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Rational solutions of certain Diophantine equations involving norms

by

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1. Introduction. Let k be a number field and K/k be an algebraic extension of degree n. There are a lot of papers devoted to the study of k-rational solutions of Diophantine equations of the form

$$N_{K/k}(X_1\omega_1 + \cdots + X_n\omega_n) = f(t), \tag{1.1}$$

where $N_{K/k}$ is a full norm form for the extension K/k, $\{\omega_1, \ldots, \omega_n\}$ is a fixed basis of the extension and $f(t)$ is a polynomial over k. The main problem here is whether the Hasse principle, or in other words the local-to-global principle, holds for the smooth proper model of the hypersurface given by (1.1). For example, if $f(t)$ is constant and K/k is cyclic or of prime degree, then the local-to-global principle holds for (1.1) (Hasse).

If $n = 2$ and $\deg f = 3$ or 4 then the variety defined by (1.1) is called a *Châtelet surface*. The arithmetic of these surfaces is well understood. In particular, in [2, 3] it is proved that the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only one. Moreover, the existence of a k-rational solution implies k-unirationality. These results are unconditional. However, the most general result in this area is obtained under Schinzel's hypothesis (H) and says that if K is a cyclic extension of a number field k, and $f(t)$ is a separable polynomial of arbitrary degree, then the Brauer–Manin obstruction to the Hasse principle and weak approximation is the only one for the smooth and projective model X of the variety given by (1.1). Moreover, if there is no Brauer–Manin obstruction to the Hasse principle then the k-rational points are Zariski dense in X.

Most of the results in this area were proved using algebraic considerations (via the computation of the Brauer–Manin obstructions) or a combination of algebraic methods together with analytic techniques (see for example [5]).

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However, only a few papers present constructions which allow producing new solutions from a given k-rational solution of (1.1). As mentioned in [5, p. 162], this is usually a rather difficult problem.

We work with a field k of characteristic 0 and an algebraic extension K/k of degree n. We take $\omega_i = \alpha^{i-1}$ for $i = 1, \dots, n$, where $\alpha \in K$ is chosen in such a way that $K = k(\alpha)$. We are thus interested in the equation

$$\mathrm{N}_{K/k}(X_1, \dots, X_n) = f(t), \tag{1.2}$$

where to shorten notation we put

$$\mathrm{N}_{K/k}(X_1, \dots, X_n) := N_{K/k}(X_1 + \alpha X_2 + \dots + \alpha^{n-1} X_n),$$

i.e. $\mathrm{N}_{K/k}$ will denote a norm form, and $N_{K/k}$ the corresponding field norm. In what follows, by a *non-trivial* solution of (1.2) we mean a solution $(X_1, \dots, X_n, t)$ which satisfies $f(t) \neq 0$. We show that in some cases the existence of one k-rational solution of (1.2) implies the existence of infinitely many k-rational solutions. This is obtained mainly by constructing a parametric solution of the corresponding equation, or, in a more geometric language, by constructing a k-rational curve lying on the corresponding algebraic variety. Of course, we are only interested in the existence of k-rational curves which are not contained in the fiber of the map $\Phi : \mathcal{S}_f \ni (X_1, \dots, X_n, t) \mapsto t \in \mathbb{P}^1(k)$. Our argument is based on a similar approach to the one proposed by Mestre in a series of papers [6, 7, 8] devoted to the study of the existence of rational points on (generalized) Châtelet surfaces, i.e. surfaces defined by (1.2) with $n = 2$ and $\deg f \geq 5$.

Let us describe the content of the paper in some detail. In Section 2, we prove that if K/k is a pure cubic extension generated by a root of $h(x) = x^3 + b \in k[x]$, $f \in k[t]$ is of degree 4, and the variety $\mathcal{S}_f$ defined by (1.2) contains a non-trivial k-rational point, then $\mathcal{S}_f$ is unirational over k. In particular, the set of k-rational points on $\mathcal{S}_f$ is Zariski dense. We prove a similar result for $f \in k[t]$ of degree 5, provided that f satisfies some mild conditions. In particular, if f is an irreducible polynomial, then $\mathcal{S}_f$ is k-unirational. We also prove that if $f \in k[t]$ is monic of degree 6 and $\mathcal{S}_f$ contains a non-trivial k-rational point, and f is not equivalent to a polynomial $h \in k[t]$ satisfying $h(t) \neq h(\zeta_3 t)$, then $\mathcal{S}_f$ is k-unirational. This result is particularly interesting in the light of recent work of Várilly-Alvarado and Viray [9]. Indeed, in the case under consideration the variety $\mathcal{S}_f$ is a so called *Châtelet threefold* (in the terminology of [9]). The authors of [9] asked whether the existence of a k-rational point on $\mathcal{S}_f$ implies k-unirationality [9, Problem 6.2]. Our result shows that $\mathcal{S}_f$ is k-unirational for a broad class of polynomials. Moreover, if k is a number field with a real embedding, we prove that for each polynomial $f(t) = a_0 t^6 + \sum_{i=0}^{4} a_{6-i} t^i \in k[t]$ and any given $\epsilon > 0$ there exists a polynomial $g(t) = c_0 t^6 + \sum_{i=0}^{4} c_{6-i} t^i \in k[t]$ which is close to f, i.e.

$|a_i - c_i| < \epsilon$ for $i = 0, 2, \ldots, 6$, and such that for any $b \in k \setminus k^3$ and a pure cubic extension K/k generated by a root of $h(x) = x^3 + b$, the variety $\mathcal{S}_g$ is unirational over k.

In Section 3, we consider the variety $\mathcal{S}_f$ defined by (1.2) involving a norm form of an extension K/k generated by a root of an irreducible polynomial $h(x) = x^3 + ax + b \in k[x]$. We prove that if $f(t) = t^6 + a_4t^4 + a_1t + a_0 \in k[t]$ with $a_1a_4 \neq 0$ then $\mathcal{S}_f$ is unirational over k. Moreover, we make a remark concerning unirationality of slightly more general varieties defined by equations of the form $F(x, y, z) = f(t)$, where F is a homogeneous form of degree 3 and f is a polynomial.

2. Solutions of $\mathrm{N}_{K/k}(X_1, X_2, X_3) = f(t)$ with K/k pure cubic and f of degree ≤ 6. Let k be a field of characteristic 0 and K/k be an extension of degree 3 generated by a root, say α, of the irreducible polynomial $h(x) = x^3 + ax + b$ defined over k. We are interested in the rational points lying on the variety defined by the equation

$$\mathcal{S}_f : \ \mathrm{N}_{K/k}(X_1, X_2, X_3) = f(t), \tag{2.1}$$

where $f \in k[t]$. In this section we consider the case of f of degree ≤ 6. Since we are interested in k-unirationality of $\mathcal{S}_f$, we assume that the set of k-rational points on $\mathcal{S}_f$ is nonempty. To be more precise, we assume that there is a nontrivial k-rational point lying on $\mathcal{S}_f$, i.e. there is a $P = (x_0, y_0, z_0, t_0) \in \mathcal{S}_f(k)$ such that $f(t_0) \neq 0$. In particular, P is a smooth point on $\mathcal{S}_f$. In this section we consider the case of a pure cubic extension K/k, i.e. K is generated by a root of a polynomial h as above with $a = 0$. Let us recall that in this case

$$\mathrm{N}_{K/k}(X_1, X_2, X_3) = X_1^3 - bX_2^3 + b^2X_3^3 + 3bX_1X_2X_3.$$

Before we state our results, we note that $\mathcal{S}_f$ is isomorphic to $\mathcal{S}_g$, where $g(t) = \sum_{i=1}^{6} c_it^i + 1$. Indeed, making a change of variables $t \mapsto t + t_0$ we can assume that $f(0) = c_0 = \mathrm{N}_{K/k}(u, v, w) \neq 0$ for some $u, v, w \in k$. Multiplying this equation by $c_0^{-1} = \mathrm{N}_{K/k}(u', v', w')$ with u', v', w' such that $\mathrm{N}_{K/k}(u, v, w)\,\mathrm{N}_{K/k}(u', v', w') = 1$, and using the multiplicative property of the norm form, we get the desired form of our equation. It is clear that $\mathcal{S}_f$ is k-unirational if and only if $\mathcal{S}_g$ is.

We are ready to prove the following result.

THEOREM 2.1. *Let k be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Put $g(t) = 1 + \sum_{i=1}^{6} c_it^i \in k[t]$ and suppose that*

$$(c_2, c_4, c_5) \neq \left(\frac{5c_1^2}{12}, -\frac{1}{144}c_1(5c_1^3 - 72c_3), -\frac{1}{144}c_1^2(c_1^3 - 12c_3)\right). \tag{2.2}$$

Then the variety $\mathcal{S}_g$ is k-unirational.

Proof. Let $G = G(X_1, X_2, X_3, t)$ be a polynomial defining $\mathcal{S}_g$. We note that $\mathcal{S}_g$ contains the k-rational point $(1,0,0,0)$. We use it in order to construct a k-rational curve lying on $\mathcal{S}_g$. More precisely, we are looking for a rational curve, say $\mathcal{L}$, lying on $\mathcal{S}_g$. We assume that $\mathcal{L}$ can be parameterized by rational functions with parameter u in the following way:

$$\mathcal{L}:\ X_1 = pT^2 + qT + 1, \quad X_2 = rT^2, \quad X_3 = sT^2 + uT, \quad t = T, \tag{2.3}$$

where p, q, r, s, T need to be determined. With X_i and t defined above, we get $G(X_1, X_2, X_3, t) = \sum_{i=1}^{6} C_i T^i$, where

$$\begin{aligned} C_1 &= 3q - c_1, & C_3 &= b^2u^3 + 3bru + 6pq + q^3 - c_3,\\ C_2 &= 3p + 3q^2 - c_2, & C_4 &= 3(b^2su^2 + bqru + brs + p^2 + pq^2) - c_4, \end{aligned}$$

and $C_5, C_6 \in k[p,q,r,s,u]$ depend on c_i for $i = 1, \ldots, 6$. The system $C_1 = C_2 = C_3 = C_4 = 0$ has exactly one solution in p, q, r, s:

$$\begin{aligned} p &= \tfrac{1}{9}(3c_2 - c_1^2), & r &= \frac{-27b^2u^3 + 5c_1^3 - 18c_1c_2 + 27c_3}{81bu},\\ q &= \tfrac{1}{3}c_1, & s &= \frac{u(27b^2c_1u^3 - 5c_1^4 + 27c_2c_1^2 - 27c_3c_1 - 27c_2^2 + 81c_4)}{3(54b^2u^3 + 5c_1^3 - 18c_1c_2 + 27c_3)}. \end{aligned}$$

For these p, q, r, s we get $C_i = A_i/D$, $i = 5, 6$, and

$$DG(X_1, X_2, X_3, T) = A_5T^5 + A_6T^6$$

for $A_5, A_6 \in k[u]$ and $D = 3^{12}b^2u^3(54b^2u^3 + 5c_1^3 - 18c_1c_2 + 27c_3)^3$. We note that $\deg_u A_6 = 18$ and the leading coefficient of A_6 is $2^3 3^{18} b^{12}$. In particular $A_6 \neq 0$ as an element of $k[u]$. Moreover, $\deg_u A_5 = 15$, and $A_5 \neq 0$ as an element of $k[u]$ if and only if condition (2.2) is satisfied. In this case, we get a unique non-zero solution in T of the equation $T^5(A_5 + A_6T) = 0$. Indeed,

$$T = -\frac{A_5}{A_6} = \varphi(u) = \frac{2 \cdot 3^{19} b^{10}(5c_1^2 - 12c_2)u^{15} + \text{lower order terms in } u}{2^3 3^{18} b^{12} u^{18} + \text{lower order terms in } u}.$$

Summing up, the existence of a k-rational point P with $f(t_0) \neq 0$ implies that $\mathcal{S}_g$ contains a k-rational curve $\mathcal{L}$ which is not contained in any hyperplane defined by $t = t_0$ with $t_0 \in k$. This allows us to define the base change $t = \varphi(u)$ which gives the cubic surface $\mathcal{S}_{g\circ\varphi}$ defined over the field $k(u)$ with a smooth $k(u)$-rational point. This immediately implies the $k(u)$-unirationality of $\mathcal{S}_{g\circ\varphi}$ by [1, Proposition 1.3], and thus the k-unirationality of $\mathcal{S}_g$. Indeed, the map Ψ which guarantees the unirationality of $\mathcal{S}_{g\circ\varphi}$ extends to a dominant rational map (Ψ, φ), which gives the unirationality of $\mathcal{S}_g$ and thus of $\mathcal{S}_f$. ∎

COROLLARY 2.2. *Let k be a field of characteristic* 0 *and let K/k be a pure cubic extension. Let $f \in k[t]$ be of degree* 4 *and suppose that $\mathcal{S}_f$ contains a nontrivial k-rational point. Then $\mathcal{S}_f$ is k-unirational.*

Proof. We work with $\mathcal{S}_g$ where $g(t) = 1+\sum_{i=1}^{4} c_i t^i$ with $c_4 \neq 0$. We have $\mathcal{S}_g \simeq \mathcal{S}_f$. We need to check whether condition (2.2) is satisfied for all $c_i \in k$ for $i = 1, 2, 3, 4$. We see that (2.2) is not satisfied if and only if $(c_2, c_4, c_5) = (5c_1^2/12, c_1^4/144, 0)$. In particular $c_1 \neq 0$. Making the (invertible) substitution $t \mapsto 6t/c_1$ we are left with the problem of proving the unirationality of $\mathcal{S}_h$ with $h(t) = (3t^2 + 2t + 1)^2$. We assume that $\mathcal{L}$ can be parameterized by rational functions with parameter u in the following way:

$$\mathcal{L}: \; X_1 = T + 1, \quad X_2 = uT, \quad X_3 = pT, \quad t = qT, \tag{2.4}$$

where the parameters p, q, T still need to be determined. For X_1, X_2, X_3, t defined in this way we get $F = \sum_{i=1}^{4} C_i T^i$, where

$$C_1 = 3 - 6q, \quad C_2 = 3 + 3bpu - 15q^2,$$
$$C_3 = 1 + b^2p^3 + 3bpu - bu^3 - 18q^3, \quad C_4 = -9q^4.$$

We solve the system $C_1 = C_2 = 0$ with respect to p, q and get $p = 1/(4bu)$, $q = 1/2$. This substitution allows us to find an expression for T:

$$T = \frac{-64b^2u^6 - 32bu^3 + 1}{36bu^3}.$$

Together with the expressions for p, q, this gives equations (2.4) defining the rational parametric curve $\mathcal{L}$ lying on $\mathcal{S}_h$. Using now the same reasoning as at the end of the proof of Theorem 2.1, we get the result. ∎

REMARK 2.3. We have tried to prove the k-unirationality of $\mathcal{S}_g$ in the case when $g \in k[t]$ is of degree 5 and does not satisfy (2.2). Among other things we tried to replace $g(t)$ by $h(Y) = (1+vY)^6 g(Y/(1+vY))$. In this way we got the variety $\mathcal{S}_h$ via the substitution $X_i = Y_i/(1+vY)^2$ for $i = 1, 2, 3$ and $t = Y/(1+vY)$. Unfortunately, one can check that if g does not satisfy (2.2), then $h(T)$ does not satisfy it either. Because all our efforts failed, we state the following:

QUESTION 2.4. *Let k be a field of characteristic* 0 *and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Put $g(t) = 1 + \sum_{i=1}^{5} c_i t^i \in k[t]$ with $c_5 \neq 0$ and suppose that condition* (2.2) *is not satisfied. Is the variety $\mathcal{S}_g$ unirational over k?*

Note that if g does not satisfy (2.2), then g is reducible, namely

$$g(t) = -\frac{1}{144}(c_1^2t^2 + 6c_1t + 12)\big((c_1^3 - 12c_3)t^3 - c_1^2t^2 - 6c_1t - 12\big).$$

In particular, Theorem 2.1 implies that if g is irreducible of degree 5 then $\mathcal{S}_g$ is k-unirational and thus the set of k-rational points on $\mathcal{S}_g$ is Zariski dense. It is clear that the same is true for a polynomial f corresponding to g.

In a recent paper Várilly-Alvarado and Viray [9] introduced the notion of a Châtelet threefold, which is a variety defined by (1.2) with $n = 3$ and

$f \in k[t]$ of degree 6. They asked whether the existence of a k-rational point on $\mathcal{S}_f$ implies the k-unirationality of $\mathcal{S}_f$ [9, Problem 6.2]. The statement of Theorem 2.1 gives a broad family of polynomials f such that $\mathcal{S}_f$ is k-unirational. In the next corollary we make this result more explicit.

Before stating our result, we recall that two polynomials $f_1, f_2 \in k[t]$ are *equivalent* if $\deg f_1 = \deg f_2$ and there exist $\alpha, \beta \in k$ such that $f_2(t) = f_1(\alpha t + \beta)$.

COROLLARY 2.5. *Let k be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Let $f \in k[t]$ be of degree 6 and suppose that f is not equivalent to a polynomial $h \in k[t]$ satisfying $h(t) = h(\zeta_3 t)$, where ζ_3 is the primitive third root of unity. Suppose moreover that $\mathcal{S}_f$ contains a nontrivial k-rational point. Then $\mathcal{S}_f$ is k-unirational.*

Proof. First of all, note that the existence of a non-trivial k-rational point on $\mathcal{S}_f$ with f of degree 6 and the fact that the norm form is multiplicative imply that $\mathcal{S}_f \simeq \mathcal{S}_h$, where $h(t) = t^6 + \sum_{i=0}^{4} c_{6-i}t^i$ for some $c_j \in k$, $j = 2, 3, \ldots, 6$. This follows by a reasoning similar to the one just before Theorem 2.1. From our assumption on f we know that at least one of c_2, c_4, c_5 is non-zero. Making the change of variables $X_i = Y_i/T^2$ for $i = 1, 2, 3$ and $t = 1/T$ we get $\mathcal{S}_h \simeq \mathcal{S}_g$ with $g(T) = 1 + \sum_{i=2}^{6} c_i T^i$. We can now apply Theorem 2.1 to the variety $\mathcal{S}_g$. It is k-unirational provided that (2.2) is satisfied. In our case we have $c_1 = 0$ and thus (2.2) is not satisfied if and only if $c_2 = c_4 = c_5 = 0$, which is not the case. ∎

Using the corollary above in the case of a number field k with a real embedding in $\mathbb{R}$, we deduce the following interesting result.

THEOREM 2.6. *Let k be a number field with $k \subset \mathbb{R}$ and put $f(t) = a_0 t^6 + \sum_{i=0}^{4} a_{6-i}t^i \in k[t]$ with $a_0 \neq 0$. Then for each $\epsilon > 0$ there exists a polynomial $g(t) = c_0 t^6 + \sum_{i=0}^{4} c_{6-i}t^i \in k[t]$ such that $|a_i - c_i| < \epsilon$ for $i = 0, 2, \ldots, 6$ and for each pure cubic extension K/k of degree 3, the variety $\mathcal{S}_g$ given by the equation $\mathrm{N}_{K/k}(X_1, X_2, X_3) = g(t)$ is k-unirational.*

Proof. We work with $\mathcal{S}_h \simeq \mathcal{S}_f$, where $h(t) = t^6 f(1/t)$. We note that for any given $a_0 \in k^*$ we can find a triple $u, v, w \in k$ with $|\mathrm{N}_{K/k}(u, v, w) - a_0| < \epsilon$ and $\mathrm{N}_{K/k}(u, v, w) \neq 0$, which is a consequence of the density of the image of the norm map $\mathrm{N}_{K/k} : k^3 \to k$. Indeed, $\mathrm{N}_{K/k}(x, 0, 0) = x^3$ is a continuous function and thus $\overline{\mathrm{N}_{K/k}(k, 0, 0)} = \mathbb{R}$, where the closure is taken in the Euclidean topology. Then we take $c_0 = \mathrm{N}_{K/k}(u, v, w)$. If $h(t) \neq h(\zeta_3 t)$ we take $c_i = a_i$ for $i = 2, \ldots, 6$. If $h(t) = h(\zeta_3 t)$, we take $c_i = a_i$ for $i = 3, 6$, and $c_2 = c_4 = c$ for any $c \in k$ with $|c| < \epsilon$. Then we put $g(t) = c_0 t^6 + \sum_{i=0}^{4} c_{6-i}t^i$ and note that $\mathcal{S}_g$ contains a k-rational point at infinity. Moreover, $\mathcal{S}_g \simeq \mathcal{S}_{h'}$, where $h'(t) = t^6 g(1/t)$. From Corollary 2.5 we get the result. ∎

The above results motivate the following:

CONJECTURE 2.7. *Let k be a number field and K/k be a cyclic extension of degree 3. Let $f \in k[t]$ be of degree 6 and suppose that there exists a non-trivial k-rational point on $\mathcal{S}_f$. Then $\mathcal{S}_f$ is k-unirational.*

We finish this section with the following simple result.

THEOREM 2.8. *Let k be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Put $f(t) = t^{3m} + a_2t^m + a_1t + a_0 \in k[t]$ with $a_1 \neq 0$. Then the variety $\mathcal{S}_f$ is k-unirational.*

Proof. Let $F = F(X_1, X_2, X_3, t)$ be a polynomial defining $\mathcal{S}_f$. We put

$$X_1 = t^m, \quad X_2 = u, \quad X_3 = \frac{a_2}{3bu}.$$

For X_i defined in this way the polynomial F (in t) is of degree 1 with the root

$$t = \varphi(u) = -\frac{27b^2u^6 + 27ba_0u^3 - a_2^3}{27ba_1u^3},$$

which under the assumption $a_1 \neq 0$ is a non-constant element of $k(u)$. Thus the cubic surface $\mathcal{S}_{f\circ\varphi}$ is $k(u)$-unirational, which implies the k-unirationality of $\mathcal{S}_f$. ∎

3. Solutions of $\mathrm{N}_{K/k}(X_1, X_2, X_3) = f(t)$ for a general cubic extension and f of degree 6. We now consider the variety $\mathcal{S}_f$ given by (2.1) for a general extension K/k of degree 3 and a monic polynomial $f \in k[t]$ of degree 6. We assume that $K = k(\alpha)$, where α is a root of an irreducible polynomial $h(x) = x^3 + ax + b \in k[x]$ with $a \neq 0$. Unfortunately, in this case we have been unable to prove the k-unirationality of $\mathcal{S}_f$ for all f which satisfy $f(t) \neq f(\zeta_3 t)$. However, we prove the following result.

THEOREM 3.1. *Let k be a field of characteristic 0 and put $K = k(\alpha)$, where $\alpha^3 + a\alpha + b = 0$ and $f(t) = t^6 + a_4t^4 + a_1t + a_0 \in k[t]$ with $a_1a_4 \neq 0$. Then the variety $\mathcal{S}_f$ given by (2.1) is unirational over k.*

Proof. In this case $\mathrm{N}_{K/k} = \mathrm{N}_{K/k}(X_1, X_2, X_3)$, where

$$\begin{aligned}\mathrm{N}_{K/k} = X_1^3 - bX_2^3 + b^2X_3^3 + (aX_2 + 3bX_3)X_1X_2 \\ - (2aX_1^2 - a^2X_1X_3 - abX_2X_3)X_3.\end{aligned}$$

Let $G = G(X_1, X_2, X_3, t)$ be the polynomial defining $\mathcal{S}_f$. We use exactly the same approach as in the proof of Theorem 2.1. This time we just take $X_1 = t^2 + p$, where p needs to be determined. We thus get $G(X_1, X_2, X_3, t) = \sum_{i=0}^4 C_it^i$, where

$$C_2 = a^2X_3^2 - 4apX_3 + aX_2^2 + 3bX_2X_3 + 3p^2, \quad C_3 = 0, \quad C_4 = 3p - a_4 - 2aX_3.$$

Eliminating p from the equation $C_4 = 0$ we are left with the equation $C_2 = 0$ defining a curve, say $\mathcal{C}$, in the plane (X_2, X_3). The equation for $\mathcal{C}$ can be rewritten in the form

$$\mathcal{C}:\ (2a^2X_3 - 9bX_2)^2 = 4a^2a_4^2 + 3(4a^3 + 27b^2)X_2^2.$$

The curve $\mathcal{C}$ is of genus 0 and has a rational point $(X_2, X_3) = (0, a_4/a)$ and thus can be parameterized by rational functions. A parameterization of $\mathcal{C}$ together with the expression for p is given by

$$X_2 = \frac{4aa_4u}{3(4a^3+27b^2) - u^2}, \qquad X_3 = \frac{a_4(12a^3 + 81b^2 + 18bu + u^2)}{a(3(4a^3+27b^2) - u^2)},$$
$$p = \frac{a_4 + 2aX_3}{3}.$$

For X_2, X_3 and p chosen in this way we have the equality

$$DG(X_1, X_2, X_3, t) = A_0 + A_1 t,$$

where $\deg A_0 = 6$ and $D = A_1 = -27a^3a_1(12a^3 + 81b^2 - u^2)^3$. From the assumption on a_1 we know that $DA_1 \neq 0$. A careful analysis of the coefficients of the polynomial A_0 shows that if the coefficients of f satisfy $a_1a_4 \neq 0$ then the function $t = \varphi(u) = -A_0/A_1$ satisfies $\varphi \in k(u) \setminus k$. Thus, we have found a rational curve on $\mathcal{S}_f$. Finally, the same argument as at the end of the proof of Theorem 2.1 gives the k-unirationality of $\mathcal{S}_f$. ∎

Remark 3.2. It is natural to ask whether the method we employed to get k-unirationality can be used in other situations. More precisely, one can ask the following.

Question 3.3. *Let $f \in k[t]$. How general an indecomposable form $F \in k[X_1, X_2, X_3]$ of degree 3 can be for the variety defined by $F(X_1, X_2, X_3) = f(t)$ to be unirational over k for most choices of f of fixed degree?*

For example, consider the case of a monic $f \in k[t]$ of degree 6. It would be rather unexpected if taking the form

$$F(X_1, X_2, X_3) = X_1^3 + aX_2^3 + bX_3^3 + (cX_1 + dX_2 + eX_3)X_2X_3,$$

we could prove the k-unirationality of the hypersurface

$$\mathcal{S}:\ F(X_1, X_2, X_3) = f(t),$$

where $f(t) = t^6 + \sum_{i=0}^{4} a_i t^i \in k[t]$ and $a, b, c, d, e \in k$ satisfy certain conditions. We note that for a generic choice of $a, b, c, d, e \in k$ the form F is absolutely irreducible, i.e. irreducible as a polynomial in $\bar{k}[X_1, X_2, X_3]$. Let $G(X_1, X_2, X_3, t) = F(X_1, X_2, X_3) - f(t)$ be the polynomial defining $\mathcal{S}$. To verify the k-unirationality of $\mathcal{S}$, it is enough to take

$$X_1 = t^2 + \frac{a_4}{3}, \qquad X_2 = \frac{a_3 - bu^3}{cu},$$
$$X_3 = ut + \frac{u(3beu^4 - 3a_3eu - a_4^2c + 3a_2c)}{3c(2bu^3 + a_3)}. \tag{3.1}$$

Indeed, for X_1, X_2, X_3 chosen in this way we have

$$DG(X_1, X_2, X_3, t) = C_1t + C_0,$$

where $C_0, C_1 \in k[u]$ depend on the coefficients a, b, c, d, e and a_i for $i = 0, \ldots, 4$. Moreover, we have $D = 27c^3u^3(2bu^3 + a_3)^3$. If $C_0C_1 \neq 0$ as a polynomial in $k[u]$, we get a solution $t = \varphi(u) = -C_0/C_1$. We have $\deg C_1 = 17$ and $\deg C_0 = 18$. The expression for t together with the expressions for X_1, X_2, X_3 given by (3.1) yield a parameterization (with parameter u) of a rational curve on $\mathcal{S}$ with $f(\varphi(u)) \neq 0$. The existence of a rational curve lying on $\mathcal{S}$ allows us to define a rational base change $t = \varphi(u)$. Then the (cubic) surface $\mathcal{S}_\varphi$ defined by $F(X_1, X_2, X_3) = f(\varphi(u))$ (treated as a surface over the field $k(u)$) contains a smooth $k(u)$-rational point P with coordinates given by (3.1), and thus $\mathcal{S}_\varphi$ is k-unirational over $k(u)$. As an immediate consequence we get the k-unirationality of $\mathcal{S}$ over k.

It is possible to give explicit conditions on the coefficients of the polynomial f and the form F which will guarantee that $\varphi \in k(u) \setminus k$. For example, if $abcea_3 \neq 0$ then $\varphi \in k(u) \setminus k$.

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