

## Counting rational points near planar curves

by

AYLA GAFNI (University Park, PA)

**1. Introduction.** In this paper, we give an explicit asymptotic formula for the number of rational points with bounded denominator near a sufficiently smooth planar curve. This result expands on Theorem 3 of [6], and it may serve to provide quantitative information about Khinchin-type manifolds.

Our results are motivated by the convergence side of Khinchin theory, and so we will begin with an overview of the relevant points therein. We say that  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an *approximating function* if it is decreasing and satisfies  $\psi(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Given an approximating function  $\psi$ , we say that a point  $(y_1, \dots, y_n) \in \mathbb{R}^n$  is *simultaneously  $\psi$ -approximable* if there exist infinitely many  $q \in \mathbb{N}$  such that

$$(1) \quad \max_{1 \leq i \leq n} \|qy_i\| \leq \psi(q).$$

Here  $\|x\| = \min_{m \in \mathbb{Z}} |x - m|$ . We denote by  $\mathcal{S}(\psi)$  the set of all simultaneously  $\psi$ -approximable points in  $\mathbb{R}^n$ . Khinchin's theorem gives a criterion for the  $n$ -dimensional Lebesgue measure  $|\cdot|_{\mathbb{R}^n}$  of  $\mathcal{S}(\psi)$ , namely

$$|\mathcal{S}(\psi)|_{\mathbb{R}^n} = \begin{cases} 0 & \text{if } \sum_{q \geq 1} \psi(q)^n < \infty, \\ \text{FULL} & \text{if } \sum_{q \geq 1} \psi(q)^n = \infty, \end{cases}$$

where “FULL” means that the complement of the set has measure 0.

Current research in metric Diophantine approximation focuses on extending this theorem to  $m$ -dimensional manifolds in  $\mathbb{R}^n$ . Let  $\mathcal{M} \subset \mathbb{R}^n$  be a manifold and denote the induced Lebesgue measure on  $\mathcal{M}$  by  $|\cdot|_{\mathcal{M}}$ . We say that  $\mathcal{M}$  is of *Khinchin type for convergence* if  $|\mathcal{M} \cap \mathcal{S}(\psi)|_{\mathcal{M}} = 0$  for any approximating function  $\psi$  with  $\sum_{q \geq 1} \psi(q)^n < \infty$ . Similarly, we say that  $\mathcal{M}$  is of *Khinchin type for divergence* if  $|\mathcal{M} \cap \mathcal{S}(\psi)|_{\mathcal{M}} = \text{FULL}$  for any approximating function  $\psi$  with  $\sum_{q \geq 1} \psi(q)^n = \infty$ .

---

2010 *Mathematics Subject Classification*: 11J83; 11K60; 11J13.

*Key words and phrases*: metric Diophantine approximation, Khinchin theory, Selberg functions.

In this paper we are specifically concerned with curves in  $\mathbb{R}^2$ . Beresnevich et al. [1] established that any  $C^{(3)}$  non-degenerate planar curve is of Khinchin type for divergence. Vaughan and Velani [6] showed that such curves are also of Khinchin type for convergence. The proof of the convergence case relies on an upper bound on the number of rational points near the curve. The present paper provides an asymptotic formula for the number of rational points near a curve. These results may lead to information about the growth of the number of solutions to (1) with  $q \leq Q$ , as  $Q \rightarrow \infty$ , which in turn would hopefully yield a quantitative version of the Khinchin-type theorem. The aim of the quantitative theorem would be to obtain a result similar to that of Schmidt [3], which was a sharpening of the classical Khinchin theorem.

## 2. Statement of results

DEFINITION 1. Let  $\eta, \xi \in \mathbb{R}$ ,  $\eta < \xi$ ,  $I = [\eta, \xi]$  and  $f : I \rightarrow \mathbb{R}$  be such that  $f''$  is continuous and bounded away from 0 on  $I$ . For  $Q \geq 1$  and  $0 < \delta < 1/2$ , define

$$N(Q, \delta) := \text{card}\{(a, q) \in \mathbb{Z} \times \mathbb{N} : 1 \leq q \leq Q, \eta q < a \leq \xi q, \|qf(a/q)\| < \delta\}.$$

When dealing with rational points in  $\mathbb{R}^n$ , we consider the “denominator” of the point to be the least common denominator of the coordinates of the point. Then  $N(Q, \delta)$  counts the number of rational points with denominator  $q \leq Q$  that lie within a  $(\delta/q)$ -neighborhood of the curve that graphs  $f$ . When we apply our results to Khinchin theory, the parameter  $\delta$  will be replaced by a suitable approximating function  $\psi(q)$ . It is therefore reasonable, when finding asymptotic formulae, to bound  $\delta$  from below in terms of  $Q$ .

The computations are easier when all values of  $q$  are of the same order of magnitude, so we will in fact be working with a slightly different object, namely

$$\tilde{N}(Q, \delta) := \text{card}\{(a, q) \in \mathbb{Z} \times \mathbb{N} : Q < q \leq 2Q, \eta q < a \leq \xi q, \|qf(a/q)\| < \delta\}.$$

Theorem 1 gives an explicit asymptotic formula for  $\tilde{N}(Q, \delta)$ . We translate this back to  $N(Q, \delta)$  in Theorem 2.

THEOREM 1. *Suppose that  $0 < \theta < 1$  and  $f'' \in \text{Lip}_\theta([\eta, \xi])$ . If  $Q^{-\frac{1+\theta}{3-\theta}+\epsilon} \leq \delta < 1/2$ , then*

$$\tilde{N}(Q, \delta) = 3(\xi - \eta)\delta Q^2 + E(Q, \delta),$$

where the error term satisfies <sup>(1)</sup>

$$(2) \quad E(Q, \delta) \ll \begin{cases} \delta^{2/3} Q^{5/3} (\log Q)^{2/3} & \text{if } \delta \gg Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}, \\ \delta^{\frac{2}{5-\theta}} Q^{\frac{3(3-\theta)}{5-\theta}} & \text{if } \delta \ll Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}. \end{cases}$$

---

<sup>(1)</sup> The first range of  $\delta$  will not occur when  $\theta \leq 1/2$ .

**THEOREM 2.** *For  $\theta$ ,  $f$ , and  $\delta$  as above, we have*

$$N(Q, \delta) = (\xi - \eta)\delta Q^2 + F(Q, \delta),$$

where  $F(Q, \delta)$  satisfies the bound given by (2).

**COROLLARY 3.** *For  $\theta$ ,  $f$ , and  $\delta$  as above, we have*

$$\tilde{N}(Q, \delta) \sim 3(\xi - \eta)\delta Q^2.$$

**COROLLARY 4.** *For  $\theta$ ,  $f$ , and  $\delta$  as above, we have*

$$N(Q, \delta) \sim (\xi - \eta)\delta Q^2.$$

**3. Proof of Theorem 1.** For convenience we extend the definition of  $f$  to  $\mathbb{R}$  by defining  $f(\beta)$  to be  $\frac{1}{2}(\beta - \xi)^2 f''(\xi) + (\beta - \xi)f'(\xi) + f(\xi)$  when  $\beta > \xi$  and  $\frac{1}{2}(\beta - \eta)^2 f''(\xi) + (\beta - \eta)f'(\xi) + f(\xi)$  when  $\beta < \eta$ . Note that then  $f'' \in \text{Lip}_\theta(\mathbb{R})$  and  $f''$  is still bounded away from 0 and is bounded.

We follow the methods of the proof of [6, Theorem 3]. Let  $K$  be a sufficiently large integer that will be determined later. Let  $S_K^+(\alpha), S_K^-(\alpha)$  be the Selberg functions for the interval  $J = (-\delta, \delta)$ . These functions are trigonometric polynomials of degree at most  $K$  with the properties that  $S_K^-(\alpha) \leq \chi_J(\alpha) \leq S_K^+(\alpha)$  for all  $\alpha$  and  $\int_{\mathbb{T}} S_K^\pm(\alpha) d\alpha = 2\delta \pm \frac{1}{K+1}$ . See [2, Section 7.2] for more details about these functions.

From the definition of  $\tilde{N}(Q, \delta)$  and the properties of the Selberg functions, we see that

$$\begin{aligned} \tilde{N}(Q, \delta) &= \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} \chi_J(\|qf(a/q)\|) \\ &\leq \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} S_K^+(qf(a/q)) \\ &= \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} \sum_{k=-K}^K \widehat{S}_K^+(k) e(kqf(a/q)) = N_0^+ + N_1^+, \end{aligned}$$

where

$$\begin{aligned} N_0^+ &:= \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} \widehat{S}_K^+(0), \\ N_1^+ &:= \sum_{0 < |k| \leq K} \widehat{S}_K^+(k) \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} e(kqf(a/q)). \end{aligned}$$

We wish to find a suitable upper bound for  $N_0^+$ . Recall that  $\widehat{S}_K^+(0) = \int_{\mathbb{T}} S_K^+(\alpha) d\alpha = 2\delta + \frac{1}{K+1}$ . Since there are at most  $(\xi - \eta)q + 1$  integers

in the interval  $(\eta q, \xi q]$ , we have

$$\begin{aligned} N_0^+ &\leq \left(2\delta + \frac{1}{K+1}\right) \left( (\xi - \eta) \frac{Q(3Q+1)}{2} + Q \right) \\ &= 3(\xi - \eta)\delta Q^2 + (\xi - \eta + 2)\delta Q + \frac{3(\xi - \eta)Q^2}{2(K+1)} + \frac{(\xi - \eta + 2)Q}{2(K+1)} \\ &= 3(\xi - \eta)\delta Q^2 + O(\delta Q + K^{-1}Q^2). \end{aligned}$$

Using  $S_K^-$  in place of  $S_K^+$ , we similarly find that

$$\tilde{N}(Q, \delta) \geq N_0^- + N_1^-$$

where

$$\begin{aligned} N_0^- &:= \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} \widehat{S}_K^-(0) \geq 3(\xi - \eta)\delta Q^2 + O(\delta Q + K^{-1}Q^2), \\ N_1^- &:= \sum_{0 < |k| \leq K} \widehat{S}_K^-(k) \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} e(kqf(a/q)). \end{aligned}$$

It can easily be shown that  $|\widehat{S}_K^\pm(k)| \leq |\widehat{S}_K^\pm(0)| \ll \delta + K^{-1}$ . For convenience we define

$$N_1 := \sum_{0 < |k| \leq K} (\delta + K^{-1}) \left| \sum_{Q < q \leq 2Q} \sum_{\eta q < a \leq \xi q} e(kqf(a/q)) \right|.$$

It then follows that  $N_1^+, N_1^- \ll N_1$ . Thus from the above analysis, we see that

$$\tilde{N}(Q, \delta) = 3(\xi - \eta)\delta Q^2 + O(N_1 + \delta Q + K^{-1}Q^2).$$

In other words,

$$(3) \quad E(Q, \delta) := \tilde{N}(Q, \delta) - 3(\xi - \eta)\delta Q^2 \ll N_1 + \delta Q + K^{-1}Q^2.$$

In order to find an upper bound for  $E(Q, \delta)$ , we need to compute an upper bound for  $N_1$  in terms of  $\delta$ ,  $K$ , and  $Q$ . This part of the proof is entirely similar to the proof of [6, Theorem 3], and many of the details are omitted here.

Consider the function  $F(\alpha) = kqf(a/q)$ , which has derivative  $kf'(a/q)$ . Given  $k$  with  $0 < |k| \leq K$ , we define

$$\begin{aligned} H_- &= \lfloor \inf kf'(\beta) \rfloor - 1, & H_+ &= \lceil \sup kf'(\beta) \rceil + 1, \\ h_- &= \lceil \inf kf'(\beta) \rceil + 1, & h_+ &= \lfloor \sup kf'(\beta) \rfloor - 1, \end{aligned}$$

where the extrema are taken over the interval  $[\eta, \xi]$ . By [5, Lemma 4.2], we have

$$\sum_{\eta q < a \leq \xi q} e(kqf(a/q)) = \sum_{H_- \leq h \leq H_+} \int_{\eta q}^{\xi q} e(kqf(\alpha/q) - h\alpha) d\alpha + O(\log(2 + H)),$$

where  $H = \max(|H_-|, |H_+|)$ . Therefore

$$(4) \quad N_1 = N_2 + O\left(\sum_{0 < |k| \leq K} (\delta + K^{-1}) \sum_{Q < q \leq 2Q} \log(2 + H)\right),$$

where

$$N_2 = \sum_{0 < |k| \leq K} (\delta + K^{-1}) \left| \sum_{Q < q \leq 2Q} \sum_{H_- \leq h \leq H_+} \int_{\eta}^{\xi} e(kqf(\alpha/q) - h\alpha) d\alpha \right|.$$

Since  $H \ll |k| \leq K$ , the error term in (4) satisfies

$$(5) \quad \sum_{0 < |k| \leq K} (\delta + K^{-1}) \sum_{Q < q \leq 2Q} \log(2 + H) \ll (\delta + K^{-1})KQ \log K.$$

By a change of variables, the integral in the expression for  $N_2$  can be written as

$$q \int_{\eta}^{\xi} e(q(kf(\beta) - h\beta)) d\beta.$$

The function  $g(\beta) = q(kf(\beta) - h\beta)$  has second derivative  $qkf''(\beta)$ , which has modulus lying between constant multiples of  $q|k|$ . Thus, by [4, Lemma 4.4], for any subinterval  $\mathcal{I}$  of  $[\eta, \xi]$ ,

$$(6) \quad \int_{\mathcal{I}} e(q(kf(\beta) - h\beta)) d\beta \ll \frac{1}{\sqrt{q|k|}}.$$

Thus the contribution to  $N_2$  from any  $h$  with  $H_- \leq h \leq h_-$  or  $h_+ \leq h \leq H_+$  is

$$\ll \sum_{0 < |k| \leq K} (\delta + K^{-1}) \sum_{Q < q \leq 2Q} q \frac{1}{\sqrt{q|k|}} \ll \delta K^{1/2} Q^{3/2} + K^{-1/2} Q^{3/2},$$

and so

$$(7) \quad N_2 = N_3 + O(\delta K^{1/2} Q^{3/2} + K^{-1/2} Q^{3/2}),$$

where

$$N_3 = \sum_{0 < |k| \leq K} (\delta + K^{-1}) \left| \sum_{Q < q \leq 2Q} q \sum_{h_- < h < h_+} \int_{\eta}^{\xi} e(q(kf(\beta) - h\beta)) d\beta \right|.$$

Since  $f'$  is continuous and  $\inf kf'(\beta) < h_- < h < h_+ < \sup kf'(\beta)$ , and since  $f''$  is continuous and non-zero, it follows by the intermediate value theorem that there is a unique  $\beta_h = \beta_{k,h} \in [\eta, \xi]$  such that  $kf'(\beta_h) = h$ . Let

$$\lambda_h = \lambda_{k,h} = \|kf(\beta_h) - h\beta_h\|.$$

By (6), the terms of  $N_3$  with  $\lambda_h \leq Q^{-1}$  contribute

$$\ll (\delta + K^{-1}) \sum_{0 < |k| \leq K} \sum_{\substack{h_- < h < h_+ \\ \lambda_h \leq Q^{-1}}} \sum_{Q < q \leq 2Q} q^{1/2} |k|^{-1/2}.$$

By [6, Lemma 2.3] this is

$$\ll (\delta + K^{-1})Q^{3/2}(K^{3/2}Q^{\varepsilon-1} + K^{1/2}\log K),$$

where  $\varepsilon$  is any positive real number. Thus

$$(8) \quad N_3 = N_4 + O((\delta + K^{-1})Q^{3/2}(K^{3/2}Q^{\varepsilon-1} + K^{1/2}\log K)),$$

where

$$N_4 = \sum_{0 < |k| \leq K} (\delta + K^{-1}) \left| \sum_{Q < q \leq 2Q} q \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}}^{\xi} \int e(q(kf(\beta) - h\beta)) d\beta \right|.$$

Let  $\beta_h = \beta_{k,h}$  be as above and let  $\mu = (\xi - \eta)/2$ . Define

$$\mathcal{A}_1 := [\eta, \xi] \setminus [\beta_h - \mu, \beta_h + \mu], \quad \mathcal{A}_2 := [\beta_h - \mu, \beta_h + \mu] \setminus [\eta, \xi].$$

From the proof of [6, Theorem 3], we see that for  $i = 1, 2$ ,

$$\int_{\mathcal{A}_i} e(q(kf(\beta) - h\beta)) d\beta \ll \frac{1}{q(h - h_-)} + \frac{1}{q(h_+ - h)}.$$

Therefore

$$(9) \quad N_4 = N_5 + O\left((\delta + K^{-1})Q \sum_{0 < |k| \leq K} \sum_{h_- < h < h_+} \frac{1}{(h - h_-)} + \frac{1}{(h_+ - h)}\right),$$

where

$$N_5 = \sum_{0 < |k| \leq K} (\delta + K^{-1}) \left| \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}} \sum_{\substack{Q < q \leq 2Q \\ \beta_{h-\mu}}}^{\beta_{h+\mu}} q \int e(q(kf(\beta) - h\beta)) d\beta \right|.$$

Note that the error term in (9) is

$$(10) \quad \ll (\delta + K^{-1})Q \sum_{0 < |k| \leq K} \log K \ll (\delta + K^{-1})QK \log K.$$

We are left to deal with  $N_5$ . Again by the proof of [6, Theorem 3], we have

$$\sum_{Q < q \leq 2Q} q \int_{\beta_{h-\mu}}^{\beta_{h+\mu}} e(q(kf(\beta) - h\beta)) d\beta \ll Q^{1/2} \lambda_h^{-1} |k|^{-1/2} + Q^{(3-\theta)/2} |k|^{(-1-\theta)/2}.$$

Using [6, Lemma 2.3] it then follows that

$$(11) \quad N_5 \ll (\delta + K^{-1}) \sum_{0 < |k| \leq K} \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}} (Q^{1/2} \lambda_h^{-1} |k|^{-1/2} + Q^{(3-\theta)/2} |k|^{(-1-\theta)/2}) \\ \ll (\delta + K^{-1})(Q^{1/2+\varepsilon} K^{3/2} + Q^{3/2} K^{1/2} \log K + Q^{(3-\theta)/2} K^{(3-\theta)/2}).$$

We now have our upper bound for  $N_1$ . Combining the error terms in (5), (7), (8), (10), and (11) leads to

$$(12) \quad N_1 \ll (\delta K + 1)Q \left( \frac{Q^{1/2} \log K}{K^{1/2}} + \log K + \frac{K^{1/2}}{Q^{1/2-\varepsilon}} + (KQ)^{(1-\theta)/2} \right).$$

Thus we see that

$$(13) \quad \begin{aligned} E(Q, \delta) &\ll \frac{Q^2}{K} + (\delta K + 1)Q \left( \frac{Q^{1/2} \log K}{K^{1/2}} + \log K + \frac{K^{1/2}}{Q^{1/2-\varepsilon}} + (KQ)^{(1-\theta)/2} \right). \end{aligned}$$

The goal now is to find the choice of  $K$  that minimizes  $E(Q, \delta)$ . To simplify the computations, we allow  $K \in \mathbb{R}$  for the time being. We will take the floor function of our choice later to get back to  $K \in \mathbb{N}$ . If  $K > Q^{1-\frac{2}{3}\varepsilon}$ , then

$$\delta K Q \left( \frac{K}{Q} \right)^{1/2} Q^\varepsilon > \delta Q^2,$$

and hence is too big to give an asymptotic formula. Thus we may suppose that  $K \leq Q^{1-2\varepsilon/3}$ . Then, since  $\theta < 1$ , we obtain

$$(14) \quad E(Q, \delta) \ll K^{-1}Q^2 + (\delta K + 1)Q \left( \left( \frac{Q}{K} \right)^{1/2} \log K + (KQ)^{(1-\theta)/2} \right).$$

If  $\delta K \leq 1$  then  $K^{-1}Q^2 \geq \delta Q^2$ , and we do not get our asymptotic formula. So we assume that  $\delta K > 1$  and (14) simplifies to

$$(15) \quad E(Q, \delta) \ll K^{-1}Q^2 + \delta K^{1/2}Q^{3/2} \log Q + \delta(KQ)^{(3-\theta)/2}.$$

We replaced  $\log K$  by  $\log Q$  in the above bound to simplify our computations. This is valid because the restrictions we have placed on  $\delta$  and  $K$  so far require that  $\log K \ll \log Q$ . The optimal choice for  $K$  will occur when two of the three terms in (15) are equal. So we may reduce our analysis to three cases:  $K = \delta^{-2/3}Q^{1/3}(\log Q)^{-2/3}$ ,  $K = \delta^{\frac{-2}{5-\theta}}Q^{\frac{1+\theta}{5-\theta}}$ , and  $K = Q^{\frac{\theta}{2-\theta}}(\log Q)^{\frac{\theta}{2-\theta}}$ . These cases will yield three upper bounds for  $E(Q, \delta)$ . We will then compare them to find the least upper bound.

CASE 1:  $K = \delta^{-2/3}Q^{1/3}(\log Q)^{-2/3}$ . With this choice of  $K$  we have

$$\begin{aligned} K^{-1}Q^2 &= \delta K^{1/2}Q^{3/2} \log Q = \delta^{2/3}Q^{5/3}(\log Q)^{2/3}, \\ \delta(KQ)^{(3-\theta)/2} &= \delta^{\theta/3}Q^{\frac{2}{3}(3-\theta)}(\log Q)^{-\frac{1}{3}(3-\theta)}. \end{aligned}$$

Straightforward computations to find the dominating terms show that

$$E(Q, \delta) \ll \begin{cases} \delta^{2/3}Q^{5/3}(\log Q)^{2/3} & \text{if } \delta \gg Q^{\frac{1-2\theta}{2-\theta}}(\log Q)^{-\frac{5-\theta}{2-\theta}}, \\ \delta^{\theta/3}Q^{\frac{2}{3}(3-\theta)}(\log Q)^{-\frac{1}{3}(3-\theta)} & \text{if } \delta \ll Q^{\frac{1-2\theta}{2-\theta}}(\log Q)^{-\frac{5-\theta}{2-\theta}}. \end{cases}$$

CASE 2:  $K = \delta^{\frac{-2}{5-\theta}} Q^{\frac{1+\theta}{5-\theta}}$ . In this case we have

$$\begin{aligned} K^{-1}Q^2 &= \delta(KQ)^{(3-\theta)/2} = \delta^{\frac{2}{5-\theta}} Q^{\frac{3(3-\theta)}{5-\theta}}, \\ \delta K^{1/2}Q^{3/2} \log Q &= \delta^{\frac{4-\theta}{5-\theta}} Q^{\frac{8-\theta}{5-\theta}} (\log Q). \end{aligned}$$

Thus,

$$E(Q, \delta) \ll \begin{cases} \delta^{\frac{4-\theta}{5-\theta}} Q^{\frac{8-\theta}{5-\theta}} (\log Q) & \text{if } \delta \gg Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}, \\ \delta^{\frac{2}{5-\theta}} Q^{\frac{3(3-\theta)}{5-\theta}} & \text{if } \delta \ll Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}. \end{cases}$$

CASE 3:  $K = Q^{\frac{\theta}{2-\theta}} (\log Q)^{\frac{\theta}{2-\theta}}$ . We now have

$$\begin{aligned} \delta K^{1/2}Q^{3/2} \log Q &= \delta(KQ)^{(3-\theta)/2} = \delta Q^{\frac{3-2\theta}{2-\theta}} (\log Q)^{\frac{4-3\theta}{2(2-\theta)}}, \\ K^{-1}Q^2 &= Q^{\frac{4-\theta}{2-\theta}} (\log Q)^{\frac{\theta}{2(2-\theta)}}. \end{aligned}$$

We obtain

$$E(Q, \delta) \ll Q^{\frac{4-\theta}{2-\theta}} (\log Q)^{\frac{\theta}{2(2-\theta)}}.$$

Comparing the bounds from each of the three cases, we find that the least upper bound is given by

$$E(Q, \delta) \ll \begin{cases} \delta^{2/3} Q^{5/3} (\log Q)^{2/3} & \text{if } \delta \gg Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}, \\ \delta^{\frac{2}{5-\theta}} Q^{\frac{3(3-\theta)}{5-\theta}} & \text{if } \delta \ll Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}. \end{cases}$$

Hence we will choose  $K = \lfloor \delta^{-2/3} Q^{1/3} (\log Q)^{-2/3} \rfloor$  when  $\delta \gg Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}$  and  $K = \lfloor \delta^{\frac{-2}{5-\theta}} Q^{\frac{1+\theta}{5-\theta}} \rfloor$  when  $\delta \ll Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}$ . Since we have the additional assumption that  $\delta < 1/2$ , the first range for  $\delta$  will only occur if  $\theta > 1/2$ . This completes the proof of Theorem 1. ■

**4. Proof of Theorem 2.** We obtain  $N(Q, \delta)$  from  $\tilde{N}(Q, \delta)$  by a dyadic sum. That is,

$$N(Q, \delta) = \sum_{r=1}^{\infty} \tilde{N}\left(\frac{Q}{2^r}, \delta\right).$$

It is easy to see that this sum converges since  $\tilde{N}(Q/2^r, \delta) = 0$  if  $2^{r-1} > Q$ . To avoid restrictions on  $\delta$  in terms of  $Q/2^r$ , we will use the estimate for  $E(Q, \delta)$  given by (13). We have

$$\begin{aligned} N(Q, \delta) &= \sum_{r=1}^{\infty} \tilde{N}\left(\frac{Q}{2^r}, \delta\right) = \sum_{r=1}^{\infty} \left(3(\xi - \eta)\delta \left(\frac{Q}{2^r}\right)^2 + E\left(\frac{Q}{2^r}, \delta\right)\right) \\ &= \sum_{r=1}^{\infty} 3(\xi - \eta)\delta \frac{Q^2}{4^r} + \sum_{r=1}^{\infty} F_r(Q, \delta), \end{aligned}$$



where

$$F_r(Q, \delta) \ll \frac{Q^2}{4^r K} + (\delta K + 1) \left( \frac{Q^{3/2} \log K}{2^{3r/2} K^{1/2}} + \frac{Q \log K}{2^r} + \frac{K^{1/2} Q^{1/2+\varepsilon}}{2^{r(1/2+\varepsilon)}} + \frac{K^{1-\theta} Q^{3-\theta}}{2^{r\frac{3-\theta}{2}}} \right).$$

Since  $r$  only appears as an exponent of  $(1/2)^\alpha$  for various values of  $\alpha > 0$ , it is clear by the convergence of the geometric series that

$$\begin{aligned} F(Q, \delta) &:= \sum_{r=1}^{\infty} F_r(Q, \delta) \\ &\ll \frac{Q^2}{K} + (\delta K + 1) Q \left( \frac{Q^{1/2} \log K}{K^{1/2}} + \log K + \frac{K^{1/2}}{Q^{1/2-\varepsilon}} + (KQ)^{(1-\theta)/2} \right). \end{aligned}$$

Note that this is the same estimate that is given for  $E(Q, \delta)$  in (13). Thus the proof of Theorem 1 gives the bound for  $F(Q, \delta)$ . We now return to the main term of  $N(Q, \delta)$ . We have

$$\begin{aligned} N(Q, \delta) &= \sum_{r=1}^{\infty} 3(\xi - \eta) \delta \frac{Q^2}{4^r} + F(Q, \delta) = 3(\xi - \eta) \delta Q^2 \frac{1/4}{1 - 1/4} + F(Q, \delta) \\ &= (\xi - \eta) \delta Q^2 + F(Q, \delta), \end{aligned}$$

as desired. ■

**5. Proof of the corollaries.** Denote the piecewise upper bound given in (2) by  $E_1(Q, \delta)$ . To prove both corollaries, it is clearly enough to show that

$$\frac{E_1(Q, \delta)}{\delta Q^2} \rightarrow 0$$

as  $Q \rightarrow \infty$  and  $\delta \rightarrow 0$ . We will call upon the assumption that  $\delta \geq Q^{-\frac{1+\theta}{3-\theta}+\varepsilon}$ . When  $\delta \gg Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}$ , we have

$$\begin{aligned} \frac{E_1(Q, \delta)}{\delta Q^2} &\ll \frac{\delta^{2/3} Q^{5/3} (\log Q)^{2/3}}{\delta Q^2} \\ &= (\delta Q)^{-1/3} (\log Q)^{2/3} \leq Q^{-\frac{2-2\theta}{3(3-\theta)} - \frac{\varepsilon}{3}} (\log Q)^{2/3}, \end{aligned}$$

which tends to 0 as  $Q \rightarrow \infty$ . On the other hand, when  $\delta \ll Q^{\frac{1-2\theta}{2-\theta}} (\log Q)^{-\frac{5-\theta}{2-\theta}}$ , we have

$$\frac{E_1(Q, \delta)}{\delta Q^2} \ll \frac{\delta^{\frac{2}{5-\theta}} Q^{\frac{3(3-\theta)}{5-\theta}}}{\delta Q^2} = \delta^{-\frac{3-\theta}{5-\theta}} Q^{-\frac{1+\theta}{5-\theta}} \leq Q^{-\varepsilon \frac{3-\theta}{5-\theta}},$$

which also tends to 0 as  $Q \rightarrow \infty$ . ■

**Acknowledgements.** The author would like to thank her adviser, Robert C. Vaughan, for his guidance throughout this research.

## References

- [1] V. Beresnevich, D. Dickinson, S. Velani, and R. C. Vaughan, *Diophantine approximation on planar curves and the distribution of rational points*, Ann. of Math. 166 (2007), 367–426.
- [2] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS Reg. Conf. Ser. Math. 84, Amer. Math. Soc., 1994.
- [3] W. Schmidt, *A metrical theorem in diophantine approximation*, Canad. J. Math. 12 (1960), 619–631.
- [4] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford Univ. Press, 1986.
- [5] R. C. Vaughan, *The Hardy–Littlewood Method*, 2nd ed., Cambridge Tracts in Math. 125, Cambridge Univ. Press, 1997.
- [6] R. C. Vaughan and S. Velani, *Diophantine approximation on planar curves: the convergence theory*, Invent. Math. 166 (2006), 103–124.

Ayla Gafni  
Department of Mathematics  
Pennsylvania State University  
109 McAllister Bldg  
University Park, PA 16802, U.S.A.  
E-mail: gafni@math.psu.edu

*Received on 18.11.2013  
and in revised form on 22.5.2014*

(7647)