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On a problem posed by R. Salvati Manni

by

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1. Introduction. Throughout the paper we consider only positive definite even unimodular lattices.

The following theorem was proven in [26, p. 293] by R. Salvati Manni:

THEOREM 1.1. *The theta series of degree* 3 *associated to two* 56- (*resp.* 72-) *dimensional even unimodular extremal lattices differ by a multiple, possibly* 0, *of* χ_{28} (*resp.* χ_{36}).

For 40-*dimensional lattices, if two extremal theta series are equal in degree* 2, *then in degree* 3 *they differ by a multiple, possibly* 0, *of* χ_{20}.

Here χ_{28} (resp. χ_{36}, χ_{20}) is a Siegel cusp form of degree 3 and weight 28 (resp. 36, 20). Note that χ_{28} and χ_{20} were first introduced and studied by Tsuyumine [28], who wrote γ_{20} for χ_{20}. The form χ_{18} was studied in Igusa [7], and $\chi_{36} = \chi_{18}^2$.

Salvati Manni then states the following problem, suggested by his paper's referee: find two even unimodular extremal lattices L_1 and L_2 of rank 40 whose theta series coincide in degree 2 and differ in degree 3.

In the present paper we show that there are two 40-dimensional even unimodular extremal lattices coming from two doubly even self-dual extremal codes, whose theta series of degree 2 coincide and theta series of degree 3 differ definitely. We also exhibit two even unimodular extremal lattices coming from another pair of doubly even self-dual extremal codes, whose theta series of degree 2 and degree 3 coincide. These results are shown by computing some beginning Fourier coefficients of the theta series of the lattices in question combined with some facts on the dimensions of the linear spaces of Siegel modular forms already proved by other people.

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2. Basic facts on Siegel theta series and other tools

2.1. Definition of Siegel modular forms. The *symplectic group* $\mathrm{Sp}_g(\mathbb{R})$ of degree g over $\mathbb{R}$ is defined to be

$$\mathrm{Sp}_g(\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) \,\middle|\, {}^tMJM = J,\, J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}.$$

The *Siegel modular group* $\mathrm{Sp}_g(\mathbb{Z})$ of degree g is the subgroup of $\mathrm{Sp}_g(\mathbb{R})$ consisting of those elements in $\mathrm{Sp}_g(\mathbb{R})$ whose entries are in $\mathbb{Z}$. Let $\mathbb{H}_g$ be the *Siegel upper half-space* of degree g:

$$\mathbb{H}_g = \{\tau = {}^t\tau \in M_g(\mathbb{C}) \mid \mathrm{Im}(\tau) \text{ is positive definite}\}.$$

A *Siegel modular form* of degree g ($g \geq 2$) and weight k is a holomorphic complex valued function $f(\tau)$ defined on $\mathbb{H}_g$ satisfying the condition:

$$f((A\tau+B)(C\tau+D)^{-1}) = (\det(C\tau+D))^k f(\tau) \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{Z}).$$

The set $M(g,k)$ of all Siegel modular forms of degree g and even weight k is a linear space of certain dimension.

Table 1. The dimension of $M(g,k)$

$g \setminus k$	4	6	8	10	12	14	16	18	20
1	1	1	1	1	2	1	2	2	2
2	1	1	1	2	3	2	4	4	5
3	1	1	1	2	4	3	7	8	11
4	1	1	2	3	6	6	14	?	?

The dimensions of $M(1,k)$ are classical. The dimensions of $M(2,k)$ were found by Igusa [6], those of $M(3,k)$ by Tsuyumine [28], and those of $M(4,k)$ by Poor–Yuen [22], Duke–Imamoḡlu [3] and Oura–Poor–Yuen [16]. The spots marked by ? are not known.

2.2. Lattice. A *lattice* L of rank n (or dimension n) is a $\mathbb{Z}$-module generated by vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^n$ that are linearly independent over $\mathbb{R}$. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are called a *basis* of L. The lattice L is *integral* if the inner product $(\mathbf{x}, \mathbf{y})$ belongs to $\mathbb{Z}$ for all pairs $\mathbf{x}$ and $\mathbf{y}$ in L. The *dual lattice* $L^\#$ of L is defined to be

$$L^\# = \{\mathbf{y} \in \mathbb{R}^n \mid (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}\ \forall \mathbf{x} \in L\}.$$

A lattice L is *unimodular* if $L = L^\#$. A lattice L is *even* if any element $\mathbf{x}$ of L has even norm $(\mathbf{x}, \mathbf{x})$. Even unimodular lattices exist only when $n \equiv 0 \pmod 8$.

The *minimal norm* of a lattice is

$$\operatorname{Min}(L) = \min_{\mathbf{x} \in L \setminus \{0\}} (\mathbf{x}, \mathbf{x}).$$

If L is even unimodular of rank n then (cf. [11])

$$\operatorname{Min}(L) \leq 2\left[\frac{n}{24}\right] + 2.$$

A lattice which attains the above maximum is called *extremal.*

Let L be an even unimodular lattice of rank n. For $m \geq 1$ we let $\Lambda_{2m}(L)$ be the set of $\mathbf{x}$ in L with $(\mathbf{x}, \mathbf{x}) = 2m$. A relatively tractable class of unimodular lattices are the lattices constructed from root sublattices. A root lattice is an integral lattice which has a strong connection with root systems in the theory of Lie algebras. Basic root lattices are A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 and E_8. For precise definitions the readers may refer to Chapter 4 of Conway–Sloane's book [2].

Let N be an orthogonal sum of some copies of the above basic root lattices. Then the quotient $N^{\#}/N$ is well described (cf. [2] or [15]). In nice cases N plus some representatives of $N^{\#}/N$ form an integral lattice, and sometimes an even unimodular lattice. The added representatives of $N^{\#}/N$ are now called *glue vectors* of N. For instance let

$$\begin{aligned} D_{28} \oplus D_{12} = [&\mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{27} - \mathbf{e}_{28}, \mathbf{e}_{27} + \mathbf{e}_{28}] \\ &\oplus [\mathbf{f}_1 - \mathbf{f}_2, \ldots, \mathbf{f}_{11} - \mathbf{f}_{12}, \mathbf{f}_1 + \mathbf{f}_{12}] \end{aligned}$$

be an orthogonal sum of two root lattices D_{28} and D_{12}, where $\mathbf{e}_1, \ldots, \mathbf{e}_{28}$, $\mathbf{f}_1, \ldots, \mathbf{f}_{12}$ are orthonormal vectors in the 40-dimensional Euclidean space. Then the vectors $\mathbf{h}_1 = \frac{1}{2}\sum_{i=1}^{28} \mathbf{e}_i + \mathbf{f}_1$ and $\mathbf{h}_2 = \mathbf{e}_1 + \frac{1}{2}\sum_{i=1}^{12} \mathbf{f}_i$ are glue vectors for the lattice $D_{28} \oplus D_{12}$. The lattice $D_{28} \oplus D_{12} + \mathbb{Z}\mathbf{h}_1 + \mathbb{Z}\mathbf{h}_2$ is verified to be a 40-dimensional even unimodular lattice. An extensive account of gluing theory is given in Chapter 4 of [2].

2.3. Siegel theta series. The *Siegel theta series* of degree g attached to the lattice L is defined by

$$\vartheta_g(\tau, L) = \sum_{\mathbf{x}_1, \ldots, \mathbf{x}_g \in L} \exp(\pi i \sigma([\mathbf{x}_1, \ldots, \mathbf{x}_g]\tau)),$$

where τ is the variable on the Siegel upper-half space of degree g, $[\mathbf{x}_1, \ldots, \mathbf{x}_g]$ is a g by g square matrix whose (i, j) entry is $(\mathbf{x}_i, \mathbf{x}_j)$, and σ is the trace of the matrix.

The Siegel theta series of degree g can be expanded as

$$\vartheta_g(\tau, L) = \sum_T a(T, L) e^{2\pi i \sigma(T\tau)}.$$

Here T runs over the set of positive semi-definite semi-integral symmetric square matrices of degree g, and $a(T,L) = \#\{\langle \mathbf{x}_1,\ldots,\mathbf{x}_g\rangle \in L^g \mid [\mathbf{x}_1,\ldots,\mathbf{x}_g] = 2T\}$.

Fact. *A Siegel theta series of degree g associated with an even integral unimodular lattice L of rank $2k$ (where $2k$ is a multiple of 8) is a modular form of degree g and weight k.*

2.4. Binary linear codes. Let $\mathbb{F}_2 = \mathrm{GF}(2)$ be the field of two elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. A *linear* $[n,k]$ *code* $\mathbf{C}$ is a vector subspace of V of dimension k; we then say that $\mathbf{C}$ has *length* n. An element $\mathbf{u}$ in $\mathbf{C}$ is called a *codeword* of $\mathbf{C}$. In V, the inner product, denoted by $\mathbf{u}\cdot\mathbf{v}$ for $\mathbf{u},\mathbf{v}$ in V, is defined as usual. Two codes $\mathbf{C}_1$ and $\mathbf{C}_2$ are said to be *equivalent* if they coincide after a permutation of coordinates.

The *dual code* $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^\perp = \{\mathbf{u} \in V \mid \mathbf{u}\cdot\mathbf{v} = \mathbf{0}\ \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called *self-orthogonal* if $\mathbf{C} \subseteq \mathbf{C}^\perp$, and *self-dual* if $\mathbf{C} = \mathbf{C}^\perp$. Self-dual $[n,k]$ codes exist only if $n \equiv 0 \pmod 2$ and $k = n/2$.

Let $\mathbf{u} = (u_1,\ldots,u_n)$ be a vector in V. Then the *Hamming weight* $\mathrm{wt}(\mathbf{u})$ of $\mathbf{u}$ is defined to be the number of i's such that $u_i \neq 0$. The *Hamming distance* d on V is defined by $d(\mathbf{u},\mathbf{v}) = \mathrm{wt}(\mathbf{u}-\mathbf{v})$. Let $\mathbf{C}$ be a code. Then the *minimum distance* d of the code $\mathbf{C}$ is defined by

$$d(\mathbf{C}) = \min_{\mathbf{u},\mathbf{v}\in\mathbf{C},\mathbf{u}\neq\mathbf{v}} d(\mathbf{u},\mathbf{v}) = \min_{\mathbf{u}\in\mathbf{C},\mathbf{u}\neq\mathbf{0}} \mathrm{wt}(\mathbf{u}).$$

An $[n,k]$ code $\mathbf{C}$ with $d = d(\mathbf{C})$ is denoted by $[n,k,d]$. Let $\mathbf{C}$ be a self-dual binary $[n,n/2]$ code. Then the weight $\mathrm{wt}(\mathbf{u})$ of each codeword $\mathbf{u}$ in $\mathbf{C}$ is an even number. Further, if the weight of each codeword $\mathbf{u}$ in $\mathbf{C}$ is divisible by 4, then the code is said to be *doubly even*. It is known that doubly even self-dual binary codes $\mathbf{C}$ exist only when n is a multiple of 8. If $\mathbf{C}$ is a self-dual doubly even code, it is known that (cf. [12])

$$d(\mathbf{C}) \le 4\left[\frac{n}{24}\right] + 4.$$

A self-dual doubly even code $\mathbf{C}$ satisfying $d(\mathbf{C}) = 4\left[\frac{n}{24}\right]+4$ is called *extremal*.

Let $\mathbf{C}$ be a self-dual doubly even code of length n, which is embedded in $\mathbb{F}_2^n$. Let $\mathbf{u} = (u_1,\ldots,u_n), \mathbf{v} = (v_1,\ldots,v_n)$ be any pair of vectors in $\mathbb{F}_2^n$. Then the number of common 1's in the corresponding coordinates of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}*\mathbf{v}$. This is called the *intersection number* of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}*\mathbf{u}$ is nothing other than $\mathrm{wt}(\mathbf{u})$.

2.5. Multiple weight enumerator. Let $\mathbf{C}$ be a doubly even self-dual code of length n, let g be a positive integer and let α run over the set $\mathbb{F}_2^g$ of g-vectors. Let X_α for $\alpha \in \mathbb{F}_2^g$ be 2^g variables algebraically independent over $\mathbb{C}$.

Let $\mathbf{u}_1 = (u_1^1, \dots, u_1^n), \dots, \mathbf{u}_g = (u_g^1, \dots, u_g^n)$ be a g-tuple of codewords of $\mathbf{C}$. For each $\alpha \in \mathbb{F}_2^g$ the *generalized weight* $\mathrm{wt}_\alpha(\mathbf{u}_1, \dots, \mathbf{u}_g)$ is defined to be the number of coordinates j $(1 \le j \le n)$ such that $\alpha = (u_1^j, \dots, u_g^j)$.

The *multiple weight enumerator* $\mathbf{W}_g(X_\alpha; \mathbf{C})$ of degree g for the code $\mathbf{C}$ is defined by

$$\mathbf{W}_g(X_\alpha; \mathbf{C}) = \sum_{(\mathbf{u}_1, \dots, \mathbf{u}_g) \in \mathbf{C}^g} \prod_{\alpha \in \mathbb{F}_2^g} X_\alpha^{\mathrm{wt}_\alpha(\mathbf{u}_1, \dots, \mathbf{u}_g)}.$$

The multiple weight enumerator of degree 2 is called a *biweight enumerator*, and the multiple weight enumerator of degree 3 is called a *triweight enumerator*.

2.6. From binary codes to lattices. Let $\mathbf{C}$ be a binary self-orthogonal $[n, k]$ code. Let

$$\rho : \mathbb{Z}^n \to \mathbb{F}_2^n, \qquad \mathbf{x} \mapsto \mathbf{x} \bmod 2.$$

Then

$$\mathcal{M}(\mathbf{C}) = \frac{1}{\sqrt{2}} \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \rho^{-1}(\mathbf{C}) \;\middle|\; \sum_{i=1}^n x_i \equiv 0 \pmod 4 \right\}$$

defines an even lattice. Suppose that $\mathbf{C}$ is a doubly even self-dual binary extremal $[n, n/2]$ code. Then the so called *density doubling process* is described as follows. Put

$$\gamma = \begin{cases} \frac{1}{\sqrt{8}}(1, \dots, 1, -3) & \text{if } n \equiv 8 \pmod{16}, \\ \frac{1}{\sqrt{8}}(1, \dots, 1, 1) & \text{if } n \equiv 0 \pmod{16}. \end{cases}$$

Then

$$\mathcal{N}(\mathbf{C}) = \mathcal{M}(\mathbf{C}) \cup (\gamma + \mathcal{M}(\mathbf{C}))$$

is an even unimodular extremal lattice of rank n for $n = 8, 16, 24, 32, 40$.

2.6.1. *40-dimensional case.* We are particularly concerned with the set of minimal vectors $\Lambda_4(\mathcal{N}(\mathbf{C}))$ in an extremal even unimodular lattice constructed from a binary self-dual extremal $[40, 20, 8]$ code.

When $\mathbf{C}$ is a doubly even self-dual binary $[40, 20, 8]$ code, $\Lambda_4 = \Lambda_4(\mathcal{N}(\mathbf{C}))$ consists of two kinds of vectors:

$$A = \Lambda_4^1 = \left\{ \frac{1}{\sqrt{2}}((\pm 2)^2, 0^{38}) \right\}, \qquad B = \Lambda_4^2 = \left\{ \frac{1}{\sqrt{2}}((\pm 1)^8, 0^{32}) \right\}.$$

The set A forms a root system of type D_{40} scaled by a factor $\sqrt{2}$, and the vectors in the set B come from codewords of weight 8 in the code $\mathbf{C}$. Further the product of nonzero integers is 1 for each element of B. The cardinalities of these sets are

$$|A| = 4 \cdot \binom{40}{2} = 3120, \qquad |B| = 285 \cdot 2^7 = 36480.$$

In a later section we will use these two sets extensively.

We pick up some specific codes. We denote by $\mathcal{C}_1$ (respectively $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$) the second code in [19], Yorgov's code C_5, Yorgov's code C_2 and Yorgov's code C_4 [33] respectively. The lattices constructed by the above density doubling process are denoted by $M_{11} = \mathcal{N}(\mathcal{C}_1), M_{12} = \mathcal{N}(\mathcal{C}_2), M_{13} = \mathcal{N}(\mathcal{C}_3)$ and $M_{14} = \mathcal{N}(\mathcal{C}_4)$.

3. Preliminary results. We give tables of some beginning indices T in the Fourier coefficients $a(T, L)$ of Siegel theta series that should determine the Fourier expansion of the series uniquely. The conclusion will be summarized as Proposition 3.1 at the end of this subsection.

Table 2.1. Case $g = 1$

numbered $T \setminus 2k$	8	16	24	32	40
T_0	0	0	0	0	0
T_1			1	1	1

Table 2.2. Case $g = 2$

numbered $T \setminus 2k$	8	16	24	32	40
T_0	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
T_1			$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
T_2			$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
T_3				$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$
T_4					$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Table 2.3. Case $g = 3$

numbered $T \setminus 2k$	24	32	40
T_0	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
T_1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 2.3 [cont.]

numbered $T \setminus 2k$	24	32	40
T_2		$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
T_3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
T_4			$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
T_5		$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$
T_6		$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
T_7	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
T_8			$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 2 \end{pmatrix}$
T_9			$\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$
T_{10}			$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Even unimodular lattices K_i of rank 24, with underlying root lattices

$$K_1 : 3E_8, \quad K_2 : D_{24}, \quad K_3 : A_{24}, \quad K_4 : A_{17} \oplus E_7,$$

are picked up from 24 Niemeier lattices [15]. We must compute some Fourier coefficients of their Siegel theta series. When the lattice is an over-lattice of a root lattice of full rank, namely the rank of the even unimodular lattice equals the rank of the underlying root lattice, the computing of some Fourier coefficients is not very difficult. When the lattice is extremal of rank greater than 16, computing the Fourier coefficients is in general hard. However if the lattice comes from an extremal binary code, some Fourier coefficients are well controlled by the multiple weight enumerators of the code. Some instances of this fact are developed in Section 5. Here we exhibit results in tables.

Table 3.1. The Fourier coefficients $a(T_j, K_m)$ in degree 3

$m \setminus j$	0	1	3	7
1	1	720	436320	219024000
2	1	1104	1022304	781393536
3	1	600	303600	127512000
4	1	432	158112	48263040

Even unimodular lattices L_i of rank 32 have underlying root lattices as follows:

$$L_1 : 4E_8, \quad L_2 : D_{24} \oplus E_8, \quad L_3 : A_{24} \oplus E_8, \quad L_4 : E_7 \oplus A_{17} \oplus E_8,$$
$$L_5 : D_{32}, \quad L_6 : A_1 \oplus A_{31}, \quad L_7 : A_{16} \oplus A_{16}.$$

The glue vectors of the lattices L_4, L_6 and L_7 are well described in [8]. The lattices containing D_{32}, D_{40} were explained in [17]. Other lattices L_j are simply enlargements of some of Niemeier lattices by E_8.

Table 3.2. The Fourier coefficients $a(T_j, L_m)$ in degree 3

$m \setminus j$	0	1	2	3	5	6	7
1	1	960	53760	812160	1451520	42577920	600307200
2	1	1344	110592	1582464	4540416	120729600	1619421696
3	1	840	41040	621840	970080	28406880	402350400
4	1	672	27264	395712	570240	14736384	203109120
5	1	1984	238080	3456128	14046720	386641920	5240378880
6	1	994	59520	867008	1726080	8449280	657636480
7	1	544	16320	262208	228480	7409280	111041280

Even unimodular lattices M_i of rank 40 have underlying root lattices as follows:

$$M_1 : E_8^5, \quad M_2 : D_{24} \oplus E_8^2, \quad M_3 : A_{24} \oplus E_8^2, \quad M_4 : E_7 \oplus A_{17} \oplus E_8^2,$$
$$M_5 : D_{32} \oplus E_8, \quad 'M_6 : A_1 \oplus A_{31} \oplus E_8, \quad M_7 : A_{16}^2 \oplus E_8, \quad M_8 : D_{20}^2,$$
$$M_9 : D_{40}, \quad M_{10} : D_{28} \oplus D_{12}.$$

Glue vectors of M_8 are given as follows. Let

$$D_{20}^2 = [\mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{19} - \mathbf{e}_{20}, \mathbf{e}_{19} + \mathbf{e}_{20}] \oplus [\mathbf{f}_1 - \mathbf{f}_2, \ldots, \mathbf{f}_{19} - \mathbf{f}_{20}, \mathbf{f}_{19} + \mathbf{f}_{20}],$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_{20}, \mathbf{f}_1, \ldots, \mathbf{f}_{20}$ are orthonormal vectors in a 40-dimensional Euclidean space. Then the vectors $\mathbf{h}_1 = \frac{1}{2}\sum_{i=1}^{20} \mathbf{e}_i + \mathbf{f}_1$ and $\mathbf{h}_2 = \mathbf{e}_1 + \frac{1}{2}\sum_{i=1}^{20} \mathbf{f}_i$ are glue vectors for the lattice M_8. Glue vectors for the lattice M_{10} are found in Section 2.2. The lattices $M_{11}, M_{12}, M_{13}, M_{14}$ were introduced in Section 2.6.1.

Table 3.3. The Fourier coefficients $a(T_j, M_m)$ in degree 3

$m \setminus j$	0	1	2	3	4	5	6
1	1	1200	67200	1303200	226106935200	1814400	69350400
2	1	1584	124032	2257824	383278632864	4903296	166302720
3	1	1080	54480	1055280	185566037280	1332960	50513760
4	1	912	40704	748512	135668986272	933120	31279104
5	1	2224	251520	4438688	734060529568	14409600	471413760
6	1	1234	72960	1374368	238658059648	2088960	77061120
7	1	784	29760	553568	102348818848	591360	19605120
8	1	1520	109440	2088480	357220647840	3830400	142709760
9	1	3120	474240	8779680	1422569435040	35568000	1263375360
10	1	1776	167808	2815008	474354791328	8220288	247698432
11	1	0	0	0	994281120	0	0
12	1	0	0	0	994281120	0	0
13	1	0	0	0	1035568800	0	0
14	1	0	0	0	1035568800	0	0

$m \setminus j$	7	8	9	10
1	1273968000	5378688000	4410980582400	
2	2882537856	11702768640	9419495777280	
3	928094400	3945926400	16132530547200	
4	550800000	2366742528	1991438493696	
5	7910593920	31503139840	24945629414400	
6	1373972320	5677661440	4703505845760	
7	350997120	1516810240	1343172216960	
8	2579975040	10705566720	8231241262080	
9	22161709440	88857400320	67244838036480	
10	3964519296	15670729728	12915085943808	
11	0	0	0	15596332778880
12	0	0	0	15596205376896
13	0	0	0	17448486307200
14	0	0	0	17448486307200

The blanks in the above table are not necessary to know for the present purpose.

Proposition 3.1. *A Siegel theta series $\vartheta_g(\tau, L)$ of degree g $(1 \le g \le 3)$ associated with an even unimodular lattice L of rank $2k$ $(k = 4, 8, 12, 16, 20)$ is uniquely determined if the Fourier coefficients $a(T, L)$ are known for the indices T given in Tables 2.1–2.3.*

Proof. From Table 1 in Section 2.1 we know the dimensions of $M(g, k)$ $(1 \le g \le 3,\ k = 4, 8, 12, 16, 20)$. If we could find the theta series of degree g associated with appropriate even unimodular lattices of rank $2k = 8, 16, 24, 32, 40$ that are uniquely determined by the selected Fourier coefficients and show that the vector space spanned by these theta series has the dimension of $M(g, k)$, then the proof will be complete. The case $g = 1$ is classical, and we omit the proof. The case $g = 2$ is treated in [17]. Therefore we only have to treat the case $g = 3$. When the rank of the lattice is 8 or 16 the dimension of the space $M(3, 4)$ (resp. $M(3, 8)$) is one and the proof is trivial.When the rank of the lattice is 24, one verifies that the determinant

$$\begin{vmatrix} 1 & 720 & 436320 & 219024000 \\ 1 & 1104 & 1022304 & 781393536 \\ 1 & 600 & 303600 & 127512000 \\ 1 & 432 & 158112 & 48263040 \end{vmatrix} \neq 0$$

(cf. Table 3.1), therefore the series $\vartheta_3(\tau, K_m)$ $(1 \le m \le 4)$ form a basis of $M(3, 12)$, and it is enough to determine any element in $M(3, 12)$ from the Fourier coefficients at T_j for $j = 0, 1, 3, 7$.

When the rank of the lattice is 32, by forming appropriate linear combinations of the theta series $\vartheta_3(\tau, L_m)$ one obtains $\psi_i(\tau) = \sum_T b_i(T) e^{2\pi i \sigma(T\tau)}$ $(1 \le i \le 7)$ with $b_1(T_0) = 1$, $b_1(T_j) = 0$ for $j = 1, 2, 3, 5, 6, 7$, $b_2(T_1) = 1$, $b_2(T_j) = 0$, $j \ne 1$, $b_3(T_2) = 1$, $b_3(T_j) = 0$, $j \ne 2$, $b_4(T_3) = 1$, $b_4(T_j) = 0$, $j \ne 3$, $b_5(T_5) = 1$, $b_4(T_j) = 0$, $j \ne 5$, $b_6(T_6) = 1$, $b_6(T_j) = 0$, $j \ne 6$, $b_7(T_7) = 1$, $b_7(T_j) = 0$, $j \ne 7$. This shows that $\psi_i(\tau)$ are linearly independent, and consequently $\vartheta_3(\tau, L_m)$ $(1 \le m \le 7)$ are linearly independent. This implies that the values $c(T_j)$, $j = 0, 1, 2, 3, 5, 6, 7$, are enough to determine the series $\sum_T c(T) e^{2\pi i \sigma(T\tau)}$ in $M(3, 16)$.

When the rank of the lattice is 40, by forming appropriate linear combinations of $\vartheta_3(\tau, M_m)$ $(1 \le m \le 10)$ we obtain $\phi_h(\tau) = \sum_T d_h(T) e^{2\pi i \sigma(T\tau)}$ $(0 \le h \le 9)$ such that $d_h(T_j) = \delta_{h,j}$, $0 \le j \le 9$, where $\delta_{h,j}$ is Kronecker's delta. This shows that $\vartheta_3(\tau, M_m)$ $(1 \le m \le 10)$ span a 10-dimensional subspace of the 11-dimensional space $M(3, 20)$. By Table 3.3 the difference $\vartheta_3(\tau, M_{11}) - \vartheta_3(\tau, M_{12})$ has nonzero Fourier coefficient at T_{10}. This difference cannot be expressed as a linear combination of $\phi_h(\tau) = \sum_T d_h(T) e^{2\pi i \sigma(T\tau)}$ $(0 \le h \le 9)$, since the difference has zero value for each Fourier coefficient at T_j for $0 \le j \le 9$. This implies that $\vartheta_3(\tau, M_m)$ $(1 \le m \le 10)$ and $\vartheta_3(\tau, M_{11}) - \vartheta_3(\tau, M_{12})$ span the full space $M(3, 20)$. It is easy to see that it is enough to determine any element in $M(3,20)$ from the Fourier coefficients at T_j for $0 \le j \le 10$. ∎

4. Main results

THEOREM 4.1. *There are even unimodular* 40*-dimensional extremal lattices L_1 and L_2 whose Siegel theta series of degrees* 1 *and* 2 *coincide, but whose theta series of degree* 3 *differ.*

THEOREM 4.2. *There are even unimodular* 40*-dimensional nonisometric lattices L_3 and L_4 whose Siegel theta series of degrees* 1, 2 *and* 3 *coincide.*

THEOREM 4.3. *If two even unimodular* 40*-dimensional extremal lattices L_3 and L_4 coming from extremal binary doubly even self-dual codes have identical triweight enumerators, then they have identical Siegel theta series of degrees* 1, 2 *and* 3.

The proofs of these theorems will be given after a description of the computation of the crucial Fourier coefficients.

5. How to compute the Fourier coefficients of Siegel theta series

5.1. The Fourier coefficients of $\vartheta_2(\tau, L)$ for even unimodular extremal 40-dimensional lattices L. We recall the sets A and B introduced in Section 2.6. To each $\mathbf{y} \in \Lambda_4$ we associate a binary vector $\mathbf{v} = \operatorname{supp} \mathbf{y} \in \mathbb{F}_2^{40}$ which corresponds to nonzero positions of $\mathbf{y}$.

By Proposition 3.1, to determine $\vartheta_2(Z, L)$ it is enough to compute $a(T_4, L)$ with $T_4 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

Computation of $a(T_4, L)$. We put

$$N(2,2,0) = \{\langle \mathbf{x}, \mathbf{y} \rangle \in \Lambda_4 \times \Lambda_4 \mid (\mathbf{x}, \mathbf{y}) = 0\}.$$

Then $a(T_4, L) = \# N(2,2,0)$. We divide the set $N(2,2,0)$ into mutually disjoint subsets:

$$N(2,2,0) = N_{A,A}(2,2,0) \cup N_{A,B}(2,2,0) \cup N_{B,A}(2,2,0) \cup N_{B,B}(2,2,0),$$

where $N_{A,A}(2,2,0) = \{\langle \mathbf{x}, \mathbf{y} \rangle \in A \times A \mid (\mathbf{x}, \mathbf{y}) = 0\}$, $N_{A,B}(2,2,0) = \{\langle \mathbf{x}, \mathbf{y} \rangle \in A \times B \mid (\mathbf{x}, \mathbf{y}) = 0\}$, $N_{B,A}(2,2,0) = \{\langle \mathbf{x}, \mathbf{y} \rangle \in B \times A \mid (\mathbf{x}, \mathbf{y}) = 0\}$, $N_{B,B}(2,2,0) = \{\langle \mathbf{x}, \mathbf{y} \rangle \in B \times B \mid (\mathbf{x}, \mathbf{y}) = 0\}$. According to this decomposition we put $\nu_{A,A} = \# N_{A,A}(2,2,0)$, $\nu_{A,B} = \# N_{A,B}(2,2,0)$, $\nu_{B,A} = \# N_{B,A}(2,2,0)$, $\nu_{B,B} = \# N_{B,B}(2,2,0)$. We observe that $\nu_{A,B} = \nu_{B,A}$, and so

$$a(T_4, L) = \nu_{A,A} + 2\nu_{A,B} + \nu_{B,B}. \tag{1}$$

Proposition 5.1. *One has*

$$\nu_{A,A} = 4 \cdot \binom{40}{2} \cdot \left[2 + 4 \cdot \binom{40}{2}\right] = 8779680,$$

which is independent of the code employed (but the code should be extremal).

Since the proof is easy we omit it.

Computation of $\nu_{B,A}$. With each $\mathbf{x} \in B$ we associate $\operatorname{supp}\mathbf{x}$, a codeword in $\mathbf{C}$ of weight 8. Also with each $\mathbf{y} \in A$ we associate a binary vector $\operatorname{supp}\mathbf{y}$ of weight 2.

The number of $\mathbf{y}$ with $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{y} = 0$ is $4 \cdot \binom{32}{2} = 1984$. The number of $\mathbf{y}$ with $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{y} = 2$ is $2 \cdot \binom{8}{2} = 56$. Therefore for each $\mathbf{x} \in B$ there are $1984 + 56 = 2040$ $\mathbf{y}$'s in A satisfying $(\mathbf{x}, \mathbf{y}) = 0$. Consequently,

$$\nu_{B,A} = 285 \cdot 2^7 \cdot 2040 = 74419200.$$

Computation of $\nu_{B,B}$. To compute $\nu_{B,B}$ we need to know the intersections of $\operatorname{supp}\mathbf{x}$ with $\operatorname{supp}\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in B$. First we have

Proposition 5.2. *For* $\mathbf{x}, \mathbf{y} \in B$ *one has*

$$\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{y} \in \{0, 2, 4, 8\}.$$

Since this proposition is easy to prove we skip the proof. We want to know a portion of the biweight enumerator of the binary $[40, 20, 8]$ doubly even code. Here we give the biweight enumerators of the codes $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, introduced before.

The biweight enumerator of a linear code of length n is defined to be

$$\mathcal{BW}(\mathbf{C}, X_{11}, X_{10}, X_{01}, X_{00}) = \sum_{\mathbf{u},\mathbf{v} \in \mathbf{C}} X_{11}^{w_{11}(\mathbf{u},\mathbf{v})} X_{10}^{w_{10}(\mathbf{u},\mathbf{v})} X_{01}^{w_{01}(\mathbf{u},\mathbf{v})} X_{00}^{w_{00}(\mathbf{u},\mathbf{v})},$$

where X_{11}, X_{10}, X_{01}, X_{00} are algebraically independent variables over the field of complex numbers, and $w_{ij}(\mathbf{u}, \mathbf{v})$ $(0 \le i, j \le 1)$ is the number of coordinates k $(1 \le k \le n)$ such that the kth component of $\mathbf{u}$ takes the value i and the kth component of $\mathbf{v}$ takes the value j. We exhibit the biweight enumerators of the codes $\mathcal{C}_i$ $(1 \le i \le 4)$:

$$\begin{aligned}
\mathcal{BW}(\mathcal{C}_1, X_{11}, X_{10}, X_{01}, X_{00}) &= \mathcal{BW}(\mathcal{C}_2, X_{11}, X_{10}, X_{01}, X_{00}) \\
&= \cdots + 285 X_{11}^8 X_{00}^{32} + 5040 X_{11}^4 X_{10}^4 X_{01}^4 X_{00}^{28} \\
&\quad + 53760 X_{11}^2 X_{10}^6 X_{01}^6 X_{00}^{26} + 22140 X_{10}^8 X_{01}^8 X_{00}^{24} + \cdots, \\
\mathcal{BW}(\mathcal{C}_3, X_{11}, X_{10}, X_{01}, X_{00}) &= \mathcal{BW}(\mathcal{C}_4, X_{11}, X_{10}, X_{01}, X_{00}) \\
&= \cdots + 285 X_{11}^8 X_{00}^{32} + 11760 X_{11}^4 X_{10}^4 X_{01}^4 X_{00}^{28} \\
&\quad + 40320 X_{11}^2 X_{10}^6 X_{01}^6 X_{00}^{26} + 28860 X_{10}^8 X_{01}^8 X_{00}^{24} + \cdots.
\end{aligned}$$

In the above we display all the terms for which both $\mathbf{u}$ and $\mathbf{v}$ are of weight 8. Now we show

PROPOSITION 5.3. *Let* $\mathbf{C}$ *be an extremal binary self-dual* $[40, 20, 8]$ *code, and* α_1 *(resp.* α_2, α_3, α_4*) be the coefficient of* $X_{11}^8 X_{00}^{32}$ *(resp.* $X_{11}^4 X_{10}^4 X_{01}^4 X_{00}^{28}$, $X_{11}^2 X_{10}^6 X_{01}^6 X_{00}^{26}$, $X_{10}^8 X_{01}^8 X_{00}^{24}$*) in the biweight enumerator* $\mathcal{BW}(\mathbf{C}, X_{11}, X_{10}, X_{01}, X_{00})$*. Then*

$$\nu_{B,B} = 2^7 (70 \cdot 285 + 48 \cdot \alpha_2 + 64 \cdot \alpha_3 + 128 \cdot \alpha_4).$$

Proof. The number of $\mathbf{y}$'s in B such that $\operatorname{supp} \mathbf{x} = \operatorname{supp} \mathbf{y}$ is computed to be 70. This is because $\mathbf{x}$ and $\mathbf{y}$ have four nonzero coordinates of identical signs and four of opposite signs. The coordinates of the same signs can be chosen in $\binom{8}{4} = 70$ ways.

For each $\mathbf{x} \in B$ we look for $\mathbf{y}$'s in B such that

$$(*) \quad \begin{aligned} &w_{11}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 4, \quad w_{10}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 4, \\ &w_{01}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 4, \quad w_{00}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 28, \quad (\mathbf{x}, \mathbf{y}) = 0. \end{aligned}$$

Let $i_1, \dots, i_4$ be the nonzero positions common to $\mathbf{x}$ and $\mathbf{y}$. There are $\binom{4}{2} = 6$ choices of signs of $\mathbf{y}$ in these positions so as to satisfy $(\mathbf{x}, \mathbf{y}) = 0$. Other signs of nonzero positions of $\mathbf{y}$ have 8 possibilities (three of four are arbitrary and the remaining one is unique). Therefore there are $6 \cdot 8 = 48$ $\mathbf{y}$'s satisfying $(*)$. There are $2^7 \cdot 48 \cdot \alpha_2$ pairs of $\mathbf{x}$ and $\mathbf{y}$ in B satisfying $(*)$. Finding $\mathbf{y}$'s with

$$(**) \quad \begin{aligned} &w_{11}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 2, \quad w_{10}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 6, \\ &w_{01}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 6, \quad w_{00}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 26, \quad (\mathbf{x}, \mathbf{y}) = 0 \end{aligned}$$

or

$$(***) \quad \begin{aligned} &w_{11}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 0, \quad w_{10}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 8, \\ &w_{01}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 8, \quad w_{00}(\operatorname{supp} \mathbf{x}, \operatorname{supp} \mathbf{y}) = 24, \quad (\mathbf{x}, \mathbf{y}) = 0 \end{aligned}$$

is similar. ∎

In summary we obtain

$$\begin{aligned} a(T_4, \mathcal{N}(\mathcal{C}_1)) &= a(T_4, \mathcal{N}(\mathcal{C}_2)) = 994281120, \\ a(T_4, \mathcal{N}(\mathcal{C}_3)) &= a(T_4, \mathcal{N}(\mathcal{C}_4)) = 1035568800. \end{aligned}$$

Before closing this section we prove

THEOREM 5.4. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two extremal binary doubly even self-dual codes of length* 40*. Let* $\mathcal{N}(\mathbf{C}_1)$ *(resp.* $\mathcal{N}(\mathbf{C}_2)$*) be the even unimodular extremal lattices of rank* 40 *constructed from* $\mathbf{C}_1$ *and* $\mathbf{C}_2$*. Then a necessary and sufficient condition that the Siegel theta series* $\vartheta_2(\tau, \mathcal{N}(\mathbf{C}_1))$ *and* $\vartheta_2(\tau, \mathcal{N}(\mathbf{C}_2))$ *coincide is that the biweight enumerators* $\mathcal{BW}(\mathbf{C}_1, X_{11}, X_{10}, X_{01}, X_{00})$ *and* $\mathcal{BW}(\mathbf{C}_2, X_{11}, X_{10}, X_{01}, X_{00})$ *coincide.*

Proof. By the work by H. Maschke [13] the biweight enumerator of any doubly even self-dual binary linear code can be expressed as a polynomial in

F_8, F_{12}, F_{20} and F_{24}. Note that $F_8, \ldots, F_{24}$ are polynomials in the variables z_1, z_2, z_3, z_4, and we should use the variables $X_{11}, X_{10}, X_{01}, X_{00}$ instead. By computer algebra we find that the biweight enumerator of a doubly even self-dual binary linear extremal code of length 40 has the shape $\mathcal{F}_1 + \alpha\mathcal{F}_2$, where

$$\mathcal{F}_1 = \frac{19}{54}F_8^5 + \frac{35}{54}F_8^2F_{12}^2 - \frac{1960}{3}F_8F_{12}F_{20} - \frac{2660}{3}F_{24}F_8^2, \qquad \mathcal{F}_2 = F_{20}^2,$$

with a constant α depending only the code. By Proposition 3.1 the Siegel theta series $\vartheta_2(\tau, \mathcal{N}(\mathbf{C}))$ is completely determined if we know the values $a(T_j, \mathcal{N}(\mathbf{C}))$, $j = 0, 1, 2, 3, 4$ (cf. Table 2.2). Since we consider the 40-dimensional even unimodular extremal lattice $\mathcal{N}(\mathbf{C})$ we know $a(T_0, \mathcal{N}(\mathbf{C})) = 1$, $a(T_j, \mathcal{N}(\mathbf{C})) = 0$, $j = 1, 2, 3$. By Proposition 5.3 the value $a(T_4, \mathcal{N}(\mathbf{C}))$ is controlled by some terms of the biweight enumerator. Therefore the equality $a(T_4, \mathcal{N}(\mathbf{C}_1)) = a(T_4, \mathcal{N}(\mathbf{C}_2))$ holds if and only if $\mathcal{F}_1 + \alpha\mathcal{F}_2 = \mathcal{BW}(\mathbf{C}_1, X_{11}, X_{10}, X_{01}, X_{00}) = \mathcal{BW}(\mathbf{C}_2, X_{11}, X_{10}, X_{01}, X_{00}) = \mathcal{F}_1 + \alpha'\mathcal{F}_2$. This completes the proof of the theorem. ∎

5.2. The Fourier coefficients of $\vartheta_3(\tau, L)$ for even unimodular extremal 40-dimensional lattices L. We compute

$$a(T, L) = \#\{\langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle \in L^3 \mid [\mathbf{x}, \mathbf{y}, \mathbf{z}] = 2T\}$$

for the case when L is an even unimodular 40-dimensional extremal lattice constructed from a binary code. We need to compute $a(T, L)$ for the matrix T_{10} given in Table 2.3.

In a similar way to $\vartheta_2(\tau, L)$ this quantity is expressed as

$$\begin{aligned} a(T_{10}, L) = \mu_{A,A,A} &+ \mu_{A,A,B} + \mu_{A,B,A} + \mu_{B,A,A} \\ &+ \mu_{A,B,B} + \mu_{B,A,B} + \mu_{B,B,A} + \mu_{B,B,B}, \end{aligned}$$

where

$$\begin{aligned} \mu_{A,A,A} &= \#\{\langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle \in A^3 \mid [\mathbf{x}, \mathbf{y}, \mathbf{z}] = 2T\}, \\ \mu_{A,A,B} &= \#\{\langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle \in A \times A \times B \mid [\mathbf{x}, \mathbf{y}, \mathbf{z}] = 2T\}, \\ &\vdots \\ \mu_{B,B,B} &= \#\{\langle \mathbf{x}, \mathbf{y}, \mathbf{z}\rangle \in B^3 \mid [\mathbf{x}, \mathbf{y}, \mathbf{z}] = 2T\}. \end{aligned}$$

We can easily prove

PROPOSITION 5.5.

(i) $\mu_{A,A,B} = \mu_{A,B,A} = \mu_{B,A,A}$,
(ii) $\mu_{A,B,B} = \mu_{B,A,B} = \mu_{B,B,A}$.

By the above proposition we get

$$a(T_{10}, L) = \mu_{A,A,A} + 3\mu_{B,A,A} + 3\mu_{B,B,A} + \mu_{B,B,B}. \tag{2}$$

Computation of $\mu_{A,A,A}$. There are $\binom{40}{2}\cdot 4 = 3120$ elements in the set A. For each $\mathbf{x}\in A$ there are $2+\binom{38}{2}\cdot 4 = 2814$ elements $\mathbf{y}\in A$ that are perpendicular to $\mathbf{x}$. For each pair $\langle\mathbf{x},\mathbf{y}\rangle\in A^2$ with $(\mathbf{x},\mathbf{y})=0$ two cases are possible: (i) $\operatorname{supp}\mathbf{x}=\operatorname{supp}\mathbf{y}$, (ii) $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=0$. In the first case there are $\binom{38}{2}\cdot 4 = 2812$ $\mathbf{z}$'s in A that are perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. In the second case there are $2+2+\binom{36}{2}\cdot 4 = 2524$ such $\mathbf{z}$'s. Therefore

$$\mu_{A,A,A} = 3120(2\cdot 2812 + 2812\cdot 2524) = 22161709440. \tag{3}$$

Computation of $\mu_{B,A,A}$. There are 36480 vectors in B. For each $\mathbf{x}\in B$ there are $\binom{8}{2}\cdot 2+\binom{32}{2}\cdot 4 = 2040$ vectors $\mathbf{y}\in A$ satisfying $(\mathbf{x},\mathbf{y})=0$. If $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=2$, then there are $\binom{6}{2}\cdot 2+\binom{32}{2}\cdot 4 = 2014$ $\mathbf{z}$'s in B such that $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$. If $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=0$, then there are $\binom{8}{2}\cdot 2+2+\binom{30}{2}\cdot 4 = 1798$ $\mathbf{z}$'s in B such that $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$. Therefore

$$\mu_{B,A,A} = 36480\cdot(56\cdot 2014 + 1984\cdot 1798) = 134246983680. \tag{4}$$

Computation of $\mu_{B,B,A}$. Let $\mathbf{x},\mathbf{y}\in B$. Let $C_8(i)$ $(1\le i\le 4)$ be the set of codewords of weight 8 in the code $\mathcal{C}_i$. There are 285 pairs $\langle\mathbf{u},\mathbf{v}\rangle$ in $C_8(i)\times C_8(i)$ $(1\le i\le 4)$ such that $\mathbf{u}*\mathbf{v}=8$. For a fixed $\mathbf{x}\in B$ there are 70 vectors $\mathbf{y}\in B$ satisfying $(\mathbf{x},\mathbf{y})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=8$. There are $\binom{4}{2}\cdot 2\cdot 2+\binom{32}{2}\cdot 4 = 24+1984 = 2008$ such pairs $\langle\mathbf{x},\mathbf{y}\rangle$. The number $24\cdot 70 = 1680$ is the number of pairs $\mathbf{y}\in B,\mathbf{z}\in A$ such that $(\mathbf{x},\mathbf{y})=(\mathbf{x},\mathbf{z})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=8$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{z}=2$ for a fixed $\mathbf{x}\in B$. This reflects the first row of Table 6.1 below. The number $24\cdot 1984 = 138880$ is the number of pairs $\mathbf{y}\in B$, $\mathbf{z}\in A$ such that $(\mathbf{x},\mathbf{y})=(\mathbf{x},\mathbf{z})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=8$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{z}=0$ for a fixed $\mathbf{x}\in B$. This yields the third row of Table 6.1.

There are 11760 (resp. 5040) pairs of codewords $\langle\mathbf{u},\mathbf{v}\rangle$ in $C_8(i)\times C_8(i)$ $(1\le i\le 2)$ (resp. $3\le i\le 4$) such that $\mathbf{u}*\mathbf{v}=4$. For a fixed $\mathbf{x}\in B$ there are 48 vectors $\mathbf{y}\in B$ satisfying $(\mathbf{x},\mathbf{y})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=4$. For each pair $\langle\mathbf{x},\mathbf{y}\rangle$ in B^2 with the above conditions there are $2\cdot\binom{4}{2}=12$ $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$, $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{z}=2$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=0$; there are 4 $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$, $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{z}=2$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=2$; and 12 $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$, $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{z}=0$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=2$. There are $4\cdot\binom{28}{2}=1512$ $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$, $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{z}=0$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=0$. The numbers $48\cdot 12 = 576, 48*4 = 192, 576, 48\cdot 1512 = 72576$ yield some rows in Table 6.1.

There are 40320 (resp. 53760) pairs $\langle\mathbf{u},\mathbf{v}\rangle$ in $C_8(i)\times C_8(i)$ $(1\le i\le 2)$ (resp. $3\le i\le 4$) such that $\mathbf{u}*\mathbf{v}=2$. For a fixed $\mathbf{x}\in B$ there are 64 vectors $\mathbf{y}\in B$ satisfying $(\mathbf{x},\mathbf{y})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=2$. There are 30 $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z})=(\mathbf{y},\mathbf{z})=0$, $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{z}=2$, $\operatorname{supp}\mathbf{y}*\operatorname{supp}\mathbf{x}=0$; there are 30 $\mathbf{z}$'s in A with the additional conditions

$(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z}) = 0$, $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{z} = 0$, $\operatorname{supp}\mathbf{y} * \operatorname{supp}\mathbf{x} = 2$; and there are $4\cdot\binom{26}{2} = 1300$ $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z}) = 0$, $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{z} = 0$, $\operatorname{supp}\mathbf{y} * \operatorname{supp}\mathbf{x} = 0$. The numbers $64\cdot 30$, $64\cdot 30$, $64\cdot 1300 = 83200$ yield another row in Table 6.1.

There are 28860 (resp. 22140) pairs $\langle\mathbf{u},\mathbf{v}\rangle$ in $C_8(i)\times C_8(i)$ $(1\le i\le 2)$ (resp. $3\le i\le 4$) such that $\mathbf{u}*\mathbf{v}=0$. For a fixed $\mathbf{x}\in B$ there are 128 vectors $\mathbf{y}\in B$ satisfying $(\mathbf{x},\mathbf{y})=0$ and $\operatorname{supp}\mathbf{x}*\operatorname{supp}\mathbf{y}=0$. For each pair $\langle\mathbf{x},\mathbf{y}\rangle$ in B^2 with the above conditions there are $2\cdot\binom{8}{2}=56$ $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z}) = 0$, $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{z} = 2$, $\operatorname{supp}\mathbf{y} * \operatorname{supp}\mathbf{x} = 0$; there are 56 $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z}) = 0$, $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{z} = 0$, $\operatorname{supp}\mathbf{y} * \operatorname{supp}\mathbf{x} = 2$; and there are $4\cdot\binom{24}{2} = 1104$ $\mathbf{z}$'s in A with the additional conditions $(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z}) = 0$, $\operatorname{supp}\mathbf{x} * \operatorname{supp}\mathbf{z} = 0$, $\operatorname{supp}\mathbf{y} * \operatorname{supp}\mathbf{x} = 0$. The numbers $128\cdot 56 = 7168$, $7168, 128\cdot 1104 = 141312$ reflect some of the last rows in Table 6.1. Altogether we get

$$\mu_{B,B,A} = \begin{cases} 2^7\cdot(285\cdot 70\cdot 2008 + 5040\cdot 48\cdot 1540 \\ \qquad + 53760\cdot 64\cdot 1360 + 22140\cdot 128\cdot 1216) \\ \quad = 1092855490560 \quad \text{for the lattices } \mathcal{N}(\mathcal{C}_1), \mathcal{N}(\mathcal{C}_2), \\ 2^7\cdot(285\cdot 70\cdot 2008 + 11760\cdot 48\cdot 1540 \\ \qquad + 40320\cdot 64\cdot 1360 + 28860\cdot 128\cdot 1216) \\ \quad = 1140584048640 \quad \text{for the lattices } \mathcal{N}(\mathcal{C}_3), \mathcal{N}(\mathcal{C}_4). \end{cases} \tag{5}$$

Table 6.1

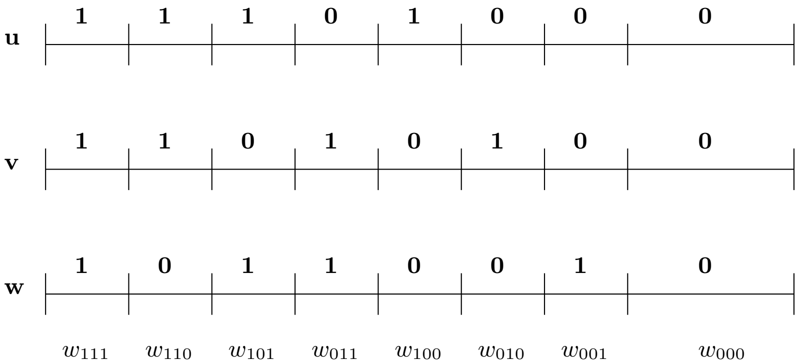
11	10	01	00	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	freq
8	0	0	32	285	285	285	285	14056
4	4	4	28	5040	5040	11760	11760	73920
2	6	6	26	53760	53760	40320	40320	87040
0	8	8	24	22140	22140	28860	28860	155648

Computation of $\mu_{B,B,B}$. Explaining every detail of the computation would take too much space, therefore we only describe the inner product relations of the vectors $\mathbf{x},\mathbf{y}$ and $\mathbf{z}$ in B. The description is well-controlled by some terms of the triweight enumerator of a code $\mathbf{C}$:

$$\begin{aligned} &\mathcal{TW}(\mathbf{C}, X_{111}, X_{110}, X_{101}, X_{011}, X_{100}, X_{010}, X_{001}, X_{000}) \\ &\quad = \sum_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{C}} X_{111}^{w_{111}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{110}^{w_{110}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{101}^{w_{101}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{011}^{w_{011}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{100}^{w_{100}(\mathbf{u},\mathbf{v},\mathbf{w})} \\ &\qquad\qquad \cdot X_{010}^{w_{010}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{001}^{w_{001}(\mathbf{u},\mathbf{v},\mathbf{w})} X_{000}^{w_{000}(\mathbf{u},\mathbf{v},\mathbf{w})}, \end{aligned}$$

where $X_{111}, X_{110}, X_{101}, X_{011}, X_{100}, X_{010}, X_{001}$ and X_{000} are algebraically independent variables over the field of complex numbers, and $w_{ijh}(\mathbf{u},\mathbf{v},\mathbf{w})$

$(0 \le i, j, h \le 1)$ is the number of coordinates k $(1 \le k \le n)$ such that the kth component of $\mathbf{u}$ takes the value i, the kth component of $\mathbf{v}$ takes the value j, and the kth component of $\mathbf{w}$ takes the value h. Here we give a picture of generalized weights:



For our present computation we only need the terms coming from the codewords $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of weight 8; for instance, in the case of $\mathcal{C}_1$, terms such as

$$11760 X_{111}^2 X_{110}^2 X_{101}^2 X_{011}^2 X_{100}^2 X_{010}^2 X_{001}^2 X_{000}^{26}$$

and

$$42000 X_{111}^4 X_{111}^4 X_{100}^4 X_{010}^4 X_{001}^4 X_{000}^{24}.$$

There are 51 types of terms that correspond to triples of codewords of weight 8.

For a fixed $\mathbf{x} \in B$ we want to count the vectors $\mathbf{y}, \mathbf{z} \in B$ such that $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{z}) = (\mathbf{y}, \mathbf{z}) = 0$. However the frequencies of the pairs $\langle \mathbf{y}, \mathbf{z} \rangle$ vary according to the intersection relation among $\operatorname{supp} \mathbf{x}$, $\operatorname{supp} \mathbf{y}$, $\operatorname{supp} \mathbf{z}$. We give a typical instance of counting the number of pairs $\mathbf{y}$ and $\mathbf{z}$ when $\operatorname{supp} \mathbf{x}$, $\operatorname{supp} \mathbf{y}$, $\operatorname{supp} \mathbf{z}$ are specified. Consider the triple of codewords $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of weight 8 corresponding to the term $X_{111}^2 X_{110}^2 X_{101}^2 X_{011}^2 X_{100}^2 X_{010}^2 X_{001}^2 X_{000}^{26}$. For a fixed $\mathbf{x} \in B$ we look for $\mathbf{y}, \mathbf{z} \in B$ satisfying $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{z}) = (\mathbf{y}, \mathbf{z}) = 0$ and $\operatorname{supp} \mathbf{x} = \mathbf{u}$, $\operatorname{supp} \mathbf{y} = \mathbf{v}$, $\operatorname{supp} \mathbf{z} = \mathbf{w}$. Without loss of generality we may suppose that the nonzero coordinate positions of $\mathbf{x} \in B$ are $I_1 = \{i_1, \ldots, i_8\}$ and the coordinates x_i, $i \in I_1$, take the value 1. Further we may suppose that the nonzero coordinate positions of $\mathbf{y}$ are $I_2 = \{i_1, i_2, \ldots, i_4, i_9, \ldots, i_{12}\}$. The nonzero coordinate values y_i $(i \in I_2)$ are ± 1 with the additional condition that the product of those 8 values should be 1. Let $I_3 = \{i_1, i_2, i_5, i_6, i_9, i_{10}, i_{13}, i_{14}\}$ be the nonzero coordinate positions of $\mathbf{z}$ together with the conditions that $z_i = \pm 1$ $(i \in I_3)$ and the product of those 8 values should be 1. We understand that indices with different numbers are different. We seek all the solutions of the simultaneous equations

$$\sum_{i \in I_1 \cap I_2} y_i = 0, \quad \sum_{i \in I_1 \cap I_3} z_i = 0, \quad \sum_{i \in I_2 \cap I_3} y_i z_i = 0, \quad \prod_{i \in I_2} y_i = \prod_{i \in I_3} z_i = 1.$$

The number of solutions, computed by hand or by computer programming, is 832. This is the value in the last column and the ninth row in Table 6.2. The other entries of the last column are similarly computed.

Table 6.2

111	110	101	011	100	010	001	000	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	freq
8	0	0	0	0	0	0	32	285	285	285	285	2520
4	4	0	0	0	0	4	28	5040	5040	11760	11760	1248
4	0	4	0	0	4	0	28	5040	5040	11760	11760	1248
4	0	0	4	4	0	0	28	5040	5040	11760	11760	1248
0	4	4	4	0	0	0	28	5040	5040	11760	11760	1728
2	6	0	0	0	0	6	26	53760	53760	40320	40320	1920
2	0	6	0	0	6	0	26	53760	53760	40320	40320	1920
2	0	0	6	6	0	0	26	53760	53760	40320	40320	1920
2	2	2	2	2	2	2	26	13440	9408	137088	115584	832
3	1	1	1	3	3	3	25	0	16128	32256	21504	768
1	3	1	3	3	1	3	25	0	16128	32256	21504	1152
1	3	3	1	1	3	3	25	0	16128	32256	21504	1152
1	1	3	3	3	3	1	25	0	16128	32256	21504	1152
0	8	0	0	0	0	8	24	22140	22140	28860	28860	8960
0	0	8	0	0	8	0	24	22140	22140	28860	28860	8960
0	0	0	8	8	0	0	24	22140	22140	28860	28860	8960
4	0	0	0	4	4	4	24	7440	1680	42000	52752	1536
0	4	0	4	4	0	4	24	7440	1680	42000	52752	2304
0	0	4	4	4	4	0	24	7440	1680	42000	52752	2304
0	4	4	0	0	4	4	24	7440	1680	42000	52752	2304
2	2	2	0	2	4	4	24	65280	53760	201600	223104	1024
2	2	0	2	4	2	4	24	65280	53760	201600	223104	1024
2	0	2	2	4	4	2	24	65280	53760	201600	223104	1024
0	2	2	4	4	2	2	24	65280	53760	201600	223104	1536
0	4	2	2	2	2	4	24	65280	53760	201600	223104	1536
0	2	4	2	2	4	2	24	65280	53760	201600	223104	1536
1	1	1	3	5	3	3	23	445440	456960	483840	365568	1536
1	1	3	1	3	5	3	23	445440	456960	483840	365568	1536
1	3	1	1	3	3	5	23	445440	456960	483840	365568	1536
2	2	0	0	4	4	6	22	172800	165312	346752	389760	1024
2	0	2	0	4	6	4	22	172800	165312	346752	389760	1024
2	0	0	2	6	4	4	22	172800	165312	346752	389760	1024

Table 6.2 [cont.]

111	110	101	011	100	010	001	000	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	freq
0	4	0	2	4	2	6	22	172800	165312	346752	389760	3072
0	4	2	0	2	4	6	22	172800	165312	346752	389760	3072
0	0	4	2	4	6	2	22	172800	165312	346752	389760	3072
0	2	4	0	2	6	4	22	172800	165312	346752	389760	3072
0	2	0	4	6	2	4	22	172800	165312	346752	389760	3072
0	0	2	4	6	4	2	22	172800	165312	346752	389760	3072
0	2	2	2	4	4	4	22	2730240	2733696	1161216	1290240	2048
1	1	1	1	5	5	5	21	3763200	3827712	1193472	1053696	2048
2	0	0	0	6	6	6	20	218880	193536	225792	311808	0
0	2	0	2	6	4	6	20	2749440	2735040	2153088	2131584	4096
0	2	2	0	4	6	6	20	2749440	2735040	2153088	2131584	4096
0	0	2	2	6	6	4	20	2749440	2735040	2153088	2131584	4096
0	0	4	0	4	8	4	20	225840	244272	827568	773808	6144
0	4	0	0	4	4	8	20	225840	244272	827568	773808	6144
0	0	0	4	8	4	4	20	225840	244272	827568	773808	6144
0	2	0	0	6	6	8	18	1224960	1236480	1532160	1510656	8192
0	0	2	0	6	8	6	18	1224960	1236480	1532160	1510656	8192
0	0	0	2	8	6	6	18	1224960	1236480	1532160	1510656	8192
0	0	0	0	8	8	8	16	261540	236772	559332	634596	16384

We explain how to compute $\mu_{B,B,B}$ by using Table 6.2. Let a_i $(1 \le i \le 51)$ be the ith entry of the 9th column, and m_i $(1 \le i \le 51)$ be the ith entry in the last column. Then the quantity $\mu_{B,B,B}$ for the lattice $\mathcal{N}(\mathcal{C}_1)$ is given by

$$\mu_{B,B,B} = 2^7 \cdot \sum_{i=1}^{51} a_i m_i. \tag{6}$$

The result is $\mu_{B,B,B} = 11892863646720$. In the same way other three cases are computed:

$$\mu_{B,B,B} = \begin{cases} 11892736244736 & \text{for the code } \mathcal{C}_2, \\ 13601831500800 & \text{for the code } \mathcal{C}_3, \\ 13601831500800 & \text{for the code } \mathcal{C}_4. \end{cases} \tag{7}$$

Using the formula (2) together with (3)–(7) we get

$$a\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathcal{N}(\mathcal{C}_1)\right) = 15596332778880,$$

$$a\left(\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}, \mathcal{N}(\mathcal{C}_2)\right) = 15596205376896,$$

$$a\left(\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}, \mathcal{N}(\mathcal{C}_3)\right) = 17448486307200,$$

$$a\left(\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}, \mathcal{N}(\mathcal{C}_4)\right) = 17448486307200.$$

These values are the basis of our theorems in Section 4.

Remark 1. Along with the present method of computation we can give a method to compute

$$a\left(\begin{pmatrix}2&0&0\\0&2&1/2\\0&1/2&2\end{pmatrix}, \mathcal{N}(\mathcal{C}_i)\right), \quad a\left(\begin{pmatrix}2&0&1/2\\0&2&1/2\\1/2&1/2&2\end{pmatrix}, \mathcal{N}(\mathcal{C}_i)\right) \quad (1 \leq i \leq 4).$$

These values will serve as checks for the present computations.

6. Proofs of Theorems 4.1, 4.2 and 4.3

Proof of Theorem 4.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the lattices M_{11} and M_{12} respectively in Section 2.6.1. Since the two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ have identical biweight enumerators by Theorem 5.4, $\mathcal{L}_1$ and $\mathcal{L}_2$ have identical Siegel theta series of degree 2. By Table 3.3 we know that $a(T_{10}, \mathcal{L}_1) \neq a(T_{10}, \mathcal{L}_2)$, therefore their Siegel theta series of degree 3 differ. ∎

Proof of Theorem 4.2. Let $\mathcal{L}_3$ and $\mathcal{L}_4$ be the lattices M_{13} and M_{14} respectively in Section 2.6.1. By Table 3.3 we see that $a(T_j, \mathcal{L}_3) = a(T_j, \mathcal{L}_4)$ for $0 \leq j \leq 10$. Then by Proposition 3.1 we deduce that $\vartheta_3(\tau, \mathcal{L}_3) = \vartheta_3(\tau, \mathcal{L}_4)$. In particular, the theta series of degrees 1 and 2 coincide. ∎

Proof of Theorem 4.3. Suppose that two doubly even self-dual binary extremal codes $\mathbf{C}, \mathbf{C}'$ of length 40 have identical triweight enumerators. Then they also have identical biweight enumerators. The Fourier coefficients $a(T_j, \mathcal{N}(\mathbf{C}))$ may be written in terms of coefficients of the weight enumerators as shown by Proposition 5.3, equations (5) and (6). From this we see that $a(T_j, \mathcal{N}(\mathbf{C})) = a(T_j, \mathcal{N}(\mathbf{C}'))$ for $0 \leq j \leq 10$. By Proposition 3.1 we have $\vartheta_3(\tau, \mathcal{N}(\mathbf{C})) = \vartheta_3(\tau, \mathcal{N}(\mathbf{C}'))$. ∎

Remark 2. The two codes $\mathcal{C}_3$ and $\mathcal{C}_4$ have different triweight enumerators as shown by Table 6.2, but they have identical Siegel theta series of degree 3. This contrasts with Theorem 5.4.

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