On sums of two \( k \)th powers: 
a mean-square asymptotics over short intervals 

by 

MANFRED KÜHLEITNER and WERNER GEORG NOWAK (Wien)

1. Introduction. For \( k \geq 2 \) a fixed integer, define the arithmetic function \( r_k(n) \) as the number of ways to write \( n \in \mathbb{N}^* \) as a sum of two \( k \)th powers of absolute values of integers, i.e.,
\[
 r_k(n) = \# \{(u_1, u_2) \in \mathbb{Z}^2 : |u_1|^k + |u_2|^k = n \}.
\]
To describe its average behaviour, one is interested in asymptotic results about the Dirichlet summatory function
\[
 R_k(u) = \sum_{1 \leq n \leq u^k} r_k(n),
\]
where \( u \) is a large real variable \(^1\).

For \( k = 2 \), the classic Gaussian circle problem, a detailed historical exposition can be found in the monograph of Krätzel [10]. The sharpest published results to date \(^2\) read
\[
\begin{align*}
(1.1) \quad & R_2(u) = \pi u^2 + P_2(u), \\
(1.2) \quad & P_2(u) = O(u^{46/73} (\log u)^{315/146}),
\end{align*}
\]
and \(^3\)
\[
\begin{align*}
(1.3) \quad & P_2(u) = \Omega_- (u^{1/2} (\log u)^{1/4} (\log \log u)^{1/4} ) \times \exp(-c\sqrt{\log \log \log u}) \quad (c > 0), \\
(1.4) \quad & P_2(u) = \Omega_+ (u^{1/2} \exp(c' (\log \log u)^{1/4} (\log \log \log u)^{-3/4}) \quad (c' > 0).
\end{align*}
\]

\(^1\) Note that, in part of the relevant literature, \( t = u^2 \) is used as the basic variable.

\(^2\) Actually, M. Huxley has meanwhile improved further this upper bound, essentially replacing the exponent \( 46/73 = 0.6301\ldots \) by \( 131/208 = 0.6298\ldots \). The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

\(^3\) We recall that \( F_1(u) = \Omega_+(F_2(u)) \) means that \( \limsup_{u \to \infty} (\ast F_1(u)/F_2(u)) > 0 \) where \( \ast \) is either + or −, and \( F_2(u) \) is positive for \( u \) sufficiently large.
While (1.2) is due to Huxley [5], [7], (1.3) has been established by Hafner [4], and (1.4) by Corrádi & Kátaï [2]. Most experts conjecture that
\[
\inf\{\theta \in \mathbb{R} : P_2(u) \ll \theta u^\theta\} = 1/2.
\]
This hypothesis is supported by the mean-square asymptotics
\[
\int_0^T (P_2(u))^2 \, du = C_2 T^2 + O(T (\log T)^2), \quad C_2 = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{(r_2(n))^2}{n^{3/2}},
\]
which in this precise form is due to Kátaï [8].

The results (1.3), (1.4), (1.6) were obtained by means of the fact that the generating function (Dirichlet series) of \( r_2(n) \) is the Epstein zeta-function of the quadratic form \( u_1^2 + u_2^2 \), which satisfies a well known functional equation and thus opens the possibility of an approach via complex integration.

For the general case \( k \geq 3 \), quite different methods must be employed. Investigations in this direction have first been undertaken by van der Corput [18] and Krätzel [9]. In Krätzel’s textbook [10], an enlightening exposition of the history of the problem including all results until 1988 can be found. It turns out that
\[
R_k(u) = \frac{2\Gamma^2(1/k)}{k \Gamma(2/k)} u^2 + B_k \Phi_k(u) u^{1-1/k} + P_k(u)
\]
where
\[
B_k = 2^{3-1/k} \pi^{-1-1/k} k^{1/k} \Gamma \left( 1 + \frac{1}{k} \right), \quad \Phi_k(u) = \sum_{n=1}^{\infty} n^{-1-1/k} \sin \left( 2\pi nu - \frac{\pi}{2k} \right),
\]
and the new remainder term \( P_k(u) \) can essentially be bounded by (1.2), i.e.,
\[
P_k(u) = O(u^{46/73} (\log u)^{315/146}).
\]
This was proved by Kuba [11], on the basis of Huxley’s method [5], [7].

For lower bounds, it was shown by the second named author [15] that, for any fixed \( k \geq 3 \),
\[
P_k(u) = \Omega_-(u^{1/2} (\log u)^{1/4}),
\]
and by Kühleitner, Nowak, Schoißengeier & Wooley [13] that
\[
P_3(u) = \Omega_+(u^{1/2} (\log \log u)^{1/4}).
\]
The analogy between these results and those for the case \( k = 2 \) might suggest extending the classic conjecture (1.5) to arbitrary \( k \geq 2 \). In fact, this is true again in mean-square: According to Nowak [14],
\[
\frac{1}{T} \int_0^T (P_k(u))^2 \, du \ll T
\]
for any fixed $k \geq 3$ and $T$ large. Kühleitner [12] refined this result, proving an asymptotic formula

$$\frac{1}{T} \int_0^T (P_k(u))^2 \, du = C_k \, T + O(T^{1 - \varepsilon_0(k)}),$$

with explicitly given $\varepsilon_0(k) > 0$ and

$$C_k := \frac{4}{\pi^2(k-1)} \sum_{(h_1, m_1, h_2, m_2) \in \mathbb{Z}^4} (h_1 m_1 h_2 m_2)^{-1 + 1/2} \|(h_1, m_1)\|_q \|h_2, m_2\|_q.$$ 

Here $q = k/(k-1)$ and $\|(h, m)\|_q = (|h|^q + |m|^q)^{1/q}$ denotes the $q$-norm in $\mathbb{R}^2$.

Inspired by a work of Huxley [6] on the lattice point discrepancy of a convex disc, the second named author recently [16] proved a localized form of (1.11), with only a logarithmic loss of accuracy, namely

$$\int_{T-1/2}^{T+1/2} (P_k(u))^2 \, du \ll T \log T.$$ 

In view of (1.9), this result seems pretty close to what might be possible. Nevertheless, our aim in the present article is to shed some more light on this short-interval behaviour of this remainder term. It will turn out that the bound in (1.14) (even refined by a factor $\log T$) remains valid for an interval up to a length of order $\log T$. In fact, it will be shown that, for any fixed $c_1 > 0$, 

$$\int_{T-c_1 \log T}^{T+c_1 \log T} (P_k(u))^2 \, du \ll T \log T.$$ 

Furthermore, we shall see that, as soon as the interval becomes a little longer, we can observe essentially the same asymptotic behaviour as stated in (1.12).

**Theorem.** Let $k \geq 3$ be a fixed integer, $T$ a large real variable, and $T \mapsto \Lambda = \Lambda(T)$ an increasing function such that $\Lambda(T) \leq \frac{1}{2} T$ throughout and

$$\lim_{T \to \infty} \frac{\log T}{\Lambda(T)} = 0.$$ 

Then, as $T \to \infty$,

$$\int_{T-A}^{T+A} (P_k(u))^2 \, du \sim 4C_k \, \Lambda T,$$

the constant $C_k$ being defined in (1.13).
2. Two pivotal lemmas

 Lemma 1 (Transition from fractional parts to trigonometric sums according to Vaaler [17]; see also Graham & Kolesnik [3], p. 116). For arbitrary \( w \in \mathbb{R} \) and \( H \in \mathbb{N}^* \), let

\[
\psi(w) = w - [w] - \frac{1}{2}, \quad \psi^*_H(w) = -\frac{1}{\pi} \sum_{h=1}^{H} \frac{\sin(2\pi hw)}{h} \tau\left(\frac{h}{H + 1}\right),
\]

where

\[
\tau(\xi) = \pi \xi (1 - \xi) \cot(\pi \xi) + \xi \quad \text{for } 0 < \xi < 1.
\]

Then

\[
|\psi(w) - \psi^*_H(w)| \leq \frac{1}{H + 1} \sum_{h=1}^{H} \left(1 - \frac{h}{H + 1}\right) \cos(2\pi hw) + \frac{1}{2H + 2}.
\]

Lemma 2. Let \( k \geq 3 \) be a positive integer, and \( q = k/(k-1) \). Then, for \( M \to \infty \),

\[
S(M) := \sum_{(h_1, m_1, h_2, m_2) \in \mathbb{Z}_+^4 \atop |(h_1, m_1)|_q = |(h_2, m_2)|_q \leq M} (h_1 h_2 m_1 m_2)^{-1+q/2} |(h_1, m_1)|_q^{1-2q} \ll M^{-1/2}.
\]

Proof. For positive integers \( h_1, h_2, m_1, m_2 \) the condition \( |(h_1, m_1)|_q = |(h_2, m_2)|_q \) is satisfied if and only if either \( (h_1, m_1) = (h_2, m_2) \) or \( h_1, h_2, m_1, m_2 \) all have the same maximal \((k-1)\)-free divisor \( r \), say, i.e.,

\[
h_1 = a^{k-1} r, \quad m_1 = b^{k-1} r, \quad h_2 = c^{k-1} r, \quad m_2 = d^{k-1} r,
\]

with \( a, b, c, d \in \mathbb{N}^* \) satisfying \( a^k + b^k = c^k + d^k \). This follows from the fact that, by a classic theorem of Besicovitch [1], the \((k-1)\)th roots of distinct \((k-1)\)-free positive integers are linearly independent over the rationals. Therefore, the sum in question is

\[
\ll R_1(M) + R_2(M)
\]

with

\[
R_1(M) = \sum_{\substack{h_1, m_1=1 \atop m_1 \gg M}}^{\infty} (h_1 m_1)^{-2+q}(h_1 m_1)^{-q+1/2},
\]

\[
R_2(M) = \sum_{\substack{a \leq b, c \leq d \leq d \leq r \gg M \atop b^{k-1} r, d^{k-1} r \gg M}} \frac{(abcd)^{(k-1)(-1+q/2)}}{r^{-3}((a^{k-1}, b^{k-1})|_q (c^{k-1}, d^{k-1})|_q)^{-q+1/2}},
\]

since

\[
|(h_1, m_1)|_q = r|(a^{k-1}, b^{k-1})|_q \quad \text{and} \quad |(h_2, m_2)|_q = r|(c^{k-1}, d^{k-1})|_q.
\]
Clearly,

\[ R_1(M) \ll \sum_{m_1 \gg M} m_1^{-3/2} \ll M^{-1/2}. \]

We estimate the contribution of \( R_2(M) \) in the cases \( k = 3, 4, \) resp. \( k \geq 5 \) in two different ways. In the first case we use

\[
\frac{1}{|(u^{k-1}, v^{k-1})|_q^{q-1/2}} \ll (uv)^{-\frac{1}{2}(k-1)(q-1/2)},
\]

to conclude that

\[
R_2(M) \ll \sum_{b^{k-1}r, d^{k-1}r \gg M} \sum_{a, c=1}^{\infty} (abcd)^{(k-1)(-1+q/2)} r^{-3} (abcd)^{-\frac{1}{2}(k-1)(q-1/2)}
\]

\[
\ll \sum_{r=1}^{\infty} r^{-3} \left( \sum_{b \gg (M/r)^{1/(k-1)}} b^{-\frac{3}{4}(k-1)} \right)^2 \ll \sum_{r=1}^{\infty} r^{-3} \left( \frac{M}{r} \right)^{2/(k-1)-3/2}
\]

\[
\ll M^{-1/2}.
\]

In the case \( k \geq 5 \) we use the fact that

\[
\sum_{a, c=1}^{\infty} (ac)^{(k-1)(-1+q/2)} \ll 1,
\]

to infer

\[
R_2(M) \ll \sum_{b^{k-1}r, d^{k-1}r \gg M} (bd)^{(k-1)(-1+q/2-q+1/2)} r^{-3}
\]

\[
\ll \sum_{r=1}^{\infty} r^{-3} \left( \sum_{b \gg (M/r)^{1/(k-1)}} b^{-k+1/2} \right)^2
\]

\[
\ll \sum_{r=1}^{\infty} r^{-3} \left( \frac{M}{r} \right)^{(3-2k)/(k-1)} \ll M^{-7/4}.
\]

### 3. Proof of the Theorem.

Throughout what follows, let \( T \) and \( M \) be large real parameters, independent of each other. All constants implied in the symbols \( O, \ll, \) or \( \asymp \) do not depend on \( M \) and \( T, \) but may depend on \( k. \)

Suppose that \( u \in [T - \Lambda, T + \Lambda] \subseteq \left[ \frac{1}{2} T, \frac{3}{2} T \right], \) thus \( u \asymp T \) as \( T \to \infty. \)

For any complex-valued function \( f : u \mapsto f(u) \) which is square-integrable on \([T - \Lambda, T + \Lambda],\) we shall write

\[
Q(f) = Q_{T,\Lambda}(f) := \int_{T - \Lambda}^{T + \Lambda} |f(u)|^2 du.
\]
By Cauchy’s inequality, for arbitrary $f_1, f_2 \in L^2[T - A, T + A],
(3.2) \quad Q(f_1 + f_2) \leq 2(Q(f_1) + Q(f_2)),$
which will be used frequently in what follows.

We start from formulae (3.57), (3.58) (and the asymptotic expansion below) of Kratzel [10], p. 148. In our notation, this reads

\begin{equation}
(3.3) \quad P_k(u) = -8 \sum_{2^{-1/k}u < n \leq u} \psi((u^k - n^k)^{1/k}) + O(1),
\end{equation}

with $\psi(w) = w - [w] - 1/2$ throughout. We define $q$ by $1/k + 1/q = 1$, i.e., $q = k/(k - 1)$, and thus $1 < q \leq 3/2$. We break up the range of summation into subintervals $N_j(u) = [N_j, N_{j+1}]$, where $N_j = u(1 + 2^{-j}q)^{-1/k}$, $j = 0, 1, \ldots, J$, with $J$ minimal such that $u - N_j < 1$ for all $u \in [T - A, T + A]$. It is clear that $J \ll \log T$. Furthermore, the length of any $N_j(u)$ is equal to $N_{j+1} - N_j \approx 2^{-j}qT$. By means of Lemma 1, $\psi$ will be approximated by $\psi^*_H$, with $H := [T]$. Thus overall $P_k(u)$ is approximated by

\begin{equation}
(3.4) \quad P_k^*(u) := -8 \sum_{j=0}^{J} \sum_{n \in N_j(u)} \psi^*_H((u^k - n^k)^{1/k}).
\end{equation}

By the definition in Lemma 1,

\begin{equation}
(3.5) \quad \sum_{n \in N_j(u)} \psi^*_H((u^k - n^k)^{1/k})
= -\frac{1}{\pi} \sum_{1 \leq h \leq T} \frac{1}{h} \left( \frac{h}{H + 1} \right) \sum_{n \in N_j(u)} \sin(2\pi h(u^k - n^k)^{1/k}).
\end{equation}

The innermost sum on the right hand side is now subject to a van der Corput transformation (“B-step”). See Kühleitner [12], Lemmas 2 and 3, for details. In particular, we use formula (3.5) from [12] which reads (with $u$ instead of $\sqrt{t}$, and $e(z) = e^{2\pi iz}$ as usual)

\begin{equation}
(3.6) \quad \sum_{n \in N_j(u)} e(-h(u^k - n^k)^{1/k})
= \frac{e(1/8)}{\sqrt{k-1}} hu^{1/2} \sum_{m \in M_j(h)} (hm)^{-1+q/2} |(h, m)|^{-q+1/2} e(-u|\langle h, m \rangle_q)
+ O(j + \log(1 + h)),
\end{equation}

\footnote{The idea of this special choice of subdivision points is that $\frac{ddw}{dw}((u^k - w^k)^{1/k})$ assumes integer values at $w = N_j$. This is useful for the van der Corput transformation of the exponential sums involved.}
Sums of two $k$th powers

where $\mathcal{M}_j(h) = [2^j h, 2^{j+1} h]$, $|(h, m)|_q = (|h|^q + |m|^q)^{1/q}$ denotes the $q$-norm in $\mathbb{R}^2$, and $\sum''$ means that the terms corresponding to $m = 2^j h$ and $m = 2^{j+1} h$ get a factor $1/2$.

Using the imaginary part of (3.6) in (3.5), we obtain

$$\sum_{n \in \mathcal{N}_j(u)} \psi^*_H((u^k - n^k)^{1/k})$$

$$= \frac{u^{1/2}}{\sqrt{k-1}} \sum_{1 \leq h \leq T} \sum''_{m \in \mathcal{M}_j(h)} \frac{(hm)^{-1+q/2}}{|(h, m)|_q^{q-1/2}} \sin(\pi/4 - 2\pi u|(h, m)|_q) \quad + O(\log T)$$

with

$$\gamma_0(h, T) := \frac{1}{\pi^2} \left( \frac{h}{|T| + 1} \right).$$

In fact, the main contribution to our mean-square asymptotics will come from a truncation of the double sum here, namely (5)

$$\sum_j(M, u) := \frac{u^{1/2}}{\sqrt{k-1}} \sum_{1 \leq h \leq T} \sum''_{m \in \mathcal{M}_j(h)} \frac{(hm)^{-1+q/2}}{|(h, m)|_q^{q-1/2}} \sin(\pi/4 - 2\pi u|(h, m)|_q).$$

What about the errors we commit by these approximations? First of all, evidently,

$$\sum_{n \in \mathcal{N}_j(u)} \psi^*_H((u^k - n^k)^{1/k}) - \sum_j(M, u)$$

$$\ll T^{1/2} \left| \sum_{1 \leq h \leq T} \sum''_{m \in \mathcal{M}_j(h)} \gamma_1(h, m, T) \frac{(hm)^{-1+q/2}}{|(h, m)|_q^{q-1/2}} \cos(2\pi u|(h, m)|_q) \right| + \log T$$

with

$$\gamma_1(h, m, T) := \begin{cases} \gamma_0(h, T) & \text{if } |(h, m)|_q > M, \\ 0 & \text{else.} \end{cases}$$

Furthermore, by Lemma 1,

$$\sum_{n \in \mathcal{N}_j(u)} (\psi((u^k - n^k)^{1/k}) - \psi^*_H((u^k - n^k)^{1/k}))$$

$$\ll \sum_{1 \leq h \leq T} \frac{1 - h/([T] + 1)}{[T] + 1} \sum_{n \in \mathcal{N}_j(u)} \cos(2\pi h(u^k - n^k)^{1/k}) + 2^{-jq}.$$
Applying again (3.6) to the cosine sum here, we see that this is

\[ T_{10} = 2X_1 h_t \]

\[ X_0 M_j \]

\[ m^2 \]

\[ (h_1)^{1+q/2} \]

\[ e(u(h,m)|q) \]

\[ + 2^{-jq} \]

with

\[ \gamma_2(h,m,T) := \frac{(1 - h/([T] + 1))h}{[T] + 1}. \]

The great similarity of the main parts of the expressions (3.9) and (3.10) enables us to estimate their mean-square by essentially the same calculation. Let

\[ R_j(u) := \sum_{1 \leq h \leq T} \sum_{m \in M_j(h)}'' \gamma(h,m,T) \frac{(hm)^{1+q/2}}{|(h,m)|^{q-1/2}} e(u(h,m)|q) \]

where \( \gamma \) is either of \( \gamma_1, \gamma_2 \).

We want to bound \( Q(R_j) \). To this end, we employ an ingenious trick due to Huxley [6] which involves the Fejér kernel

\[ \varphi(w) := \left( \frac{\sin(\pi w)}{\pi w} \right)^2. \]

By Jordan’s inequality, \( \varphi(w) \geq 4/\pi^2 \) for \( |w| \leq 1/2 \), and the Fourier transform has the simple shape

\[ \hat{\varphi}(y) = \int_{\mathbb{R}} \varphi(w)e(yw) \, dw = \max(0, 1 - |y|). \]

Therefore,

\[ Q(R_j) \]

\[ = 2A \int_{-1/2}^{1/2} |R_j(T + 2A w)|^2 \, dw \leq \frac{\pi^2}{2} A \int_{\mathbb{R}} \varphi(w)|R_j(T + 2A w)|^2 \, dw \]

\[ = \frac{\pi^2}{2} A \sum_{1 \leq h_1, h_2 \leq T} \sum_{m_1 \in M_j(h_1)}'' \gamma(h_1,m_1,T) \gamma(h_2,m_2,T) \frac{(h_1 m_1 h_2 m_2)^{-1+q/2}}{|(h_1,m_1)|^q |(h_2,m_2)|^q}^{q-1/2} \]

\[ \times e(-T(|(h_1,m_1)| - |(h_2,m_2)|)) \]

\[ \times \int_{\mathbb{R}} \varphi(w)e(-2A w(|(h_1,m_1)| - |(h_2,m_2)|)) \, dw \]

\[ \ll A \sum_{1 \leq h_1, h_2 \leq T} \sum_{m_1 \in M_j(h_1)}'' \frac{(h_1 m_1 h_2 m_2)^{-1+q/2}}{|(h_1,m_1)|^q |(h_2,m_2)|^q}^{q-1/2} \]

\[ \times \max(0, 1 - 2A |(|(h_1,m_1)| - |(h_2,m_2)|)|). \]
We recall that \( m \in \mathcal{M}_j(h) \) implies that \(|(h, m)|_q \asymp m \asymp 2^j h\). Furthermore, for a term of the last multiple sum to be nonzero it is necessary that \(|(h_1, m_1)|_q - |(h_2, m_2)|_q| < (2A)^{-1}\), hence \( h_1 \asymp h_2 \) and \( m_1 \asymp m_2 \). Therefore, the last expression in (3.11) is

\[
(3.12) \quad \ll A 2^{-j(q+1)} \sum_{1 \leq h_1 \leq T} h_1^{-3} \sum_{m_1 \in \mathcal{M}_j(h_1)} \gamma(h_1, m_1, T) \times \sum_{(h_2, m_2) \in \mathbb{Z}^2_+} \gamma(h_2, m_2, T) \quad \text{where} \quad |(h_1, m_1)|_q - |(h_2, m_2)|_q < (2A)^{-1}
\]

We now have to distinguish if we are dealing with \( \gamma_1 \) or \( \gamma_2 \), recalling the respective definitions: For \( \gamma_1(h, m, T) \), we know that this is bounded and vanishes for \(|(h, m)|_q \leq M\). Further, denote by \( A^*_q(U) \) the number of lattice points \( v \in \mathbb{Z}^2 \) with \(|v|_q \leq U\); then it is known that

\[
(3.13) \quad A^*_q(U) = \frac{2\Gamma^2(1/q)}{q\Gamma(2/q)} U^2 + O(U^{2/3})
\]

for any fixed \( q \) with \( 1 < q < 2 \). This asymptotic formula is contained in Theorem 3.6 of Krätzel [10], p. 116. From this it is immediate that, for any fixed \((h_1, m_1)\),

\[
(3.14) \quad \sum_{(h_2, m_2) \in \mathbb{Z}^2} 1 \ll \frac{|(h_1, m_1)|_q}{A} + |(h_1, m_1)|_q^{2/3} \quad \text{for} \quad |(h_1, m_1)|_q - |(h_2, m_2)|_q < (2A)^{-1}
\]

Thus, for \( \gamma = \gamma_1 \), the expression in (3.12) is

\[
(3.15) \quad \ll A 2^{-j} \sum_{1 \leq h_1 \leq T} h_1^{-2} \left( \frac{2j h_1}{A} + (2j h_1)^{2/3} \right)
\]

\[
\ll 2^{-j(q-1)} \log T + A M^{-1/6} 2^{-j(q-5/6)} \sum_{1 \leq h_1 \leq T} h_1^{-7/6}
\]

\[
\ll 2^{-j(q-1)} (\log T + A M^{-1/6}).
\]

For \( \gamma = \gamma_2 \), we may use that \( \gamma_2(h, m, T) \ll h T^{-1} \). Thus (3.12) is now, again by (3.14),

\[
(3.16) \quad \ll A 2^{-j(q+1)} T^{-2} \sum_{1 \leq h_1 \leq T} h_1^{-1} \sum_{m_1 \in \mathcal{M}_j(h_1)} \left( \frac{|(h_1, m_1)|_q}{A} + |(h_1, m_1)|_q^{2/3} \right)
\]

\[
\ll A 2^{-j(q+1)} T^{-2} \sum_{1 \leq h_1 \leq T} \left( \frac{2^j h_1}{A} + 2^{5j/3} h_1^{2/3} \right)
\]

\[
\ll 2^{-j(q-1)} (1 + A T^{-1/3}).
\]
Let us summarize what we have proved so far: The remainder term in question can be represented as

\[ P_k(u) = \sum_{j=0}^{J} (-8\Sigma_j(M, u) + \Delta_j(M, u)), \]

where \( \Sigma_j(M, u) \) has been defined in (3.8) and \( \Delta_j(M, u) \) satisfies (in view of (3.9), (3.10), (3.15), (3.16))

\[ Q(\Delta_j(M, u)) \ll 2^{-j(q-1)}(T \log T + \Lambda T^{1/6} + \Lambda T^{2/3}) + A(\log T)^2. \]

To proceed further, let \( \delta \) be a positive constant, less than \( \frac{1}{2} (q-1) \) and small compared to \( (\log T)/J \). Then, by Cauchy’s inequality,

\[
Q \left( \sum_{j=0}^{J} \Delta_j(M, u) \right) \leq \int_{T-\Lambda}^{T+\Lambda} \sum_{j=0}^{J} 2^{-j\delta} \sum_{j=0}^{J} 2^{j\delta} \left| \Delta_j(M, u) \right|^2 \, du
\]

\[
\ll \sum_{j=0}^{J} 2^{j\delta} Q(\Delta_j(M, u)) \ll T \log T + \Lambda T^{1/6} + \Lambda T^{2/3}. \]

Adding up the main terms \( \Sigma_j(M, u) \), we arrive at:

**Proposition.** Uniformly in \( T-\Lambda \leq u \leq T+\Lambda \),

\[ P_k(u) = \Sigma(M, u) + \Delta(M, u), \]

with

\[ Q(\Delta(M, u)) \ll T \log T + \Lambda T^{1/6} + \Lambda T^{2/3} \]

and

\[
\Sigma(M, u) := -8u^{1/2} \pi \sqrt{k-1} \sum_{1 \leq h \leq T} \tau \left( \frac{h}{\lceil T \rceil + 1} \right)
\]

\[
\times \sum_{|\langle h, m \rangle_q| \leq M} \sum_{h \leq m \leq h2^{J+1}} \frac{(hm)^{-1+q/2}}{|\langle h, m \rangle_q|^{-1/2}} \cos(\pi/4 + 2\pi u |\langle h, m \rangle_q|),
\]

where \( \sum' \) means that the terms corresponding to \( m = h \) and \( m = h2^{J+1} \) get a factor \( 1/2 \).

Next we infer from the definition in Lemma 1 that \( \tau(w) = 1 + O(w^2) \). Therefore, defining

\[
\Sigma^{(0)}(M, u)
\]

\[
:= -8u^{1/2} \pi \sqrt{k-1} \sum_{1 \leq h \leq T} \sum_{|\langle h, m \rangle_q| \leq M} \sum_{h \leq m \leq h2^{J+1}} \frac{(hm)^{-1+q/2}}{|\langle h, m \rangle_q|^{-1/2}} \cos(\pi/4 + 2\pi u |\langle h, m \rangle_q|),
\]
it is immediate that

\[ (3.18) \quad \Sigma(M, u) = \Sigma^{(0)}(M, u) + O(K_1(M)T^{-3/2}), \]

where \( K_1(M), K_2(M), \ldots \) will denote appropriate bounds depending on \( M \) (but not on \( T \)). If we keep \( M \) fixed and make \( T \) (and thus \( u \)) large, the summation conditions \( h \leq T \) and \( m \leq h^{2f+1} \) ultimately become meaningless, and \( \Sigma^{(0)}(M, u) \) becomes equal to

\[ \Sigma^{(1)}(M, u) := \frac{-4u^{1/2}}{\pi \sqrt{k - 1}} \sum_{\substack{|(h,m)|_q \leq M \\backslash (h,m) \in \mathbb{N}^*}} \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} \cos(\pi/4 + 2\pi u|(h,m)|_q). \]

We now square out \((\Sigma^{(1)}(M, u))^2\), using the elementary formula

\[ \cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B)), \]

and integrate over \( u \in [T - A, T + A] \). The main contribution comes from the diagonal terms, i.e. those with \(|(h_1,m_1)|_q = |(h_2,m_2)|_q\), and reads altogether

\[ \frac{16}{\pi^2(k - 1)}AT \sum_{\substack{|(h_1,m_1)|_q = |(h_2,m_2)|_q \leq M \\backslash h_1,m_1,h_2,m_2 \in \mathbb{N}^*}} \frac{(h_1m_1h_2m_2)^{-1+q/2}}{|(h_1,m_1)|_q^{2q-1}}. \]

By Lemma 2 and the definition of the constant \( C_k \) in (1.13), this is equal to

\[ 4AT(C_k + O(M^{-1/2})). \]

All the other terms are pretty small: In fact,

\[ T+\lambda \int_{T-\lambda} \cos(2\pi u|(h_1,m_1)|_q \pm |(h_2,m_2)|_q) \, du \]

\[ \ll \frac{T}{|(h_1,m_1)|_q \pm |(h_2,m_2)|_q}, \]

which contributes altogether \( \ll K_2(M)T \) to \( Q(\Sigma^{(1)}(M, u)) \). Going back to (3.18) and to the Proposition, and applying Cauchy’s inequality one more time, we end up with

\[ (3.19) \quad Q(P_k) = 4C_kAT + O(K_3(M)T) + O(T(A \log T)^{1/2}) \]

\[ + O(AM^{-1/12}) + O(AT^{5/6}). \]

Therefore, for any fixed \( M \),
\[ \limsup_{T \to \infty} \left| \frac{1}{AT} Q(P_k) - 4C_k \right| \ll M^{-1/12}, \]

if we recall our condition (1.16). Since \( M \) can be chosen arbitrarily large, the proof of our Theorem is thereby complete.

We finally establish (1.15). To this end, it suffices to choose \( M = 1/2 \) in the above argument; then all sums over \( 0 < |(h, m)|_q \leq M \) are empty, and (3.19) yields what we claimed, since now \( A \asymp \log T \).

References


Institut für Mathematik und angewandte Statistik
Universität für Bodenkultur
Peter Jordan-Straße 82
A-1190 Wien, Austria
E-mail: kleitner@edv1.boku.ac.at
nowak@mail.boku.ac.at
Web: http://www.boku.ac.at/math/nth.html

Received on 10.8.2000