9/8} \sum_{y<n\leq T} \frac{f(n)}{n^{(2\rho+5)/8}} \cos\left\{8\pi(nx)^{1/4} + \frac{1-2\rho}{4}\pi\right\} + O(T^{\rho/2+k-1+\epsilon}) \ll \left| x^{3\rho/4+k-9/8} \sum_{y<n\leq T} \frac{f(n)}{n^{(2\rho+5)/8}} e(4(nx)^{1/4}) \right| + T^{\rho/2+k-1+\epsilon},$$

which implies

(3.4)
$$\int_T^{2T} \mathcal{R}_2^2 \, dx \ll T^{\rho+2k-1+\epsilon} + \int_T^{2T} \left| x^{3\rho/4+k-9/8} \sum_{y<n\leq T} \frac{f(n)}{n^{(2\rho+5)/8}} e(4(nx)^{1/4}) \right|^2 dx$$
$$\ll T^{\rho+2k-1+\epsilon} + T^{3\rho/2+2k-5/4} \sum_{y<n\leq T} \frac{f^2(n)}{n^{(2\rho+5)/4}} + T^{3\rho/2+2k-3/2} \sum_{y<m<n\leq T} \frac{f(m)f(n)}{(mn)^{(2\rho+5)/8}(\sqrt[4]{n} - \sqrt[4]{m})}$$
$$\ll T^{\rho+2k-1+\epsilon} + T^{3\rho/2+2k-5/4} y^{-(1+2\rho)/4} \ll T^{3\rho/2+2k-5/4} y^{-(1+2\rho)/4}.$$

Here we use the bound (1.3) and the estimate

$$\sum_{y<m<n\leq T} \frac{f(m)f(n)}{(mn)^{(2\rho+5)/8}(\sqrt[4]{n} - \sqrt[4]{m})} \ll T^{(1-\rho)/2+\epsilon},$$

which can be proved in a standard way.

Now suppose y satisfies $y^{4b_\rho(H_0)} \leq T$. Then by Lemma 3.3, we obtain

$$\int_T^{2T} |\mathcal{R}_1|^{H_0} \, dx \ll T^{1+(3\rho/4+k-9/8)H_0+\epsilon},$$

which implies

$$\int_T^{2T} |\mathcal{R}_1|^{A_0}\,dx \ll T^{1+(3\rho/4+k-9/8)A_0+\epsilon} \tag{3.5}$$

since $A_0 \le H_0$. From (1.10) and (3.5) we get

$$\begin{aligned}\int_T^{2T} |\mathcal{R}_2|^{A_0}\,dx &\ll \int_T^{2T} (|D_1(x;Z_F)|^{A_0} + |\mathcal{R}_1|^{A_0})\,dx \\ &\ll T^{1+(3\rho/4+k-9/8)A_0+\epsilon}.\end{aligned} \tag{3.6}$$

For any $2 < A < A_0$, by (3.4), (3.6) and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T} |\mathcal{R}_2|^{A}\,dx &= \int_T^{2T} |\mathcal{R}_2|^{\frac{2(A_0-A)}{A_0-2}+\frac{A_0(A-2)}{A_0-2}}\,dx \\ &\ll \Big(\int_T^{2T} \mathcal{R}_2^2\,dx\Big)^{\frac{A_0-A}{A_0-2}} \Big(\int_T^{2T} |\mathcal{R}_2|^{A_0}\,dx\Big)^{\frac{A-2}{A_0-2}} \\ &\ll T^{1+(3\rho/4+k-9/8)A+\epsilon} y^{-\frac{(1+2\rho)(A_0-A)}{4(A_0-2)}}.\end{aligned}$$

Therefore we have

Lemma 3.4. *Suppose $T^{\varepsilon} \le y \le T^{\min(\frac{2\rho-1}{2\rho+1},\frac{1}{4b_\rho(H_0)})}$, $2 < A < A_0$. Then*

$$\int_T^{2T} |\mathcal{R}_2|^{A}\,dx \ll T^{1+(3\rho/4+k-9/8)A+\epsilon} y^{-\frac{(1+2\rho)(A_0-A)}{4(A_0-2)}}.$$

3.3. Upper bound of the integral $\int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2\,dx$. In this subsection we shall estimate the integral $\int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2\,dx$. We suppose $T^{\epsilon} \le y \le T^{\min(\frac{2\rho-1}{2\rho+1},\frac{1}{4b_\rho(H_0)})}$, which, combined with Lemma 3.3, implies that

$$\int_T^{2T} \mathcal{R}_1^{h-1}\,dx \ll T^{1+(3\rho/4+k-9/8)(h-1)}.$$

Thus from (3.3) we get

$$\int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2\,dx = \int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2^{*}\,dx + O(T^{(9-2\rho)/8+(3\rho/4+k-9/8)h+\epsilon}), \tag{3.7}$$

where

$$\mathcal{R}_2^{*} = (2\pi)^{-2} x^{3\rho/4+k-9/8} \sum_{y<n\le T} \frac{f(n)}{n^{(2\rho+5)/8}} \cos\Big\{8\pi(nx)^{1/4} + \frac{1-2\rho}{4}\pi\Big\}.$$

Similarly to (3.1) we can write

$$\mathcal{R}_1^{h-1}\mathcal{R}_2^{*} = \frac{1}{(2\pi)^{(\rho+1)h}2^{h-1}}(S_3(x)+S_4(x)),$$

where

$$S_3(x) := x^{(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \cos\biggl(\frac{1-2\rho}{4}\pi\beta(\mathbf{i})\biggr),$$

$$S_4(x) := x^{(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \times \cos\biggl(8\pi\sqrt[4]{x}\,\alpha(\mathbf{n};\mathbf{i}) + \frac{1-2\rho}{4}\pi\beta(\mathbf{i})\biggr).$$

By Lemma 3.1, the contribution of $S_3(x)$ is

$$\begin{aligned}
(3.8)\qquad &\int_T^{2T} S_3(x)\,dx \\
&\ll \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})=0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}} \int_T^{2T} x^{(3\rho/4+k-9/8)h}\,dx \\
&\ll \sum_{l=1}^{h-1} |s_\rho(l,h;f) - s_\rho(l,h,y;f)| \int_T^{2T} x^{(3\rho/4+k-9/8)h}\,dx \\
&\ll T^{1+(3\rho/4+k-9/8)h+\epsilon} y^{-(1+2\rho)/4}.
\end{aligned}$$

Also, using Lemma 3.2 and (3.2), the contribution of $S_4(x)$ is bounded by

$$\begin{aligned}
(3.9)\qquad &\int_T^{2T} S_4(x)\,dx \\
&\ll T^{3/4+(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{y<n_1\le T} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}|\alpha(\mathbf{n};\mathbf{i})|} \\
&\ll T^{3/4+(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{y<n_1\le h^4 y} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8}|\alpha(\mathbf{n};\mathbf{i})|} \\
&\quad + T^{3/4+(3\rho/4+k-9/8)h} \sum_{\mathbf{i}\in\mathbb{I}^{h-1}} \sum_{n_1> h^4 y} \sum_{\substack{n_j\le y,\, 2\le j\le h\\ \alpha(\mathbf{n};\mathbf{i})\ne 0}} \frac{f(n_1)\cdots f(n_h)}{(n_1\cdots n_h)^{(2\rho+5)/8} n_1^{1/4}} \\
&\ll T^{3/4+(3\rho/4+k-9/8)h} y^{b_\rho(h)}.
\end{aligned}$$

Combining the estimates in (3.7)–(3.9) we obtain

$$\int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2\,dx \ll T^{1+(3\rho/4+k-9/8)h+\epsilon}y^{-3/4} + T^{3/4+(3\rho/4+k-9/8)h}y^{b_\rho(h)} + T^{(9-2\rho)/8+(3\rho/4+k-9/8)h+\epsilon}, \tag{3.10}$$

where $T^\epsilon \le y \le T^{\min(\frac{2\rho-1}{2\rho+1},\frac{1}{4b_\rho(H_0)})}$.

Proof of Theorem 1.4. Suppose $3 \le h < A_0$ and $T^\epsilon \le y \le T^{\min(\frac{2\rho-1}{2\rho+1},\frac{1}{4b_\rho(H_0)})}$. By the elementary formula $(a+b)^h = a^h + ha^{h-1}b + O(|a^{h-2}b^2| + |b|^h)$, we get

$$\int_T^{2T} D_\rho^h(x;Z_F)\,dx = \int_T^{2T} \mathcal{R}_1^h\,dx + h\int_T^{2T} \mathcal{R}_1^{h-1}\mathcal{R}_2\,dx + O\Big(\int_T^{2T} |\mathcal{R}_1^{h-2}\mathcal{R}_2^2|\,dx\Big) + O\Big(\int_T^{2T} |\mathcal{R}_2|^h\,dx\Big). \tag{3.11}$$

By (3.5), Lemma 3.4 and Hölder's inequality we get

$$\int_T^{2T} |\mathcal{R}_1^{h-2}\mathcal{R}_2^2|\,dx \ll \Big(\int_T^{2T} |\mathcal{R}_1|^{A_0}\,dx\Big)^{\frac{h-2}{A_0}} \Big(\int_T^{2T} |\mathcal{R}_2|^{\frac{2A_0}{A_0-h+2}}\,dx\Big)^{\frac{A_0-h+2}{A_0}} \ll T^{1+(3\rho/4+k-9/8)h+\epsilon}y^{-\frac{(1+2\rho)(A_0-h)}{4(A_0-2)}}. \tag{3.12}$$

Now take $y = T^{\min(\frac{2\rho-1}{2\rho+1},\frac{1}{4b_\rho(H_0)})}$. Collecting the estimates in Lemma 3.3, Lemma 3.4, and (3.10)–(3.12), we finally obtain

$$\int_T^{2T} D_\rho^h(x;Z_F)\,dx = \frac{B_\rho(h;f)}{(2\pi)^{(\rho+1)h}2^{h-1}} \int_T^{2T} x^{(3\rho/4+k-9/8)h}\,dx + O(T^{1+(3\rho/4+k-9/8)h-\delta_\rho(h,A_0)+\epsilon}). \tag{3.13}$$

Hence, Theorem 1.4 follows from (3.13) immediately. ■

4. Proof of Theorem 1.5. The following lemma is the well-known Halász–Montgomery inequality (see [5, (A.40)]).

LEMMA 4.1. *Let S be an inner-product vector space over $\mathbb{C}$, let (a,b) denote the inner product in S and $\|a\|^2 = (a,a)$. Suppose that $\xi,\varphi_1,\dots,\varphi_R$ are arbitrary vectors in S. Then*

$$\sum_{l_1\le R} |(\xi,\varphi_{l_1})|^2 \le \|\xi\|^2 \max_{l_1\le R} \sum_{l_2\le R} |(\varphi_{l_1},\varphi_{l_2})|.$$

Lemma 4.2. *Suppose $T \le x_1 < \cdots < x_M \le 2T$ satisfy $|D_1(x_l; Z_F)| \gg VT^{k-1/2}$ $(l = 1, \ldots, M)$ and $|x_j - x_i| \gg V \gg T^{1/13}\mathcal{L}^{14/13}$ $(i \ne j)$. Then*

$$M \ll \mathcal{L}^3 T V^{-3} + \mathcal{L}^{17} T^3 V^{-17},$$

where $\mathcal{L} := \log T$.

Proof. Suppose $V < T_0$ is a parameter to be determined later. Let I be any subinterval of $[T, 2T]$ of length not exceeding T_0 and let $G = I \cap \{x_1, \ldots, x_M\}$. Without loss of generality, we may suppose $G = \{x_1, \ldots, x_{M_0}\}$, where $M_0 \le M$.

Taking $x = T$, by Lemma 1.2, we have

$$T^{k+\epsilon} N^{-1/2} \ll VT^{k-1/2-\epsilon}, \quad T^{k-1/2} N^{\epsilon} \ll VT^{k-1/2-\epsilon}.$$

Then we get

$$N \gg V^{-2} T^{1+4\epsilon}.$$

It is easy to check that the estimate of $D_1(x; Z_F)$ is much better when we take smaller N. Hence, we take $N = 2^{J+1} \asymp V^{-2}T^{1+4\epsilon}$, where $J := \left[\frac{(1+4\epsilon)\mathcal{L} - 2\log V}{\log 2}\right]$.

By Lemma 1.2, for $T \le x \le 2T$, we obtain

$$\begin{aligned} D_1(x; Z_F) &\ll T^{k-3/8} \Big| \sum_{n \le N} f(n) n^{-7/8} e(4(nx)^{1/4}) \Big| + VT^{k-1/2-\epsilon} \\ &\ll T^{k-3/8} \Big| \sum_{j=0}^{J+1} \sum_{n \sim 2^j} f(n) n^{-7/8} e(4(nx)^{1/4}) \Big| + VT^{k-1/2-\epsilon}. \end{aligned}$$

Squaring, summing over G and applying the Cauchy inequality, we get

$$\begin{aligned} \sum_{l \le M_0} D_1^2(x_l; Z_F) &\ll T^{2k-3/4} \sum_{l \le M_0} \Big| \sum_{j} \sum_{n \sim 2^j} f(n) n^{-7/8} e(4(nx_l)^{1/4}) \Big|^2 \\ &\ll T^{2k-3/4} \mathcal{L} \sum_{l \le M_0} \sum_{j} \Big| \sum_{n \sim 2^j} f(n) n^{-7/8} e(4(nx_l)^{1/4}) \Big|^2 \\ &\ll T^{2k-3/4} \mathcal{L}^2 \sum_{l \le M_0} \Big| \sum_{n \sim 2^{j_0}} f(n) n^{-7/8} e(4(nx_l)^{1/4}) \Big|^2, \end{aligned}$$

for some $0 \le j_0 \le J$.

Let $N_0 = 2^{j_0}$. Take $\xi = \{\xi_n\}_{n=1}^{\infty}$ with $\xi_n = f(n) n^{-7/8}$ for $n \sim N_0$ and zero otherwise, and take $\varphi_l = \{\varphi_{l,n}\}_{n=1}^{\infty}$ with $\varphi_{l,n} = e(4(nx_l)^{1/4})$ for $n \sim N_0$ and zero otherwise. Then we have

$$(\xi, \varphi_l) = \sum_{n \sim N_0} \frac{f(n)}{n^{7/8}} e(4(nx_l)^{1/4}), \quad (\varphi_{l_1}, \varphi_{l_2}) = \sum_{n \sim N_0} e(4n^{1/4}(x_{l_1}^{1/4} - x_{l_2}^{1/4})),$$

and

$$\|\xi\|^2 = \sum_{n\sim N_0} \frac{f^2(n)}{n^{7/4}} \ll N_0^{-7/4} \sum_{n\sim N_0} f(n)^2 \ll N_0^{-3/4}, \tag{4.1}$$

where we use the bound $\sum_{n\le x} |f(n)|^2 \ll x$.

From (4.1) and Lemma 4.1, we get

$$\begin{aligned} M_0V^2 &\ll T^{1/4}\mathcal{L}^2 \sum_{l\le M_0} \Big|\sum_{n\sim N_0} f(n)n^{-7/8}e(4(nx_l)^{1/4})\Big|^2 \\ &\ll T^{1/4}\mathcal{L}^2 N_0^{-3/4} \max_{l_1\le M_0} \sum_{l_2\le M_0} \Big|\sum_{n\sim N_0} e(4n^{1/4}(x_{l_1}^{1/4}-x_{l_2}^{1/4})\Big| \\ &\ll T^{1/4}\mathcal{L}^2 N_0^{1/4} + \frac{T^{1/4}\mathcal{L}^2}{N_0^{3/4}} \max_{l_1\le M_0} \sum_{\substack{l_2\le M_0\\ l_1\ne l_2}} \Big|\sum_{n\sim N_0} e(4n^{1/4}(x_{l_1}^{1/4}-x_{l_2}^{1/4})\Big|. \end{aligned} \tag{4.2}$$

By the Kuz'min–Landau inequality taking the exponent pair

$$(1/7, 6/7) = C_{5/7}((0,1),(1/2,1/2)),$$

we get

$$\begin{aligned} \sum_{n\sim N_0} e(4n^{1/4}(\sqrt[4]{x_{l_1}}-\sqrt[4]{x_{l_2}})) &\ll \frac{N_0^{3/4}}{|\sqrt[4]{x_{l_1}}-\sqrt[4]{x_{l_2}}|} + \left(\frac{|\sqrt[4]{x_{l_1}}-\sqrt[4]{x_{l_2}}|}{N_0^{3/4}}\right)^{1/7} N_0^{6/7} \\ &\ll \frac{(N_0T)^{3/4}}{|x_{l_1}-x_{l_2}|} + \left(\frac{|x_{l_1}-x_{l_2}|}{(N_0T)^{3/4}}\right)^{1/7} N_0^{6/7} \\ &\ll \frac{(N_0T)^{3/4}}{|x_{l_1}-x_{l_2}|} + T^{-3/28}T_0^{1/7}N_0^{3/4}, \end{aligned}$$

where we use the mean value theorem and the estimate $|x_{l_1}-x_{l_2}| \le T_0$.

Inserting this estimate back into (4.2) we conclude that

$$\begin{aligned} M_0V^2 &\ll T^{1/4}\mathcal{L}^2 N_0^{1/4} + \frac{T^{1/4}\mathcal{L}^2}{N_0^{3/4}} \max_{l_1\le M_0} \sum_{\substack{l_2\le M_0\\ l_1\ne l_2}} \left(\frac{(N_0T)^{3/4}}{|x_{l_1}-x_{l_2}|} + T^{-3/28}T_0^{1/7}N_0^{3/4}\right) \\ &\ll T^{1/4}\mathcal{L}^2 N_0^{1/4} + \mathcal{L}^3TV^{-1} + \mathcal{L}^2(TT_0)^{1/7}M_0 \\ &\ll \mathcal{L}^3TV^{-1} + \mathcal{L}^2(TT_0)^{1/7}M_0, \end{aligned}$$

where we use the facts that $\{x_l\}$ is V-spaced and $N_0 \asymp V^{-2}T^{1+4\epsilon}$.

Taking $T_0 = V^{14}T^{-1}\mathcal{L}^{-14}$, if $V \gg T^{1/13}\mathcal{L}^{14/13}$, it is easy to check that $T_0 \gg V$. For this T_0 we get

$$M_0 \ll \mathcal{L}^3TV^{-3}.$$

Now we divide the interval $[T, 2T]$ into $O(1+T/T_0)$ subintervals of length not exceeding T_0. In each interval of this type, the number of x_l's is at most $O(\mathcal{L}^3TV^{-3})$. So we have

$$M \ll \mathcal{L}^3TV^{-3}(1+TT_0^{-1}) \ll \mathcal{L}^3TV^{-3} + \mathcal{L}^{17}T^3V^{-17}.$$

This completes the proof of Lemma 4.2. ∎

Lemma 4.3. *For any $\epsilon > 0$, we have*

$$\int_1^T |D_1(x;Z_F)|^{16/3}\,dx \ll T^{16k/3-1+\epsilon}.$$

Proof. Suppose $x^\epsilon \ll y \ll x/2$. By the definition of $D_1(x;Z_F)$ we get

$$D_1(x;Z_F) = \frac{1}{\Gamma(2)}\sum_{n\le x}(x-n)c_n.$$

So we have

$$\begin{aligned}|D_1(x+y;Z_F) - D_1(x;Z_F)| &\ll \Big|\sum_{n\le x+y}(x+y-n)c_n - \sum_{n\le x}(x-n)c_n\Big| \\ &\ll y\sum_{n\le x}c_n + y\sum_{x<n\le x+y}c_n \ll x^{k-1/2}y,\end{aligned}$$

where we use the bound $\sum_{x<n\le x+y}|c_n| \ll x^{k-3/2}y$. So there exists an absolute constant $c_0 > 0$ such that

$$|D_1(x+y;Z_F) - D_1(x;Z_F)| \le c_0x^{k-1/2}y,$$

which implies that if $|D_1(x;Z_F)| \ge 2c_0x^{k-1/2}y$, then

$$|D_1(x+y;Z_F)| \ge |D_1(x;Z_F)| - |D_1(x+y;Z_F) - D_1(x;Z_F)| \ge c_0x^{k-1/2}y.$$

This together with (1.5) and an argument similar to (13.70) of [5] yields

$$\int_1^T |D_1(x;Z_F)|^A\,dx \ll T^{1+(k-3/8)A} + \sum_V V\sum_{r\le N_V}|D_1(x_r;Z_F)|^A, \tag{4.3}$$

where

$$T^{1/8} \le V \le T^{1/5}, \quad VT^{k-1/2} < |D_1(x_r;Z_F)| \le 2VT^{k-1/2} \quad (r = 1,\dots,N_V),$$

and

$$|x_r - x_s| \ge V \quad \text{for } r \ne s \le N = N_V.$$

Then by Lemma 4.2, we get

$$\begin{aligned}V\sum_{r\le N_V}|D_1(x_r;Z_F)|^A &\ll T^{(k-1/2)A}N_VV^{A+1} \\ &\ll \mathcal{L}^3T^{1+(k-1/2)A}V^{A-2} + \mathcal{L}^{17}T^{3+(k-1/2)A}V^{A-16} \\ &=: I_1 + I_2, \quad \text{say.}\end{aligned}$$

Suppose $A > 2$. From

$$I_1 \ll \mathcal{L}^3T^{1+(k-1/2)A}T^{(A-2)/5} \ll T^{1+(k-3/8)A},$$

we can deduce $A \leq 16/3$. It is easy to check that $I_2 \ll T^{1+(k-3/8)A}\mathcal{L}^{17}$ for $0 \leq A \leq 16/3$. Back to (4.3), we complete the proof of Lemma 4.3. ∎

Proof of Theorem 1.5. Suppose $T^\epsilon < y \leq T^{1/3}$. Taking $\rho = 1$ and applying the arguments of Lemmas 4.2 and 4.3 directly to $\mathcal{R}_1$, we obtain

$$\int_T^{2T} |\mathcal{R}_1|^{16/3}\, dx \ll T^{16/3k-1+\epsilon}.$$

Then by the definition of $\mathcal{R}_2$ and Lemma 4.3, we have

$$\int_T^{2T} |\mathcal{R}_2|^{16/3}\, dx \ll T^{16/3k-1+\epsilon}.$$

By the same argument in the last subsection, we find that for $h = 3, 4, 5$,

$$\int_T^{2T} D_1^h(x; Z_F)\, dx = \int_T^{2T} \mathcal{R}_1^h\, dx + O(T^{1+(k-3/8)h+\epsilon} y^{-\sigma_1(h,16/3)} + T^{(k-3/8)h+3/4} y^{b_1(h)}).$$

Taking $y = T^{1/(4\sigma_1(h,16/3)+4b_1(h))}$, by Lemma 3.3 we get

$$\int_T^{2T} D_1^h(x; Z_F)\, dx = \frac{B_1(h; f)}{2^{3h-1}\pi^{2h}} \int_T^{2T} x^{(k-3/8)h}\, dx + O(T^{1+(k-3/8)h-\lambda_1(h,16/3)+\epsilon}).$$

Hence, Theorem 1.5 follows by dyadic subdivision. ∎

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