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On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions

by

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1. Introduction. Let $P(a,q)$ be the least prime in an arithmetic progression $a \pmod q$ where a and q are coprime positive integers. In 1944 Linnik proved [12, 13] the impressive upper bound

$$P(a,q) \le Cq^L$$

with effectively computable constants C and L. We will refer to this last inequality as *Linnik's theorem*. The following table, taken from [8, p. 266] and supplemented by three additional references, lists some proven admissible values for L.

Table 1. Admissible values for L

L	Year of publication	Author	Reference
10000	1957	Pan	[15]
5448	1958	Pan	[16]
777	1965	Chen	[1]
630	1971	Jutila	[17, p. 370]
550	1970	Jutila	[10]
168	1977	Chen	[2]
80	1977	Jutila	[11]
36	1977	Graham	[6]
20	1981	Graham	[7]
17	1979	Chen	[3]
16	1986	Wang	[18]
13.5	1989	Chen and Liu	[4]
11.5	1991	Chen and Liu	[5]
8	1991	Wang	[19]
5.5	1992	Heath-Brown	[8]

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In [8, pp. 332–337] Heath-Brown mentions several small suggestions for improvement to his work. Making use of these suggestions we prove

Theorem 1.1. *We have*

$$P(a,q) \leq Cq^{5.18}$$

with an effectively computable constant C.

In the course of the proof of Theorem 1.1 we improve on several results in [8] concerning zero-free regions and zero-density estimates for Dirichlet L-functions. Let us cite, for example, the following result.

Theorem 1.2. *There is an effectively computable constant* q_0, *such that for* $q \geq q_0$ *the function*

$$\prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi)$$

has at most one zero in the region

$$\sigma \geq 1 - \frac{0.440}{\log q}, \qquad |t| \leq 1.$$

If this exceptional zero ("Siegel zero") exists then it is real, simple and belongs to a non-principal real character.

Remark. Heath-Brown [8, Theorem 1, p. 268] has proved this estimate with $c = 0.348$ instead of $c = 0.440$. By a small variation of Heath-Brown's argument, Liu and Wang [14, pp. 345–346] achieved $c = 0.364$.

For a motivation as well as a more detailed introduction into this topic we refer to [9, §18], [8, pp. 265–270] and [20, pp. 7–10].

In this paper we refine certain arguments of [8] in the way proposed by the nine improvement suggestions in [8, pp. 332–337]. To be precise, we use suggestions (2), (5), (7) and (9), the largest contribution to our improved estimates being due to suggestion (2). On the other hand, the improvement of the admissible value for L resulting from our use of the other five suggestions is too small and will not be discussed here.

Besides some small variations, this paper is a shorter version of our work [20] and the reader is referred to that work whenever more details are desired.

Standard notation from analytic number theory is used. For $q \in \mathbb{N} = \{1,2,3,\ldots\}$ we use χ to denote a Dirichlet character modulo q, χ_0 for the principal character modulo q and $L(s,\chi)$ to denote the corresponding Dirichlet L-function. Furthermore, we use $\operatorname{ord}\chi$ for the order of χ in the group of Dirichlet characters modulo q, and the notation $[x] = \max\{a \in \mathbb{Z} \mid a \leq x\}$ and

$$\mathscr{L} = \log q.$$

The real part of a complex number z is denoted by $\Re\{z\}$ and its imaginary part by $\Im\{z\}$. We refer the unfamiliar reader to the detailed explanations given in [20, pp. 4–5]. Generally, the results in this paper are proven for $q \geq q_0$ with q_0 being an absolute and effectively computable constant.

In analogy to [8] we need computer calculations along the way. These have been done with the computing software Maple and a standard home computer.

Remark. Some test calculations indicate that if one increased the computer calculations towards infinity one would get about $L = 5.13$.

2. Some preliminaries from [8]

2.1. An important lemma. In order to improve the admissible value for L in Linnik's theorem it turns out to be sufficient to improve the available estimates concerning the location of zeros of Dirichlet L-functions in the rectangle

$$R := R(l) \tag{2.1}$$

where $l \leq \mathscr{L}/10$ is the positive integer defined in [8, Lemma 6.1] (depending on q) and

$$R(x) := \left\{ \sigma + it \in \mathbb{C} \;\middle|\; 1 - \frac{\log\log \mathscr{L}}{3\mathscr{L}} \leq \sigma \leq 1,\ |t| \leq x \right\}. \tag{2.2}$$

We will extensively use Lemma 5.2 from [8] and want to reformulate it in order to make its application in this paper more convenient. For this purpose, let χ be a non-principal character modulo q. As in [8, Lemma 2.5] we set

$$\phi = \phi(\chi) = \begin{cases} 1/4 & \text{if } q \text{ is cube-free } (^1) \text{ or } \operatorname{ord}\chi \leq \mathscr{L}, \\ 1/3 & \text{else.} \end{cases}$$

Lemma 2.1 (variation of [8, Lemma 5.2]). *Let χ be a non-principal character modulo q and let R, l, $R(x)$ and ϕ be as above. Furthermore, let $s \in R(9l)$ and suppose the number of zeros* (2) *$\rho \in R$ of $L(s,\chi)$ with $\Re\{\rho\} > \Re\{s\}$ is at most* 10, *i.e.*

$$A_1 := \{\rho \in R \mid L(\rho,\chi) = 0,\ \Re\{\rho\} > \Re\{s\}\} \quad \textit{and} \quad \widetilde{\#} A_1 \leq 10$$

with $\widetilde{\#}$ indicating that we count the elements of the set A_1 with multiplicity.

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(1) By this we mean that for all prime numbers p we have $p^3 \nmid q$.

(2) By this we always mean the number of zeros counted with multiplicity.

Let A_2 be an arbitrary set with

$$A_2 \subseteq \{\rho \in R \mid L(\rho,\chi)=0,\ \Re\{\rho\} \le \Re\{s\}\} \quad \textit{and} \quad \widetilde{\#}A_2 \le 10.$$

If f is a function satisfying Conditions 1 *and* 2 *of* [8, *pp.* 280, 286] *then for any $\varepsilon > 0$ and $q \ge q_0(f,\varepsilon)$ we have*

$$\begin{aligned} K(s,\chi) &:= \sum_{n=1}^{\infty} \Lambda(n)\Re\Big\{\frac{\chi(n)}{n^s}\Big\} f(\mathscr{L}^{-1}\log n) \\ &\le -\mathscr{L} \sum_{\rho\in A_1\cup A_2} \Re\{F((s-\rho)\mathscr{L})\} + f(0)\frac{\phi}{2}\mathscr{L} + \varepsilon\mathscr{L}. \end{aligned}$$

Proof. Let $\varepsilon > 0$ and $s \in R(9l)$. By [8, Lemma 5.2] the statement follows from the verification of the following two inequalities:

$$\begin{aligned} -\sum_{|1+it-\rho|\le\delta} \Re\{F((s-\rho)\mathscr{L})\} &\le -\sum_{\substack{\rho\in R \\ |1+it-\rho|\le\delta \\ \Re\{\rho\}\le\Re\{s\}\Rightarrow\rho\in A_2}} \Re\{F((s-\rho)\mathscr{L})\} \\ &\le -\sum_{\substack{\rho\in R \\ \Re\{\rho\}\le\Re\{s\}\Rightarrow\rho\in A_2}} \Re\{F((s-\rho)\mathscr{L})\} + \varepsilon/2 \\ &= -\sum_{\rho\in A_1\cup A_2} \Re\{F((s-\rho)\mathscr{L})\} + \varepsilon/2. \end{aligned}$$

For the first inequality use [8, Lemma 6.1] and Condition 2. For the second use partial integration on the Laplace transform F to show that the additional ρ's contribute at most $\varepsilon/2$ (cf. the reasoning in [8, p. 287]). For more details see [20, Proof of Lemma 2.4]. ■

In Lemma 2.1 and [8, Lemma 5.3], both of which will be used extensively throughout this paper, one needs to choose some function f. We will use the following ones which appear in [8, Lemma 7.2]. Let $\gamma > 0$ be a real parameter. Set $g(x) := \gamma^2 - x^2$ and define

$$f(t) := \begin{cases} \int_{t-\gamma}^{\gamma} g(x)g(t-x)\,dx \\ \quad = -\frac{1}{30}t^5 + \frac{2\gamma^2}{3}t^3 - \frac{4\gamma^3}{3}t^2 + \frac{16\gamma^5}{15}, & t\in[0,2\gamma), \\ 0 & t \ge 2\gamma. \end{cases} \tag{2.3}$$

The function f satisfies Condition 1 in [8, p. 280] and according to [8, p. 289] the Laplace transform

$$F(z) := \int_0^{\infty} e^{-zt} f(t)\,dt$$

of f satisfies the following Condition 2 [8, p. 286]:

$$\Re\{z\} \ge 0 \ \Rightarrow\ \Re\{F(z)\} \ge 0. \tag{2.4}$$

By partial integration we get

$$F(z) = \begin{cases} \frac{16\gamma^5}{15} z^{-1} - \frac{8\gamma^3}{3} z^{-3} + 4\gamma^2(1+e^{-2\gamma z})z^{-4} \\ \quad + 4(-1+e^{-2\gamma z} + 2\gamma z e^{-2\gamma z})z^{-6}, & z \neq 0, \\ \frac{8\gamma^6}{9}, & z = 0. \end{cases} \tag{2.5}$$

We will normally refer to the above functions f and F just by giving an explicit $\gamma > 0$.

2.2. Labeling of the interesting zeros. We proceed to label some of the zeros in the rectangle R (defined in (2.1)) and their corresponding characters χ in the same way as in [8, pp. 285, 287]. Note that, whenever we write down a specific zero ρ of a Dirichlet L-function, it will be done under the implicit assumption that this zero exists.

For a fixed positive integer q we consider all zeros $\rho \in R$ of the function

$$P(s) := \prod_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} L(s, \chi). \tag{2.6}$$

First, let ρ_1 be a zero of $P(s)$ in R for which $\Re\{\rho_1\}$ is maximal and let χ_1 be a corresponding character, that is, $L(\rho_1, \chi_1) = 0$. Then in the kth step $(k \geq 2)$ consider the zeros $\rho \in R$ of the function

$$\frac{P(s)}{L(s,\chi_1)L(s,\overline{\chi_1}) \cdot \ldots \cdot L(s,\chi_{k-1})L(s,\overline{\chi_{k-1}})}$$

and choose a zero ρ_k among them with maximal real part. Write χ_k for the corresponding character. Continue until there are no more zeros to consider. Then

$$\chi_i \neq \chi_j, \overline{\chi_j} \quad \text{ for } i \neq j.$$

Also, by [8, Lemma 6.1] and for q large enough, if $L(\rho, \chi) = 0$ and $\chi \neq \chi_i, \overline{\chi_i}$ for $1 \leq i < k$ then

$$\Re\{\rho\} \leq \Re\{\rho_k\} \quad \text{or} \quad |\Im\{\rho\}| \geq 10l.$$

We set

$$\rho_k = \beta_k + i\gamma_k, \qquad \beta_k = 1 - \mathscr{L}^{-1}\lambda_k, \qquad \gamma_k = \mathscr{L}^{-1}\mu_k.$$

We proceed to label one more potential zero. Suppose $L(s, \chi_1)$ has a zero $\rho' \in R \setminus \{\rho_1\}$ or ρ_1 is a multiple zero, i.e. the zero order of ρ_1 is greater than or equal to two. Then choose a zero $\rho' \in R$ of $L(s, \chi_1)$ according to the following steps:

Case 1: If ρ_1 is a multiple zero choose $\rho' = \rho_1$.
Case 2: If we are not in Case 1 and χ_1 is real and ρ_1 is complex choose ρ' among the zeros in $R \setminus \{\rho_1, \overline{\rho_1}\}$ such that $\Re\{\rho'\}$ is maximal.

Case 3: If we are not in Case 1 or 2 choose ρ' among the zeros in $R \setminus \{\rho_1\}$ such that $\Re\{\rho'\}$ is maximal.

In analogy to the previous notation we set

$$\rho' = \beta' + i\gamma', \quad \beta' = 1 - \mathscr{L}^{-1}\lambda', \quad \gamma' = \mathscr{L}^{-1}\mu'.$$

2.3. Estimation of certain suprema. We will need estimates of the type

$$A_{\sup} := \sup_{\substack{s_1 \in [s_{11}, s_{12}] \\ s_2 \in [s_{21}, s_{22}] \\ s_2 \le s_1,\, t \in \mathbb{R}}} A(s_1, s_2, t) \le C \tag{2.7}$$

with an explicit numerical value C. Here,

$$A(s_1, s_2, t) := \Re\{k_1 F(-s_1 + it) - k_2 F(-(s_1 - s_2) + it) - k_3 F(it)\},$$

s_{ij} and k_i are non-negative constants with

$$0 \le s_{11} \le s_{12} \le 4, \quad 0 \le s_{21} \le s_{22} \le s_{12},$$

and F is given by (2.5). We also define

$$s_3 := s_1 - s_2 \in [\max\{0, s_{11} - s_{22}\}, s_{12} - s_{21}] =: [s_{31}, s_{32}]. \tag{2.8}$$

Heath-Brown [8, pp. 312–313] proves an estimate of the form (2.7) for a concrete F and $k_1 = 1$, $k_2 = 0$, $k_3 = 2$, $s_{11} = 0$, $s_{12} = (7/6 + 2\varepsilon)^{-1}$. We will proceed similarly for general parameters k_i and s_{ij} although at some points we choose to make some minor modifications in order to get sharper estimates. Since

$$\Re\{F(z)\} = \Re\{F(\overline{z})\} \tag{2.9}$$

we may assume that $t \ge 0$. We distinguish two cases.

2.3.1. *Estimates for $t \ge x_1$.* Suppose that $t \ge x_1 \ge 4$. Since F satisfies (2.4) and $k_3 \ge 0$ we have

$$\begin{aligned}\Re\{k_1 F(-s_1 + it) &- k_2 F(-(s_1 - s_2) + it) - k_3 F(it)\} \\ &\le \Re\{k_1 F(-s_1 + it) - k_2 F(-s_3 + it)\} =: \widetilde{A}(s_1, s_3, t).\end{aligned}$$

By (2.5) the function F is a sum of four terms. Accordingly, we write

$$\widetilde{A}(s_1, s_3, t) = \widetilde{A}_1(s_1, s_3, t) + \widetilde{A}_2(s_1, s_3, t) + \widetilde{A}_3(s_1, s_3, t) + \widetilde{A}_4(s_1, s_3, t).$$

For instance, $\widetilde{A}_3(s_1, s_3, t)$ is equal to

$$\Re\{4k_1\gamma^2(1 + e^{-2\gamma(-s_1+it)})(-s_1 + it)^{-4} - 4k_2\gamma^2(1 + e^{-2\gamma(-s_3+it)})(-s_3 + it)^{-4}\}.$$

Estimating in an elementary way we get (more details in [20, §3.1])

$$\widetilde{A}_1(s_1,s_3,t) \le \frac{16\gamma^5}{15}\cdot\frac{t^2\max\{0,s_{32}k_2-s_{11}k_1\}}{(s_{32}^2+t^2)(s_{11}^2+t^2)} + \frac{16\gamma^5}{15}\cdot\frac{s_{11}s_{32}\max\{0,s_{11}k_2-s_{32}k_1\}}{(s_{32}^2+t^2)(s_{11}^2+t^2)} =: A_1(t), \tag{2.10}$$

$$\widetilde{A}_2(s_1,s_3,t) \le \frac{8\gamma^3k_2s_{32}t^2}{(s_{31}^2+t^2)^3} =: A_2(t), \tag{2.11}$$

$$|\widetilde{A}_3(s_1,s_3,t)| \le 4\gamma^2k_1\frac{1+e^{2\gamma s_{12}}}{(s_{12}^2+t^2)^2} + 4\gamma^2k_2\frac{1+e^{2\gamma s_{32}}}{(s_{32}^2+t^2)^2} =: A_3(t), \tag{2.12}$$

$$|\widetilde{A}_4(s_1,s_3,t)| \le 4k_1\frac{1+e^{2\gamma s_{12}}+2\gamma\sqrt{s_{12}^2+t^2}\,e^{2\gamma s_{12}}}{t^6} + 4k_2\frac{1+e^{2\gamma s_{32}}+2\gamma\sqrt{s_{32}^2+t^2}\,e^{2\gamma s_{32}}}{t^6} =: A_4(t). \tag{2.13}$$

The functions $A_i(t)$ $(i\in\{1,2,3,4\})$ are decreasing in t since they are sums and products of non-negative decreasing functions.

2.3.2. *Estimates for* $t\in[0,x_1]$. Let Δ_1, Δ_2, Δ_t and x_1 be some arbitrary positive constants. Define a grid

$$G\subseteq M := [s_{11},s_{12}]\times[s_{21},s_{22}]\times[0,x_1]$$

by

$$G := \Big\{(s_1,s_2,t)\in\mathbb{R}^3 \;\Big|\; s_1=\min\{s_{11}+j_1\Delta_1,s_{12}\},\ j_1=0,\dots,\Big[\frac{s_{12}-s_{11}}{\Delta_1}\Big]+1,\ s_2=\min\{s_{21}+j_2\Delta_2,s_{22}\},\ j_2=0,\dots,\Big[\frac{s_{22}-s_{21}}{\Delta_2}\Big]+1,\ t=\min\{j_3\Delta_t,x_1\},\ j_3=0,\dots,\Big[\frac{x_1}{\Delta_t}\Big]+1\Big\} \tag{2.14}$$

and set

$$M_0 := \max_{(s_1,s_2,t)\in G} A(s_1,s_2,t). \tag{2.15}$$

If $s_{i1}=s_{i2}$ for an $i\in\{1,2\}$ then we also allow $\Delta_i=0$, in which case we replace the term $([(s_{i2}-s_{i1})/\Delta_i]+1)$ in the definition of G with 0.

Furthermore, for $(s_1,s_2,t)\in M$ we have

$$\left|\frac{dA(s_1,s_2,t)}{ds_1}\right| \le d\int_0^{2\gamma} xf(x)e^{s_{12}x}\,dx =: D_1, \tag{2.16}$$

$$\left|\frac{dA(s_1,s_2,t)}{ds_2}\right| \le k_2\int_0^{2\gamma} xf(x)e^{s_{32}x}\,dx =: D_2, \tag{2.17}$$

$$\left|\frac{dA(s_1,s_2,t)}{dt}\right| \le d\int_0^{2\gamma} xf(x)e^{s_{12}x}\,dx + k_3\int_0^{2\gamma} xf(x)\,dx =: D_3 \tag{2.18}$$

with

$$d := \sup_{x\in[0,2\gamma]} |k_1 - k_2 e^{-s_2 x}| = \max\{k_2-k_1, k_1-k_2e^{-2s_{22}\gamma}\}. \tag{2.19}$$

Putting everything together and using the mean value theorem of differential calculus in the case $t\in[0,x_1]$ we get

LEMMA 2.2. *Let s_{11}, s_{12}, s_{21}, s_{22} and k_i ($i\in\{1,2,3\}$) be non-negative constants and γ, Δ_1, Δ_2, Δ_t and x_1 be positive constants with*

$$0\le s_{11}\le s_{12}\le 4,\qquad 0\le s_{21}\le s_{22}\le s_{12},\qquad x_1\ge 4.$$

If $s_{i1}=s_{i2}$ for an $i\in\{1,2\}$ then $\Delta_i=0$ is allowed as well. Using the definitions (2.7)–(2.8) *and* (2.10)–(2.19) *we have*

$$\begin{aligned} A_{\sup} \le \max\{&A_1(x_1)+A_2(x_1)+A_3(x_1)+A_4(x_1),\\ &M_0+(\Delta_1/2)D_1+(\Delta_2/2)D_2+(\Delta_t/2)D_3\}.\end{aligned}$$

The inequality in Lemma 2.2 gets sharper for greater x_1 and smaller Δ_1, Δ_2 or Δ_t. However, at the same time the number of grid points for evaluation increases. If the number of grid points is kept fixed then one should choose the parameters in such a way that $\Delta_1D_1\approx\Delta_2D_2\approx\Delta_tD_3$ in order to optimize the estimate.

3. Estimates for zeros of Dirichlet L-functions

3.1. Zero-free regions and almost zero-free regions. Let $\lambda\in\{\lambda_1,\lambda_2,\lambda_3,\lambda'\}$ (cf. the notation introduced in §2.2). Our goal in this section is to prove estimates of the form

$$\lambda_1\le C_1 \;\Rightarrow\; \lambda\ge C_2. \tag{3.1}$$

Such estimates are related to zero-free regions (if $\lambda=\lambda_1$) or almost zero-free regions (in the other cases) for the function in (2.6). Note that in this section we will extensively use the improvement suggestion (2) of [8, p. 332].

3.1.1. *Estimates for $\lambda=\lambda'$ and χ_1 or ρ_1 complex.* This section improves on [8, Table 8]. In order to deduce estimates of the form (3.1) we use the following two inequalities together with monotonicity arguments.

LEMMA 3.1. *Let f be the function defined in* (2.3) *and let ε and k be positive constants.*

- *Suppose* $\operatorname{ord}\chi_1\ge 5$ *and $\lambda^\star$ is a positive number with $\lambda^\star\le\min\{\lambda',\lambda_2\}$. In case ρ_2 does not exist just assume $\lambda^\star\le\lambda'$. Then for $q\ge q_0(f,k,\varepsilon)$ we have*

$$(3.2)\qquad \begin{aligned} 0 \le{}& (k^2+1/2)(F(-\lambda^\star) - F(\lambda' - \lambda^\star)) - 2kF(\lambda_1 - \lambda^\star) \\ &+ \frac{f(0)}{6}(k^2+3k+3/2) + \varepsilon \\ &+ \sup_{t\in\mathbb{R}} \Re\{kF(-\lambda^\star + it) - (k^2+3/4)F(\lambda_1 - \lambda^\star + it)\}. \end{aligned}$$

- *Let* $\operatorname{ord}\chi_1 \in \{2,3,4\}$. *If* χ_1 *is real* ([3]) *then assume that* ρ_1 *is complex. For* $q \ge q_0(f,k,\varepsilon)$ *we have*

$$(3.3)\qquad \begin{aligned} 0 \le{}& (k^2+1/2)(F(-\lambda_1) - F(\lambda' - \lambda_1)) - 2kF(0) \\ &+ \frac{f(0)}{8}(k^2+3k+3/2) + \varepsilon + \sup_{t\in\mathbb{R}} \Re\Big\{\frac{1}{2}F(-\lambda_1+it) - 2kF(it)\Big\} \\ &+ 2\cdot \sup_{t\in\mathbb{R}} \Re\{kF(-\lambda_1+it) - (k^2+3/4)F(it)\}. \end{aligned}$$

Proof. The proof is carried out in analogy to the arguments used in [8] for similar inequalities. However, instead of working with [8, Lemma 5.2] we prefer to work with our Lemma 2.1. A complete proof can be found in [20, §3.2.1–3.2.3] which is why we skip the details in what follows.

To prove the first item start with the trigonometric inequality in [8, p. 302, 2nd line]. Set $\beta^\star := 1 - \mathscr{L}^{-1}\lambda^\star$, multiply the inequality with $n^{-\beta^\star}\Lambda(n)f(\mathscr{L}^{-1}\log n)$ and sum over n. The result is the inequality [20, (3.18)]. Applying Lemma 2.1 for each term with $\chi \ne \chi_0$ and [8, Lemma 5.3] for each term with $\chi = \chi_0$ will yield the result. The only thing one has to look closely at is how the set A_1 looks like when Lemma 2.1 is used and how to choose the set A_2:

Now, Lemma 2.1 is used for the terms 2, 4, 5, 6, 7 and 8 (counting successively) of the inequality [20, (3.18)]. For term 2 we choose A_2 in such a way that $A_1 \cup A_2 = \{\rho_1, \rho'\}$. For terms 4 and 6 we get $A_1 \cup A_2 = \{\rho_1\}$ and for the other terms we get $A_1 \cup A_2 = \emptyset$. In each of these cases one has to check which zeros lie in A_1 and that it is indeed possible to choose A_2 in such a way that we get the desired form for $A_1 \cup A_2$. The first item follows by putting everything together and noting (2.9).

For the second item, one has to deduce an inequality for each of the three cases $\operatorname{ord}\chi_1 \in \{2,3,4\}$ in the same way as the inequality for the first item was derived. One uses the same starting inequality [20, (3.18)] but with $\beta^\star$ replaced by β_1. Also, in the cases $\operatorname{ord}\chi_1 \in \{2,3\}$ one has to consider at some terms the zero $\overline{\rho_1}$. Finally, one has to show that from each of the three derived inequalities the inequality (3.3) follows. This is done using the fact that $A_{\sup}$ (see (2.7)) is always non-negative (let $t \to \infty$). ■

([3]) By this we mean that $\operatorname{ord}\chi_1 = 2$.

We now use this lemma to prove estimates for λ'. First note that $\lambda_1 \geq 0.34$ by [8, Lemma 9.5]. Now suppose

$$\lambda_1 \in [0.34, 0.36] =: [\lambda_{11}, \lambda_{12}] \quad \text{and} \quad \operatorname{ord} \chi_1 \geq 5.$$

Because of [8, Table 8] and [8, Table 10, Lemma 9.4] we have $\lambda' \geq 1.309$ and $\lambda_2 \geq 0.903$. Hence, we choose $\lambda^\star = 0.903$.

We need to estimate the supremum on the right side of (3.2), that is,

$$S := \sup_{t\in\mathbb{R}} \Re\{kF(-\lambda^\star + it) - (k^2 + 3/4)F(\lambda_1 - \lambda^\star + it)\}. \tag{3.4}$$

We use Lemma 2.2 with

$$\gamma = 1.13 - \lambda_{12}/5, \quad k = 0.75 + \lambda_{12}/7,$$
$$\Delta_1 = 0, \quad \Delta_2 = 0.004, \quad \Delta_t = 0.004, \quad x_1 = 15,$$

and

$$s_{11} = s_{12} = \lambda^\star, \quad s_{21} = \lambda_{11}, \quad s_{22} = \lambda_{12},$$
$$k_1 = k, \quad k_2 = k^2 + 3/4, \quad k_3 = 0.$$

In fact, the choice of the s_{ij} and k_i is clear from the context, which is why in future applications of Lemma 2.2 we will generally not mention these. Lemma 2.2 gives

$$S \leq 0.0172 =: C.$$

Feeding this into (3.2) and replacing λ_1 by 0.36 and λ' by 2.06 one gets for sufficiently small ε a negative value for the right side of (3.2). Since this right side without the supremum is increasing in λ_1 and λ' we conclude $\lambda' > 2.06$. We do the same for the intervals

$$[0.36, 0.38], \ \ldots, \ [0.80, 0.82], \quad [0.82, 0.827]$$

and summarize the results in Table 2. Note that for $\lambda_1 \geq 0.68$ we choose $\lambda^\star = \lambda_1$ and take

$$\Delta_1 = 0.004, \quad \Delta_2 = 0, \quad \Delta_t = 0.004, \quad x_1 = 15.$$

For the case $\operatorname{ord} \chi_1 \leq 4$ we use intervals with twice the length and the parameters

$$\gamma = 1.21 - 5\lambda_{12}/12, \quad k = 0.77 + \lambda_{12}/10,$$
$$\Delta_1 = 0.004, \quad \Delta_2 = 0, \quad \Delta_t = 0.004, \quad x_1 = 15.$$

Note that in this case we need to estimate two suprema. We denote by C_1 the upper estimate for the first supremum on the right side of (3.3), and by C_2 the upper estimate for the second. It turns out that we get better estimates in this case than in the case $\operatorname{ord} \chi_1 \geq 5$. The results are given in Tables 2 and 3.

Table 2. λ'-estimates
(χ_1 or ρ_1 complex, all cases)

$\lambda_1 \leq$	$\lambda' >$	$\lambda^\star =$	$C \leq$
0.36	2.06	0.903	0.0172
0.38	1.96	0.887	0.0134
0.40	1.86	0.871	0.0102
0.42	1.77	0.856	0.0074
0.44	1.69	0.842	0.0049
0.46	1.61	0.829	0.0032
0.48	1.53	0.816	0.0028
0.50	1.47	0.803	0.0025
0.52	1.40	0.791	0.0021
0.54	1.34	0.780	0.0018
0.56	1.28	0.769	0.0015
0.58	1.23	0.759	0.0012
0.60	1.18	0.749	0.0009
0.62	1.13	0.739	0.0008
0.64	1.09	0.730	0.0008
0.66	1.04	0.714	0.0007
0.68	1.00	0.712	0.0007
0.70	0.96		0.0012
0.72	0.93		0.0011
0.74	0.91		0.0010
0.76	0.89		0.0009
0.78	0.86		0.0008
0.80	0.84		0.0007
0.82	0.83		0.0006
0.827	0.827		0.0005

Table 3. λ'-estimates
(χ_1 or ρ_1 complex, ord $\chi_1 \in \{2,3,4\}$)

$\lambda_1 \leq$	$\lambda' >$	$C_1 \leq$	$C_2 \leq$
0.38	2.53	0.0060	0.0027
0.42	2.35	0.0051	0.0024
0.46	2.20	0.0043	0.0020
0.50	2.06	0.0035	0.0017
0.54	1.94	0.0028	0.0015
0.58	1.84	0.0021	0.0012
0.62	1.75	0.0015	0.0010
0.66	1.67	0.0010	0.0008
0.70	1.59	0.0006	0.0006
0.74	1.52	0.0006	0.0004
0.78	1.46	0.0006	0.0003
0.82	1.40	0.0006	0.0002
0.86	1.35	0.0006	0.0002
0.90	1.30	0.0006	0.0002
0.94	1.25	0.0006	0.0002
0.98	1.21	0.0006	0.0002
1.02	1.17	0.0006	0.0002
1.06	1.13	0.0006	0.0002
1.099	1.099	0.0006	0.0002

The above results in combination with [8] yield a slight improvement of the constant in [8, Theorem 2a] from $c = 0.696$ to $c = 0.702$. For more details on this see [20, pp. 38–39].

3.1.2. *Estimates for $\lambda = \lambda_2$ and χ_1 or ρ_1 complex.* This section improves on [8, Tables 9–11]. In fact, the largest contribution to the improvement from $L = 5.5$ to $L = 5.18$ in Theorem 1.1 is due to better estimates for λ_2 and λ_3; these follow from the next lemma which is a refinement of [8, Lemma 9.2].

Lemma 3.2. *Let χ_1 or ρ_1 be complex, $j \in \{2,3\}$ and $\lambda^\star > 0$ with $\lambda^\star \leq \min\{\lambda', \lambda_2\}$. If ρ' does not exist then only assume $\lambda^\star \leq \lambda_2$. Furthermore, let ε and k be positive constants. Then for $q \geq q_0(\varepsilon, f, k)$ we have*

$$0 \leq (k^2 + 1/2)(F(-\lambda^\star) - F(\lambda_j - \lambda^\star)) - 2kF(\lambda_1 - \lambda^\star) + D + \varepsilon \tag{3.5}$$

with

$$D=\begin{cases} \frac{f(0)}{6}(k^2+4k+\frac{3}{2}), & \chi_1^2,\chi_1^3\neq\chi_0,\chi_j,\overline{\chi_j}, \\ S_1+\frac{f(0)}{6}(k^2+4k+\frac{5}{4}), & \chi_1^2\in\{\chi_j,\overline{\chi_j}\} \text{ and } \operatorname{ord}\chi_1\geq 6, \\ 2S_1+\frac{f(0)}{8}(k^2+4k+1), & \chi_1^2\in\{\chi_j,\overline{\chi_j}\} \text{ and } \operatorname{ord}\chi_1=4, \\ S_1+S_2+\frac{f(0)}{8}(k^2+4k+\frac{5}{4}), & \chi_1^2\in\{\chi_j,\overline{\chi_j}\} \text{ and } \operatorname{ord}\chi_1=5, \\ S_2+\frac{f(0)}{6}(k^2+4k+\frac{3}{2}), & \chi_1^3\in\{\chi_j,\overline{\chi_j}\} \text{ and } \operatorname{ord}\chi_1\geq 7, \\ 2S_2+\frac{f(0)}{8}(k^2+4k+\frac{3}{2}), & \chi_1^3\in\{\chi_j,\overline{\chi_j}\} \text{ and } \operatorname{ord}\chi_1=6, \\ 2S_1+\frac{f(0)}{6}(k^2+\frac{7}{2}k+1), & \operatorname{ord}\chi_1=2, \\ 2S_2+\frac{f(0)}{6}(k^2+\frac{7}{2}k+\frac{11}{8}), & \operatorname{ord}\chi_1=3 \end{cases}$$

and

$$S_1=\sup_{t\in\mathbb{R}}\Re\Big\{\frac{1}{4}F(-\lambda^\star+it)-kF(\lambda_1-\lambda^\star+it)\Big\},$$
$$S_2=\sup_{t\in\mathbb{R}}\Re\Big\{-\frac{1}{4}F(\lambda_1-\lambda^\star+it)\Big\}.$$

In the definition of D all possible cases are considered.

Proof. One can check that the cases appearing in the definition of D do not overlap and do cover all possible cases. Also, by symmetry (or renaming) we can assume whenever we have the condition $\chi_1^k\in\{\chi_j,\overline{\chi_j}\}$ for a $k\in\{2,3\}$ that $\chi_1^k=\chi_j$.

As proposed in suggestion (5) of [8, p. 334] we use the inequality in [8, p. 306, 2nd line] with $\beta^\star=1-\mathscr{L}^{-1}\lambda^\star$ instead of β_1. Now proceed in the same way as in Lemma 3.1, that is: apply Lemma 2.1 for each term with $\chi\neq\chi_0$ and [8, Lemma 5.3] for each term with $\chi=\chi_0$. The lemma then follows by an adequate choice of the set A_2 (each of the eight cases has to be worked out separately). The details are written down in [20, pp. 40–43]. ∎

Using this lemma we want to first deduce estimates for λ_2. For this purpose we choose $j=2$, $\lambda^\star=\lambda_2$ and assume $\lambda_2\leq\lambda'$. If this inequality does not hold then Tables 2 and 3 give better estimates than those we will prove with this assumption.

In analogy to Tables 2 and 3, one could now set up a table for each of the eight cases in Lemma 3.2. However, with negligible costs to the results we can save some work by simultaneously covering the cases 2, 3, 4, 6 and 8. The cases 1, 5 and 7 are discussed separately.

Let us start with Case 1. Then we have $\chi_1^2,\chi_1^3\neq\chi_0,\chi_j,\overline{\chi_j}$. We use the respective inequalities in Lemma 3.2 in order to prove estimates in a similar manner as was done for λ'. However, this time we can only guarantee strict monotony of the right side of the inequality in λ_1 but not in λ_2. To overcome this problem we use the method in [8, p. 307 top]. In particular, suppose that

$$\lambda_1 \le \lambda_{12}, \qquad \lambda_2 \in [\lambda_{2,\mathrm{old}}, \lambda_{2,\mathrm{old}} + \delta] \quad \text{and} \quad 2k \ge k^2 + 1/2$$

for certain concrete values λ_{12}, $\lambda_{2,\mathrm{old}}$, k and $\delta > 0$. Then we deduce from (3.5) that

$$\begin{aligned} -\varepsilon \le{}& (k^2+1/2)(F(-\lambda_{2,\mathrm{old}}-\delta) - F(\lambda_{12}-\lambda_{2,\mathrm{old}}-\delta) - F(0)) \\ &- (2k-(k^2+1/2))F(\lambda_{12}-\lambda_{2,\mathrm{old}}) + \frac{f(0)}{6}(k^2+4k+3/2). \end{aligned} \tag{3.6}$$

Our goal is to find a $\gamma > 0$ for which the right side of (3.6) is negative, thus getting a contradiction for ε sufficiently small. Similarly for the intervals

$$\lambda_2 \in [\lambda_{2,\mathrm{old}} + j\delta, \lambda_{2,\mathrm{old}} + (j+1)\delta] \quad \text{with} \quad j \in \left\{1, 2, \ldots, \left[\frac{\lambda_{2,\mathrm{new}} - \lambda_{2,\mathrm{old}}}{\delta}\right]\right\}.$$

If in addition we know that $\lambda_2 \ge \lambda_{2,\mathrm{old}}$ by [8, Table 10, Lemma 9.4] then we have proven that

$$\lambda_1 \le \lambda_{12} \ \Rightarrow\ \lambda_2 > \lambda_{2,\mathrm{new}}.$$

Using the parameters

$$\gamma = 0.42 + \lambda_{12}, \qquad k = 0.59 + 2\lambda_{12}/5, \qquad \delta = 0.0001 \tag{3.7}$$

we indeed get Table 4 below.

Before we write this table down we want to incorporate the cases 2, 3, 4, 6 and 8 into it. For this purpose we choose the parameters as in (3.7). In order to prove estimates $S_1 \le C_1$ and $S_2 \le C_2$ we use Lemma 2.2 with

$$\begin{aligned} &s_1 = \lambda_2, && s_{11} = \lambda_{2,\mathrm{old}}, && s_{12} = \lambda_{2,\mathrm{new}}, \\ &s_2 = \lambda_1, && s_{21} = \lambda_{11}, && s_{22} = \lambda_{12}, \\ &\Delta_1 = 0.015, && \Delta_2 = 0.007, && \Delta_t = 0.015, \qquad x_1 = 7. \end{aligned}$$

Using the estimates $S_1 \le C_1$ (one for each interval $\lambda_1 \in [\lambda_{11}, \lambda_{12}]$) one checks that the bounds which were proven for Case 1 also hold for Case 2. Now for Case 3 it remains to be shown that

$$\begin{aligned} D(\text{Case 3}) &:= 2C_1 + \frac{f(0)}{8}(k^2+4k+1) \\ &\le C_1 + \frac{f(0)}{6}\left(k^2+4k+\frac{5}{4}\right) =: D(\text{Case 2}). \end{aligned}$$

If the last inequality holds, then by the inequality which according to Lemma 3.2 holds for Case 3 one gets the inequality which was used in order to prove the estimates for Case 2. Everything proven on the basis of this last inequality is then also true for Case 3.

Cases 4, 6 and 8 are proven in the same way as Case 3. Altogether we obtain Table 4.

Table 4. λ_2-estimates (χ_1 or ρ_1 complex, Cases 1–4, 6 and 8)

$\lambda_1 \le$	$\lambda_2 >$	$\lambda_{2,\text{old}} =$	$C_1 \le$	$C_2 \le$
0.36	1.69	0.903	0.0223	0.0152
0.38	1.69	0.887	0.0263	0.0181
0.40	1.69	0.871	0.0310	0.0214
0.42	1.69	0.856	0.0362	0.0252
0.44	1.67	0.842	0.0408	0.0287
0.46	1.59	0.829	0.0414	0.0297
0.48	1.52	0.816	0.0420	0.0307
0.50	1.45	0.803	0.0420	0.0315
0.52	1.39	0.791	0.0423	0.0324
0.54	1.31	0.780	0.0401	0.0317
0.56	1.23	0.769	0.0373	0.0305
0.58	1.13	0.759	0.0320	0.0274
0.60	1.04	0.749	0.0271	0.0245
0.62	0.96	0.739	0.0226	0.0216
0.64	0.88	0.730	0.0176	0.0182
0.66	0.82	0.721	0.0144	0.0156
0.68	0.76	0.712	0.0139	0.0126

Table 5. λ_2-estimates (χ_1 or ρ_1 complex, Case 5)

$\lambda_1 \le$	$\lambda_2 >$	$\lambda_{2,\text{old}} =$	$C_2 \le$
0.36	1.69	0.903	0.0664
0.38	1.69	0.887	0.0702
0.40	1.69	0.871	0.0742
0.42	1.69	0.856	0.0783
0.44	1.67	0.842	0.0799
0.46	1.56	0.829	0.0700
0.48	1.45	0.816	0.0606
0.50	1.36	0.803	0.0535
0.52	1.27	0.791	0.0465
0.54	1.19	0.780	0.0406
0.56	1.11	0.769	0.0348
0.58	1.04	0.759	0.0299
0.60	0.97	0.749	0.0249
0.62	0.91	0.739	0.0208
0.64	0.85	0.730	0.0167
0.66	0.79	0.721	0.0126
0.68	0.74	0.712	0.0092

Cases 5 and 7 are treated in complete analogy to Case 2. However, for Case 5 (Table 5) we choose the parameters

$$\gamma = 0.76 + \frac{\lambda_{12}}{2}, \qquad k = 0.84, \qquad \delta = 0.0001,$$
$$\Delta_1 = 0.01, \qquad \Delta_2 = 0.007, \quad \Delta_t = 0.01, \quad x_1 = 7$$

and for Case 7 (Table 6) the parameters

$$\gamma = 0.61 + \frac{\lambda_{12}}{2}, \qquad k = 0.81, \qquad \delta = 0.0001,$$
$$\Delta_1 = \Delta_2 = \Delta_t = 0.015, \qquad x_1 = 7.$$

Note that for $\lambda_{11} \ge 0.70$ we have no values for $\lambda_{2,\text{old}}$ from [8, Table 10] which is why we then choose $\lambda_{2,\text{old}} = \lambda_{11}$. Also, in Case 7 we have $\lambda_1 \ge 0.50$ by [8, Lemma 9.5].

The minimum of the entries in Tables 4, 5 and 6 is summarized in Table 7 which is valid whenever χ_1 or ρ_1 is complex.

Table 6. λ_2-estimates
(χ_1 real and ρ_1 complex, Case 7)

$\lambda_1 \le$	$\lambda_2 >$	$\lambda_{2,\text{old}} =$	$C_1 \le$
0.54	1.43	0.780	0.0301
0.58	1.36	0.759	0.0276
0.62	1.28	0.739	0.0242
0.66	1.20	0.721	0.0206
0.70	1.11	0.704	0.0167
0.74	1.02	0.70	0.0128
0.78	0.93	0.74	0.0090
0.82	0.82	0.78	0.0070

Table 7. λ_2-estimates
(χ_1 or ρ_1 complex, all cases)

$\lambda_1 \le$	(new) $\lambda_2 >$	(old) $\lambda_2 >$
0.36	1.69	0.903
0.38	1.69	0.887
0.40	1.69	0.871
0.42	1.69	0.856
0.44	1.67	0.842
0.46	1.56	0.829
0.48	1.45	0.816
0.50	1.36	0.803
0.52	1.27	0.791
0.54	1.19	0.780
0.56	1.11	0.769
0.58	1.04	0.759
0.60	0.97	0.749
0.62	0.91	0.739
0.64	0.85	0.730
0.66	0.79	0.721
0.68	0.74	0.712
0.70	–	0.704
0.702	–	0.702

3.1.3. *Estimates for $\lambda = \lambda_3$ if $\lambda_1 \in [0.52, 0.62]$ and χ_1 or ρ_1 complex.* Suppose χ_1 or ρ_1 is complex. This section improves on parts of [8, Table 9]. We want to deduce lower bounds for λ_3 using Lemma 3.2 with $j=3$. Let us go over the eight cases in this lemma and therefore assume for the moment that

$$\lambda_1 \in [0.54, 0.56].$$

If we are in Case 1, that is, $\chi_1^2, \chi_1^3 \ne \chi_0, \chi_3, \overline{\chi_3}$, then we take $\lambda^\star$ to be the minimum of the λ'- and λ_2-estimate from Tables 2 and 7, that is, $\lambda^\star = \min\{1.28, 1.11\} = 1.11$, and use (3.5) to get estimates for λ_3. This time, we do not need to work with a $\delta > 0$ since the right side of (3.5) without D is monotone in λ_3 and λ_1. We use the parameters γ and k as in (3.7). The result is

$$\lambda_3 > 1.160.$$

For Case 2 we again choose the parameters as in (3.7). Also, we use the estimates of S_1 and S_2 as calculated for Table 4. We get

$$\lambda_3 \ge 1.167.$$

This last estimate is valid for Cases 3 and 4 as well according to the reasoning and calculations which led to Table 4.

If we are in Case 5 or 6 then $\chi_1^3 \in \{\chi_3, \overline{\chi_3}\}$, in which case we have $\chi_1^3 \notin \{\chi_2, \overline{\chi_2}\}$. Thus, we are in the situation of Table 4 or 6 and conclude

$$\lambda_3 \geq \lambda_2 \geq \min\{1.36, 1.23\} = 1.23.$$

Apparently, this last estimate applies in Cases 7 and 8 as well.

Altogether we have

$$\lambda_3 \geq \min\{1.160, 1.167, 1.23\} = 1.160.$$

We do the same for the other λ_1-ranges and get the following table.

Table 8. λ_3-estimates
(χ_1 or ρ_1 complex)

$\lambda_1 \leq$	(all cases) $\lambda_3 >$	(Case 1) $\lambda_3 >$	(Case 2, 3, 4) $\lambda_3 >$	(Case 5, 6, 7, 8) $\lambda_3 >$	$\lambda^\star =$
0.52	1.320	1.352	1.320	1.39	1.27
0.54	1.243	1.253	1.243	1.31	1.19
0.56	1.160	1.160	1.167	1.23	1.11
0.58	1.079	1.079	1.103	1.13	1.04
0.60	1.001	1.001	1.038	1.04	0.97
0.62	0.933	0.933	0.979	0.96	0.91

3.1.4. *Estimates for $\lambda = \lambda_3$ if $\lambda_1 \in [0.62, 0.72]$ or χ_1 and ρ_1 both real.* This section refines [8, Lemma 10.3].

LEMMA 3.3. *There is an effectively computable constant q_0 such that for $q \geq q_0$ we have for χ_1 complex, respectively χ_1 and ρ_1 both real, the following Table* 9 *respectively Table* 10:

Table 9. λ_3-estimates
(χ_1 complex)

$\lambda_1 \in$	Additional condition	$\lambda_3 >$
$[0.62, 0.64]$	–	0.902
$[0.64, 0.66]$	–	0.898
$[0.64, 0.66]$	$\lambda_2 \leq 0.86$	0.938
$[0.66, 0.68]$	–	0.893
$[0.66, 0.68]$	$\lambda_2 \leq 0.83$	0.960
$[0.68, 0.72]$	–	0.883
$[0.68, 0.72]$	$\lambda_2 \leq 0.81$	0.962

Table 10. λ_3-estimates
(χ_1 and ρ_1 real)

$\lambda_1 \in$	$\lambda_3 >$
$[0.44, 0.60]$	1.175
$[0.60, 0.68]$	1.078
$[0.68, 0.78]$	0.971

Explanation of the tables: The first line in Table 9 means

$$\lambda_1 \in [0.62, 0.64] \;\Rightarrow\; \lambda_3 > 0.902$$

while the third line means

$$(\lambda_1 \in [0.64, 0.66] \text{ and } \lambda_2 \leq 0.86) \;\Rightarrow\; \lambda_3 > 0.938.$$

Similarly for the rest.

Proof. The proof is carried out in analogy to the proof of [8, Lemma 10.3]. It starts with the inequality [8, (10.2)] and then makes use of Lemma 2.1 and [8, Lemma 5.3]. More details can be found in [20, pp. 49–52]. We start with the proof of Table 9.

In this case χ_1 is complex. First suppose that none of the characters involved in Σ_3 (see [8, (10.3)]) equals χ_0. The inequality [8, (10.6)] follows:

$$0 \le F(-\lambda_1) - F(\lambda_3 - \lambda_1) - F(\lambda_2 - \lambda_1) - F(0) + \frac{7}{6} f(0) + \varepsilon. \tag{3.8}$$

We want to prove that (3.8) is always valid if χ_1 is complex and $\lambda_1 \in [0.62, 0.72]$. Therefore, we need to analyze how the inequality (3.8) is altered if one or more of the characters involved equal χ_0. By a straightforward analysis it follows that we only need to verify that

$$\sup_{t\in\mathbb{R}} \Re\{F(-\lambda_1 + it) - 2F(it)\} \le \frac{1}{6} f(0) \tag{3.9}$$

in order to prove (3.8). Putting $s_1 = \lambda_1 \in [0.62, 0.72]$ and taking

$$\gamma = 1.25, \quad \Delta_1 = 0.03, \quad \Delta_2 = 0, \quad \Delta_t = 0.03, \quad x_1 = 6$$

we confirm in the usual way that (3.9) holds and thus (3.8) is valid whenever χ_1 is complex.

We now proceed to prove the values in Table 9 using $\gamma = 1.25$. The right side of (3.8) is strictly increasing in λ_2 and λ_3. Also, $F(-\lambda_1) - F(\lambda_3 - \lambda_1)$ is strictly increasing in λ_1. Hence, if

$$\lambda_1 \in [\lambda_{11}, \lambda_{12}], \quad \lambda_2 \le \lambda_{22} \quad \text{and} \quad \lambda_3 \le \lambda_{32}$$

then

$$-\varepsilon \le F(-\lambda_{12}) - F(\lambda_{32} - \lambda_{12}) - F(\lambda_{22} - \lambda_{11}) - F(0) + \frac{7}{6} f(0). \tag{3.10}$$

To prove the statement

$$\lambda_1 \in [0.62, 0.64] \ \Rightarrow\ \lambda_3 > 0.902$$

we calculate the right side of (3.10) for $\lambda_{22} = \lambda_{32} = 0.902$ and the pairs

$$(\lambda_{11}, \lambda_{12}) = (0.62 + j\delta, 0.62 + (j+1)\delta)$$

where

$$\delta = 0.0001 \quad \text{and} \quad j = 0, \ldots, \left\lceil \frac{0.64 - 0.62}{\delta} \right\rceil.$$

A calculation shows that for each j we get something negative, thus proving the statement. In the same way and with the same parameters we prove all values of Table 9. The additional condition $\lambda_2 \le c$ is easily incorporated by putting $\lambda_{22} = c$.

For the proof of Table 10 suppose χ_1 and ρ_1 are both real. We start with the inequality [8, (10.2)] in which we replace β_1 by β_2. This time all terms

$K(\beta_2+it,\chi)$ in Σ_2 and Σ_3 with $\chi \in \{\chi_0,\chi_1\}$ need extra treatment. We can assume that $\lambda_2 \le 1.294$. Then by [8, Lemma 8.4] we have $\lambda' \ge \lambda_2$, which will be used in connection with the sets A_1 and A_2 of Lemma 2.1.

- *Case A:* Suppose no character in Σ_3 is equal to χ_0. Then

$$-\varepsilon \le F(-\lambda_2) - F(\lambda_3-\lambda_2) - F(0) - F(\lambda_1-\lambda_2) + \frac{9}{8}f(0). \tag{3.11}$$

- *Case B:* Suppose that at least one character in Σ_3 is equal to χ_0. Then a straightforward analysis yields the inequality (3.11) with the additional term

$$+\sup_{t\in\mathbb{R}} \Re\{F(-\lambda_2+it) - F(\lambda_1-\lambda_2+it) - F(it)\} - \frac{5}{48}f(0) \tag{3.12}$$

  on the right side. Using $\gamma = 1.04$ and

$$\begin{aligned} s_1 &= \lambda_2, & s_{11} &= 0.44, & s_{12} &= 1.175, \\ s_2 &= \lambda_1, & s_{21} &= 0.44, & s_{22} &= 0.80, \\ \Delta_1 &= 0.03, & \Delta_2 &= 0.03, & \Delta_t &= 0.03, & x_1 &= 6 \end{aligned}$$

  it follows that (3.12) is negative. Thus (3.11) is always valid if χ_1 and ρ_1 are both real.

The rest is proven in analogy to the proof of Table 9: We first have

$$-\varepsilon \le F(-\lambda_{22}) - F(\lambda_{32}-\lambda_{22}) - F(0) - F(\lambda_{12}-\lambda_{21}) + \frac{9}{8}f(0). \tag{3.13}$$

To prove the first line in Table 10 we take $\lambda_{11} = 0.44$, $\lambda_{12} = 0.60$, $\lambda_{32} = 1.175$ and

$$(\lambda_{21},\lambda_{22}) = (0.44 + j\delta, 0.44 + (j+1)\delta)$$

with $j \in \{0,1,\dots,[\delta^{-1}(\lambda_{32}-\lambda_{11})]\}$ and $\delta = 0.001$. For all j one gets something negative for the right side of (3.13) and hence the statement

$$\lambda_3 > \lambda_{32} = 1.175.$$

Proceed similarly for the other entries in Table 10. ■

3.1.5. *Estimates for $\lambda = \lambda_1$, proof of Theorem 1.2.* Again we assume that χ_1 or ρ_1 is complex. In this section we improve [8, Lemma 9.5]. This is done on the one hand by using the improved estimates for λ' and λ_2 from the previous sections and on the other hand by incorporating suggestion (2) of [8, p. 332] for the cases in which $\operatorname{ord}\chi_1 \le 5$. We start with the inequality [8, (9.16)] and choose $\beta = \beta^\star = 1 - \mathscr{L}^{-1}\lambda^\star$ with a

$$\lambda^\star \le \min\{\lambda_2, \lambda'\}.$$

Using the standard method we get

$$0 \le 14379F(-\lambda^\star) - 24480F(\lambda_1-\lambda^\star) + D + \varepsilon$$

with

$$D = \begin{cases} \frac{46630}{6} f(0), & \operatorname{ord}\chi_1 \geq 6, \\ \frac{46630}{8} f(0) + \sup\limits_{t\in\mathbb{R}} \Re\{-1250F(\lambda_1 - \lambda^\star + it)\}, & \operatorname{ord}\chi_1 = 5, \\ \frac{45380}{8} f(0) + \sup\limits_{t\in\mathbb{R}} \Re\{1250F(-\lambda^\star + it) - 6000F(\lambda_1 - \lambda^\star + it)\}, & \operatorname{ord}\chi_1 = 4, \\ \frac{40630}{8} f(0) + \sup\limits_{t\in\mathbb{R}} \Re\{6000F(-\lambda^\star + it) - 16150F(\lambda_1 - \lambda^\star + it)\}, & \operatorname{ord}\chi_1 = 3, \\ \frac{30480}{8} f(0) + \sup\limits_{t\in\mathbb{R}} \Re\{1250F(-\lambda^\star + it) - 6000F(\lambda_1 - \lambda^\star + it)\} & \\ \quad + \sup\limits_{t\in\mathbb{R}} \Re\{14900F(-\lambda^\star + it) - 30480F(\lambda_1 - \lambda^\star + it)\}, & \operatorname{ord}\chi_1 = 2. \end{cases}$$

The different suprema are estimated by choosing

$$s_{11} = s_{12} = \lambda^\star, \qquad s_{21} = \lambda_{1,\mathrm{old}}, \qquad s_{22} = \lambda_{1,\mathrm{assu}}$$

and

$$\Delta_1 = 0, \qquad \Delta_2 = 0.005, \qquad \Delta_t = 0.005, \qquad x_1 = 12.$$

Here, $\lambda_{1,\mathrm{old}}$ is the old lower bound for λ_1 from [8, Lemma 9.5] and $\lambda_{1,\mathrm{assu}}$ is some specific value which in the end is going to be slightly above the proven lower bound. We assume that

$$\lambda_1 \leq \lambda_{1,\mathrm{assu}}$$

and choose a corresponding $\lambda^\star$ which we get from Tables 2 and 7 (resp. Tables 3 and 6 if $\operatorname{ord}\chi_1 = 2$). The results are summarized in Table 11. There, the value C is the calculated upper estimate for the corresponding supremum, respectively the sum of the two suprema in the case $\operatorname{ord}\chi_1 = 2$.

Table 11. λ_1-estimates
(χ_1 or ρ_1 complex)

$\operatorname{ord}\chi_1$	$\lambda_1 > \lambda_{1,\mathrm{new}} =$	$\lambda_{1,\mathrm{old}} =$	$\lambda_{1,\mathrm{assu}} =$	$\lambda^\star =$	$\gamma =$	$C \leq$
≥ 6	0.440	0.364	0.44	1.67	1.00	–
$= 5$	0.493	0.397	0.50	1.36	0.90	120
$= 4$	0.478	0.348	0.48	1.45	0.82	235
$= 3$	0.498	0.389	0.50	1.36	0.82	290
$= 2$	0.628	0.518	0.66	1.20	0.70	58

As a consequence Theorem 1.2 follows for the case of χ_1 or ρ_1 complex. If χ_1 and ρ_1 are both real then the theorem follows from [8, Lemma 8.4 and Lemma 8.8].

3.2. (Weighted) zero-density estimates. Let $q \in \mathbb{N}$ and $\lambda > 0$. As in [8, p. 316] we define

$$N(\lambda) := \#\{\chi \pmod q \mid \chi \neq \chi_0, \\ L(s,\chi) \text{ has a zero in } \sigma \geq 1 - \mathscr{L}^{-1}\lambda,\ |t| \leq 1\}$$

and denote by $\chi^{(1)}, \ldots, \chi^{(N(\lambda))}$ the corresponding characters. To each of the $N(\lambda)$ characters χ we choose a corresponding zero $\rho(\chi) = \rho^{(k)}$ with maximal real part, that is,

$$\Re\{\rho^{(k)}\} = \max\{\Re\{\rho\} \mid \rho \in \mathbb{C},\ \Re\{\rho\} \geq 1 - \mathscr{L}^{-1}\lambda,\ |\Im\{\rho\}| \leq 1,\ L(\rho, \chi^{(k)}) = 0\}$$

and set

$$\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}, \qquad \beta^{(k)} = 1 - \mathscr{L}^{-1}\lambda^{(k)}.$$

3.2.1. *For large* λ. This section slightly improves [8, Lemma 11.1] as suggested in suggestion (7) of [8, pp. 336–337]. There, Heath-Brown describes a variation of the proof of [8, Lemma 11.1] by incorporating a weight function $w(t)$ into the argument. This leads to the following generalization of [8, Lemma 11.1]. By choosing $w_1(t) \equiv 1$ one recovers that lemma.

LEMMA 3.4. *Let* δ, c_1, c_2 *be positive constants,* $\lambda_0 = (1/3) \cdot \log\log \mathscr{L}$ *and*

$$u = 1/3 + 2c_1, \qquad v = 1/3 + 2c_1 + c_2, \qquad x = 2/3 + 3c_1 + c_2.$$

Furthermore let $w_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *be a continuous function with continuous first derivative on* $[0, v)$ *and* (v, ∞) *and suppose*

$$1 \ll w_1(t) \ll 1 \quad \textit{and} \quad w_1'(t) \ll 1$$

with absolute implicit constants. Then for $q \geq q_0(\delta, c_1, c_2, w_1)$ *we have*

$$\sum_{1 \leq k \leq N(\lambda_0)} \Big(\int_u^x w_1(t)^2 e^{2\lambda^{(k)} t}\, dt\Big)^{-1} \leq \frac{1+\delta}{c_1 c_2^2} \int_u^x w_1(t)^{-2} \min\{t-u, v-u\}\, dt. \tag{3.14}$$

Proof. This is proved in analogy to the proof of [8, Lemma 11.1]. The only difference lies in the definition of the parameters $a_{n\chi}$ and b_n; see [8, (11.11), (11.12)]. We introduce into these parameters a weight function $w_1(t)$ as suggested in [8, p. 335]. After this alteration, it is necessary to adjust the arguments which follow [8, (11.11), (11.12)]. This mainly involves proving the following three estimates for all $\varepsilon > 0$ and $q \geq q_0(\varepsilon)$

$$\sum_{n=1}^{\infty} a_{n\chi}\overline{a_{n\chi'}} \ll w_\chi^{1/2} w_{\chi'}^{1/2} \mathscr{L}^{-1} \qquad (\chi \neq \chi'), \tag{3.15}$$

$$\sum_{n=1}^{\infty} |a_{n\chi}|^2 \leq w_\chi \frac{1+\varepsilon}{c_1} \int_u^x w_1(t)^2 e^{2(1-\Re\{\rho(\chi)\})\mathscr{L}t}\, dt, \tag{3.16}$$

(3.17) $$\sum_{n=1}^{\infty} |b_n|^2 \le \frac{1+\varepsilon}{(v-u)^2} \int_u^x w_1(t)^{-2} \min\{t-u, v-u\}\, dt$$

and then choosing

$$w_\chi = \Big(c_1^{-1} \int_u^x w_1(t)^2 e^{2(1-\Re\{\rho(\chi)\})\mathscr{L}t}\, dt\Big)^{-1}.$$

The estimates (3.15)–(3.17) are proven in the same manner as the respective estimates in the proof of [8, Lemma 11.1]. However, to do it precisely we saw no other way than to carry out some straightforward but lengthy calculations which are written down in [20, §3.6] and which therefore do not need to be repeated here. Having these three estimates and the above choice of w_χ one gets the lemma in the same way as [8, Lemma 11.1] is deduced. ∎

3.2.2. *For small λ.* This section improves the results in [8, §12] concerning upper bounds for $N(\lambda)$ by using the remarks in suggestion (9) of [8, pp. 336–337]. The following lemma generalizes [8, Lemma 12.1]. Put $N_0 = 0$ to recover that lemma.

LEMMA 3.5. *Suppose ε, γ, λ and λ_{11} are positive constants with*

$$\lambda_{11} \le \lambda \le 2.$$

Assume that $\lambda_1 \ge \lambda_{11}$ and that for our choice of γ we have

$$F(\lambda - \lambda_{11}) > f(0)/6.$$

Also let $\lambda_0 \ge 0$ and $N_0 \in \mathbb{N}_0$ be constants with $\lambda_0 \le \lambda$, $N_0 \le 10000$ and assume that $N(\lambda_0) \ge N_0$. Then for $q \ge q_0(\varepsilon, f)$ we have

(3.18) $$\begin{aligned}((N(\lambda) - N_0)F(\lambda-\lambda_{11}) + N_0F(\lambda_0-\lambda_{11}) - N(\lambda)f(0)/6)^2 \\ \le N(\lambda)F(-\lambda_{11})(F(-\lambda_{11}) + (N(\lambda)-1)f(0)/6) + \varepsilon.\end{aligned}$$

Proof. This result is mentioned for the case $N(1) \ge 5$ without proof (and with a misprint) in [8, p. 336]. Indeed, it follows easily by incorporating the assumption $N(\lambda_0) \ge N_0$ into the proof of [8, Lemma 12.1]. The details are written down in [20, pp. 64–65]. ∎

We are interested in upper estimates for $N(\lambda)$. Suppose we have chosen some parameters λ, λ_{11}, λ_0 and N_0. In addition we choose $\varepsilon = 10^{-7}$ and

$$\gamma = 0.975 + 0.525\lambda - 0.550\lambda_{11} - 0.014N_0 \cdot (\lambda - \lambda_0).$$

The latter choice for γ turns out to give nearly optimal results. Now, by (3.18) we get

(3.19) $$h(N) \ge 0$$

where $h(N)$ is a quadratic polynomial in N. Its leading coefficient is negative if

$$(F(\lambda - \lambda_{11}) - f(0)/6)^2 > F(-\lambda_{11})f(0)/6.$$

If in addition h has two real zeros, say $h_1 \le h_2$, then

$$N(\lambda) \le [h_2].$$

This reasoning will be used in §4.2 to get concrete upper bounds for $N(\lambda)$.

4. Proof of Theorem 1.1

4.1. A variation of [8, Lemma 15.1]. Theorem 1.1 is basically proven using [8, Lemma 15.1]. However, for clarity as well as in order to incorporate Lemma 3.4 we will introduce a slight variation of [8, Lemma 15.1]. We use the notation of [8, §13, §15] which we will give here in full detail. We want to specify at once all objects that are necessary for the formulation of the lemma. Therefore, let L, K, θ, R and Λ be some positive constants. Note that in what follows the functions f and F will be different from those defined in (2.3) and (2.5). Define

$$f(x) = \begin{cases} 0, & x \le L-2K, \\ x-(L-2K), & L-2K \le x \le L-K, \\ L-x & L-K \le x \le L, \\ 0, & x \ge L \end{cases} \qquad \text{([8, p. 324])},$$

$$F(z) = \int_0^\infty e^{-zx} f(x)\,dx = e^{-(L-2K)z}\left(\frac{1-e^{-Kz}}{z}\right)^2 \qquad \text{([8, p. 324])},$$

$$F_2(z) = \left(\frac{1-e^{-Kz}}{z}\right)^2 \qquad \text{([8, p. 324])}, \tag{4.1}$$

$$\Sigma = \sum_{p\equiv a\,(\mathrm{mod}\,q)} \frac{\log p}{p} f(\mathscr{L}^{-1}\log p) \qquad \text{([8, p. 324])},$$

$$\widetilde{R} = \{\sigma + it \in \mathbb{C} \mid 1-\mathscr{L}^{-1}R \le \sigma \le 1,\ |t| \le \mathscr{L}^{-1}R\},$$

$$w_1(t) = e^{-\theta t/2}\min\{t-u+10^{-7}, v-u+10^{-7}\}^{1/4} \qquad (t\in[u,\infty)), \tag{4.2}$$

$$w(s) = \Big(\int_u^x w_1(t)^2 e^{2st}\,dt\Big)^{-1}, \tag{4.3}$$

$$B(\lambda) = \frac{1-e^{-2K\lambda}}{6K^2\lambda} + \frac{2K\lambda - 1 + e^{-2K\lambda}}{2K^2\lambda^2} \qquad \text{([8, p. 328])}, \tag{4.4}$$

$$A(\chi_1) = \begin{cases} (e^{-(L-2K)\lambda_1} - e^{-(L-2K)\lambda'})(B(\lambda_1) - \alpha(\chi_1)K^{-2}F_2(\lambda_1)) & \text{if } \rho_1 \in \widetilde{R}, \\ 0 & \text{else}, \end{cases}$$

$$\alpha(\chi_1) = \begin{cases} 2 & \text{if } \chi_1 \text{ is real and } \rho_1 \text{ complex}, \\ 1 & \text{else} \end{cases} \qquad \text{([8, p. 329])}, \tag{4.5}$$

$$n(\chi_1) = \begin{cases} 2 & \text{if } \chi_1 \text{ is complex}, \\ 1 & \text{if } \chi_1 \text{ is real} \end{cases} \qquad \text{([8, p. 329])}, \tag{4.6}$$

$$(4.7)\quad C(\lambda) = w(\lambda)\left(e^{-(L-2K)\lambda}\frac{B(\lambda)}{w(\lambda)} - e^{-(L-2K)\Lambda}\frac{B(\Lambda)}{w(\Lambda)}\right) \quad ([8, \text{p. } 329]),$$

$$\lambda_3^\star = \min(\Lambda, \lambda_3),$$

$$(4.8)\quad \Lambda_r = \Lambda - 0.001r,$$

$$s = [1000(\Lambda - \lambda_3^\star)],$$

$$\lambda_1^\star = \begin{cases} \lambda_1, & \rho_1 \in \widetilde{R}, \\ \lambda', & \text{else}, \end{cases}$$

$$T' = \max\{0, n(\chi_1)(C(\lambda_1^\star) - A(\chi_1))\} + (2 - n(\chi_1))C(\lambda_3^\star).$$

We use the convention that if ρ' does not exist then we set $\lambda' = \infty$ and $C(\infty) = 0$. Similarly we put $w(\infty) = 0$. Also, we will need estimates

$$(4.9)\qquad N(\Lambda_j) \le N_0(\Lambda_j)$$

with concrete values $N_0(\Lambda_j)$ for $j = 0, \ldots, s$. These will be derived using §3.2.2.

Now, by [8, Lemma 13.2] we have the following inequality for a constant $R = R(\varepsilon)$ and $L > 3 + 2K$:

$$(4.10)\qquad \frac{K^{-2}\varphi(q)}{\mathscr{L}}\Sigma \ge 1 - K^{-2}\sum_{\chi\ne\chi_0}\sum_{\rho\in\widetilde{R}}|F((1-\rho)\mathscr{L})| - \varepsilon.$$

This inequality forms the basis of the proof of Theorem 1.1. The goal is to estimate the right side of (4.10) from below in such a way that one is able to feed in the estimates that have been proven so far for the zeros $\rho \in \widetilde{R}$. Hopefully, we will then get $\Sigma > 0$ and will thus have proved Linnik's theorem for some constant L.

At some points we will implicitly assume the existence of certain zeros ρ. If those did not exist then it will be apparent that one would get even better estimates. For instance, if ρ_1 did not exist then (4.10) immediately yields the admissible value $L = 3 + \delta$ with any $\delta > 0$ for Linnik's theorem.

We incorporate the following three minor variations into the deduction of [8, Lemma 15.1]:

- we explicitly include the cases $\rho_1 \notin \widetilde{R}$ and "ρ' does not exist",
- instead of [8, Lemma 11.1] and the function $w(s)$ in [8, p. 329] we use Lemma 3.4 (with the weight function $w_1(t)$ from (4.2)) and $w(s)$ as in (4.3),
- we use $\Lambda_r = \Lambda - 0.001r$ instead of $\Lambda_r = \Lambda - 0.025r$.

As a result we get (more details can be found in [20, §4.1])

Lemma 4.1 (variation of [8, Lemma 15.1]). *We use the notation introduced in this section and in Lemma* 3.4. *Let $\lambda_1 \ge 0.348$ and L, K, θ, c_1, c_2, Λ and ε be positive constants with $L - 2K > \max\{3, 2x\}$. Then there* *exists an effectively computable constant q_0, depending on all of the chosen constants, such that*

$$K^{-2}\sum_{\chi\neq\chi_0}\sum_{\rho\in\widetilde{R}}|F((1-\rho)\mathscr{L})| \le \frac{e^{-(L-2K)\Lambda}B(\Lambda)}{c_1c_2^2w(\Lambda)}\int_u^x w_1(t)^{-2}\min\{t-u,v-u\}\,dt + \max\{2C(\lambda_2),\,0\}+\max\{(N_0(\Lambda_s)-4)C(\lambda_3^\star),0\} + \sum_{r=0}^{s-1}(N_0(\Lambda_r)-N_0(\Lambda_{r+1}))C(\Lambda_{r+1})+T'+\varepsilon. \tag{4.11}$$

Theorem 1.1 is deduced from (4.10) and (4.11) by showing that the right side of (4.11) is strictly smaller than 1. So suppose we have some constants $0<\lambda_{11}\le\lambda_{12}\le\infty$ and assume that

$$\lambda_1\in[\lambda_{11},\lambda_{12}]. \tag{4.12}$$

Further suppose that by the previous sections and/or [8] we have some explicit estimates

$$\lambda'\ge\lambda'_{11},\qquad \lambda_2\ge\lambda_{21},\qquad \lambda_3\ge\lambda_{31}. \tag{4.13}$$

We choose the parameters

$$L=5.18,\qquad K=0.32,\qquad \theta=1.15,\qquad c_1=0.11,\qquad c_2=0.27, \tag{4.14}$$

$$\Lambda=\max\{\lambda_{31},[1000(1.08+0.35\lambda_{11})]/1000\},\qquad \varepsilon=10^{-7}, \tag{4.15}$$

which turn out to be fairly optimal. We set

$$\lambda_{31}^\star=\min\{\lambda_{31},\Lambda\}\quad\text{and}\quad s=[1000(\Lambda-\lambda_{31}^\star)]. \tag{4.16}$$

By monotonicity (more details in [20, p. 74]) we get the following upper bound W for the right side of (4.11):

$$W=\frac{e^{-(L-2K)\Lambda}B(\Lambda)}{c_1c_2^2w(\Lambda)}\int_u^x w_1(t)^{-2}\min\{t-u,v-u\}\,dt + \max\{2C(\lambda_{21}),0\}+\max\{(N_0(\Lambda_s)-4)C(\lambda_{31}^\star),0\} + \sum_{r=0}^{s-1}(N_0(\Lambda_r)-N_0(\Lambda_{r+1}))C(\Lambda_{r+1}) + (2-n(\chi_1))C(\lambda_{31}^\star)+n(\chi_1)\cdot D+10^{-7} \tag{4.17}$$

with D being the maximum of three quantities:

$$(4.18)\qquad D = \max\Big\{0,\ C(\lambda'_{11}),\ e^{-(L-2K)\lambda'_{11}}\max\{0, B(\lambda_{11}) - \alpha(\chi_1)K^{-2}F_2(\lambda_{11})\} - e^{-(L-2K)\Lambda}B(\Lambda)\frac{w(\lambda_{12})}{w(\Lambda)} + \alpha(\chi_1)K^{-2}F_2(\lambda_{11})e^{-(L-2K)\lambda_{11}}\Big\}.$$

Let us recall what we need: Start with a case given by (4.12), and perhaps some additional assumptions (e.g. χ_1 and ρ_1 both real). Choose the parameters as in (4.14)–(4.15) and use the definitions in (4.1)–(4.8) and (4.16)–(4.18). Collect the bounds of type (4.9) with Lemma 3.5 (use the computer) and the bounds of type (4.13) given by Tables 2–11 and [8]. If $W < 1$ then Theorem 1.1 is proven for the special case which we assumed at the beginning.

4.2. Discussion of three cases

4.2.1. *Case 1: χ_1 and ρ_1 both real.* Assume that χ_1 and ρ_1 are both real. If $\lambda_1 < 0.348$ then by [8, Lemma 14.2] we have Linnik's theorem with $L = 4.9$. So let us start with the case

$$\lambda_1 \in [0.348, 0.40] =: [\lambda_{11}, \lambda_{12}].$$

Then [8, Tables 4 and 7] give $\lambda' \ge 2.108$ and $\lambda_2 \ge 1.29$. For the λ_3-estimate, we use the maximum of the last λ_2-estimate, Table 10 and 0.857 (by [8, Lemma 10.3]) which results in $\lambda_3 \ge 1.29$. If there were no estimates available for λ', λ_2 or λ_3 then we would have chosen the lower bound λ_{11}. We have $s = 0$ and $C(\Lambda) = 0$, thus no estimates for $N(\lambda)$ are needed. A calculation yields $W < 0.85 < 1$, hence Theorem 1.1 is proven for this special case.

If $\lambda_1 \in [0.40, 0.42]$ we similarly get $\lambda' \ge 2.030$, $\lambda_2 \ge 1.18$ and $\lambda_3 \ge 1.18$. We now introduce an additional split-up of this case which will be used in the same manner in all of the following cases except when χ_3 is complex and $\lambda_1 \in [0.44, 0.54]$. We choose

$$\lambda_0 = 1.19.$$

Now by Lemma 3.5 with $N_0 = 0$ and $\lambda_1 \ge \lambda_{11}$ we get $N(\lambda_0) \le 55$ in the way outlined in §3.2.2. We then separately calculate by computer the 52 cases

$$N(\lambda_0) \in [0, 4],\ N(\lambda_0) = 5,\ N(\lambda_0) = 6,\ \ldots,\ N(\lambda_0) = 54,\ N(\lambda_0) = 55$$

in the following way:

Consider a case of the form $N(\lambda_0) \in [N_{\min}, N_{\max}]$ where $N_{\min}$ and $N_{\max}$ are natural numbers and in most cases identical. We need upper estimates for $N(\Lambda_r)$. So, if $\Lambda_r \le \lambda_0$ then we take the minimum of $N_{\max}$ and the estimate derived by Lemma 3.5 with $N_0 = 0$. For $\Lambda_r > \lambda_0$ we take the estimate derived by Lemma 3.5 with $N_0 = N_{\min}$. In this way we get 52 different W's, all of which are strictly smaller than 1, thus proving Theorem 1.1 whenever $\lambda_1 \in [0.40, 0.42]$ and χ_1 and ρ_1 are both real.

For each of the λ_1-intervals

$$[0.42, 0.44], \quad [0.44, 0.60], \quad [0.60, 0.68], \quad [0.68, 0.78], \quad [0.78, \infty)$$

we do the same and always get $W < 1$. Theorem 1.1 follows for χ_1 and ρ_1 both real.

4.2.2. *Case 2: χ_1 real and ρ_1 complex.* Assume χ_1 is real and ρ_1 is complex. We proceed in exactly the same way as in Case 1, but this time taking always

$$\lambda_0 = 1.05$$

and having $\lambda_1 \geq 0.628$ according to Table 11. Thus, it is sufficient to check the cases when λ_1 is in one of the intervals

$$[0.628, 0.74], \quad [0.74, 0.78], \quad [0.78, \infty).$$

We use Tables 3 and 7 for the λ'- and λ_2-estimates. As a λ_3-estimate we take the maximum of the last λ_2-estimate and 0.857. Again we always get $W < 1$ and Theorem 1.1 follows for this case.

4.2.3. *Case 3: χ_1 complex.* By Table 11 we have $\lambda_1 \geq 0.44$. It is sufficient to distinguish the cases when λ_1 is in

$$\begin{array}{lllll} [0.44, 0.54], & [0.54, 0.58], & [0.58, 0.60], & [0.60, 0.62], & [0.62, 0.64], \\ [0.64, 0.66], & [0.66, 0.68], & [0.68, 0.72], & [0.72, 0.84], & [0.84, \infty). \end{array}$$

We choose

$$\lambda_0 = \begin{cases} 1.12 & \text{if } \lambda_{11} \in \{0.44, 0.54, 0.58\}, \\ 1.04 & \text{if } \lambda_{11} \in \{0.60, 0.62, 0.64\}, \\ 1.07 & \text{if } \lambda_{11} \in \{0.66, 0.68\}, \\ 1.02 & \text{if } \lambda_{11} \in \{0.72, 0.84\}. \end{cases}$$

Additionally, in virtue of Table 9, in the case $\lambda_1 \in [0.64, 0.66]$ we distinguish the two cases $\lambda_2 \leq 0.86$ and $\lambda_2 > 0.86$. Similarly for $\lambda_1 \in [0.66, 0.68]$ and $\lambda_1 \in [0.68, 0.72]$. Checking that $W < 1$ in all the different cases (do not forget the split-up of cases mentioned in Case 1) yields the result. Hence, the proof of Theorem 1.1 is complete. ■

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