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Determining modular forms of general level by central values of convolution L-functions

by

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1. Introduction. This note generalizes the work by Ganguly, Hoffstein and Sengupta [2] to the case of general level.

In [6], Luo and Ramakrishnan proved that two cuspidal normalized newforms f and f' must be equal if $L(1/2, f, \chi_d) = L(1/2, f', \chi_d)$ for any quadratic character. Chinta and Diaconu gave a generalization of this result to $GL(3)$-forms in [1]. By twisting all of the newforms of a fixed weight and infinitely many prime levels p, Luo [5] showed that if the L-values at $1/2$ coincide, then $f = f'$. In [2], the authors considered forms of level 1, and proved an analogous result by twisting newforms of fixed level 1 and infinitely varying weight k. They predicted that the same method should apply to the case of general level. We follow their ideas and prove

Theorem 1. *For $i = 1, 2$, let $g_i \in \mathcal{S}_{l_i}^{\mathrm{new}}(N_i, \chi_i)$ with $\chi_1 = 1$. If for infinitely many k it is true that for all $f \in \mathcal{S}_k^{\mathrm{new}}(1)$,*

$$L(1/2, f \otimes g_1) = L(1/2, f \otimes g_2),$$

then $g_1 = g_2$.

In the case of both χ_1, χ_2 non-trivial, we have the following

Theorem 2. *For $i = 1, 2$, let $g_i \in \mathcal{S}_{l_i}^{\mathrm{new}}(N_i, \chi_i)$ with $\chi_1, \chi_2 \neq 1$. If for infinitely many k it is true that for all $f \in \mathcal{S}_k^{\mathrm{new}}(1)$,*

$$L(1/2, f \otimes g_1) = L(1/2, f \otimes g_2) \quad \text{and} \quad L'(1/2, f \otimes g_1) = L'(1/2, f \otimes g_2),$$

then:

(a) $L(1, \chi_1) - \varepsilon(g_1) L(1, \overline{\chi_1}) = L(1, \chi_2) - \varepsilon(g_2) L(1, \overline{\chi_2})$,
(b) *if* $L(1, \chi_1) - \varepsilon(g_1) L(1, \overline{\chi_1}) \neq 0$, *then* $g_1 = g_2$.

The main trick in [2] is to open up the Kloosterman sums and apply the functional equations for the additive twists of our modular L-functions. We

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shall apply the same idea, combined with some results on modular forms of square-free level. Moreover for Theorem 2, we need to deal with a different integral which gives us the preferable residue but requires information on the derivatives of convolution L-functions.

A large amount of calculations will be omitted; the reader should refer to those in [2] for more details.

2. Preparation. Let $g \in \mathcal{S}_l^{\mathrm{new}}(N,\chi)$ and $f \in \mathcal{S}_k^{\mathrm{new}}(1)$. Let $e(z) = e^{2\pi i z}$. Assume

$$g(z) = \sum_{n=1}^{\infty} a_g(n) e(nz),$$

and set $\lambda_g(n) = a_g(n)/n^{(l-1)/2}$. It is now well-known that $|\lambda_g(p)| \le 2$; this is the Ramanujan conjecture for $p \nmid N$, and for $p \mid N$ see Proposition A.2 in [4]. Therefore,

$$|\lambda_g(n)| \le \tau(n),$$

where $\tau(n)$ is the number of positive divisors of n. If g is only a cusp form, the bound is $\lambda_g(n) = O(\sqrt{n})$.

We associate to g the (normalized) L-function

$$L(s,g) = \sum_{n=1}^{\infty} \lambda_g(n) n^{-s}.$$

It satisfies the functional equation

$$\Lambda(s,g) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{l-1}{2}\right) L(s,g) = i^k \Lambda(1-s, g|_{\omega_N}),$$

where

$$\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We know that $g|_{\omega_N}$ can be normalized to be a newform as follows:

$$g|_{\omega_N} := \eta_g(N)\overline{g}, \qquad \overline{g}(z) = \sum_{n=1}^{\infty} \overline{a_g(n)} e(nz) \in \mathcal{S}_l^{\mathrm{new}}(N,\overline{\chi}).$$

Here $|\eta_g(N)| = 1$.

We now consider the additive twist of $L(s,g)$,

$$L(s,g,a/c) = \sum_{n=1}^{\infty} \lambda_g(n) e(an/c) n^{-s},$$

where a, c are coprime integers, $c > 0$. In general, if our modular group has width h at ∞, the factor $e(an/ch)$ is expected instead. We shall need the following lemma:

Lemma 3. *Let $g \in S_l(N,\chi)$ and set $g^* = g|_\alpha$ for any fixed*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \qquad \text{with } c > 0.$$

Suppose at ∞,

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{(l-1)/2} e(nz), \qquad g^*(z) = \sum_{n=1}^{\infty} \lambda_{g^*}(n) n^{(l-1)/2} e(nz/h),$$

where h is the width at ∞ of $\alpha^{-1}\Gamma_1(N)\alpha$. Then we have the functional equation

$$(2.1) \qquad \left(\frac{2\pi}{c}\right)^{-s-(l-1)/2} \Gamma\left(s + \frac{l-1}{2}\right) L(s,g,a/c) = i^l \left(\frac{2\pi}{ch}\right)^{s-(l+1)/2} \Gamma\left(-s + \frac{l+1}{2}\right) L(1-s, g^*, -d/c).$$

Proof. The proof involves standard arguments and can be found, for example, in [8, Lemma 4]; the only difference is the appearance of the width h. ∎

We note here that in the case of square-free level, one can get a more accurate functional equation (see [4, (A.10)]).

For later use, let us fix a set of representatives for

$$\Gamma_1(N)\backslash SL_2(\mathbb{Z}) = \bigcup_{i=1}^{\mu} \Gamma_1(N)\alpha_i$$

where the union is disjoint, and set $g_i^* = g|_{\alpha_i}$.

Given f, g as above, the Rankin–Selberg convolution L-function is

$$L(s, f\otimes g) = L(2s,\chi) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s}.$$

It satisfies the functional equation (assuming $k > l$)

$$(2.2) \qquad \Lambda(s, f\otimes g) = \left(\frac{N}{4\pi^2}\right)^s \Gamma\left(s + \frac{k-l}{2}\right) \Gamma\left(s - 1 + \frac{k+l}{2}\right) L(s, f\otimes g) = \varepsilon(g)\Lambda(1-s, f\otimes \overline{g}),$$

where $\varepsilon(g) = \eta_g(N)^2$ and $\overline{g}(z) = \sum_{n=1}^{\infty} \overline{a_g(n)} e(nz)$. Note that $\bar{f} = f$. Moreover, this gives an analytic continuation over the whole complex s-plane.

3. Proof of Theorem 1. Let $G(u) = e^{u^2}$. For $X > 0$, let

$$I_{f\otimes g}(X,s) = \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=3/2} X^u \Lambda(s+u, f\otimes g) \frac{G(u)}{u}\, du.$$

By moving the contour to $\mathrm{Re}(u) = -3/2$, collecting the residue at $u = 0$, applying functional equation (2.2) and changing variable $u \to -u$, we obtain

$$\Lambda(s, f \otimes g) = I_{f\otimes g}(X, s) + \varepsilon(g) I_{f\otimes\overline{g}}(X^{-1}, 1-s). \tag{3.1}$$

Set

$$b_{f\otimes g}(n) = \sum_{t:\, n=mt^2} \chi(t)\lambda_f(m)\lambda_g(m),$$

and then

$$L(s, f\otimes g) = \sum_{n=1}^{\infty} b_{f\otimes g}(n) n^{-s}.$$

Straightforward calculations (see [2]) give us

$$\begin{aligned} I_{f\otimes g}(X, s) = \left(\frac{N}{4\pi^2}\right)^s \Gamma\left(s + \frac{k-l}{2}\right)\Gamma\left(s - 1 + \frac{k+l}{2}\right) \\ \times \sum_{n=1}^{\infty} \frac{b_{f\otimes g}(n)}{n^s} V_s\left(\frac{4\pi^2 n}{XN}\right), \end{aligned} \tag{3.2}$$

where for $y > 0$,

$$V_s(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=3/2} y^{-u} \frac{\Gamma\left(s+u+\frac{k-l}{2}\right)\Gamma\left(s+u-1+\frac{k+l}{2}\right)}{\Gamma\left(s+\frac{k-l}{2}\right)\Gamma\left(s-1+\frac{k+l}{2}\right)} \frac{G(u)}{u}\, du.$$

Plug (3.2) into (3.1), evaluate both sides at $s = 1/2$, $X = 1$, and cancel the same gamma factors to obtain

$$L(1/2, f\otimes g) = \sum_{n=1}^{\infty} \frac{b_{f\otimes g}(n) + \varepsilon(g) b_{f\otimes g}(n)}{\sqrt{n}} V_{1/2}\left(\frac{4\pi^2 n}{N}\right). \tag{3.3}$$

Now we fix a positive integer q which is relatively prime to N, and consider the series

$$\begin{aligned} \sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)} & L(1/2, f\otimes g)\frac{\lambda_f(q)}{\omega_f} \\ &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} V_{1/2}\left(\frac{4\pi^2 n}{N}\right) \sum_{n=mt^2} (\chi(t)\lambda_g(m) + \varepsilon(g)\overline{\chi}(t)\overline{\lambda_g(m)}) \\ &\quad \times \sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)} \frac{\lambda_f(q)\lambda_f(m)}{\omega_f}. \end{aligned}$$

By the Petersson trace formula ([3, Theorem 3.6]),

$$\sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)} \frac{\lambda_f(q)\lambda_f(m)}{\omega_f} = \delta(m, q) + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m, q; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mq}}{c}\right),$$

where $S(m,q;c)$ is the Kloosterman sum and $J_{k-1}(x)$ is the J-Bessel function. Moreover $\overline{\lambda_g(q)} = \overline{\chi}(q)\lambda_g(q)$ (see [4, (A.5)]), so

$$\sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)} L(1/2, f\otimes g)\frac{\lambda_f(q)}{\omega_f} = \frac{\lambda_g(q)}{\sqrt{q}} M_q(k,l,\chi) + E_{g,q}(k), \tag{3.4}$$

where

$$M_q(k,l,\chi) = \sum_{n=1}^{\infty} \frac{1}{n} V_{1/2}\left(\frac{4\pi^2 n^2 q}{N}\right)(\chi(n) + \varepsilon(g)\overline{\chi}(qn)) \tag{3.5}$$

and

$$\begin{aligned} E_{g,q}(k) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} V_{1/2}\left(\frac{4\pi^2 n}{N}\right) &\sum_{n=mt^2} (\lambda_g(m)\chi(t) + \varepsilon(g)\overline{\lambda_g(m)\chi(t)}) \\ &\times 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m,q;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mq}}{c}\right). \end{aligned} \tag{3.6}$$

3.1. The main term. As in [2], put $a = \frac{k+l-1}{2}$ and $b = \frac{k-l+1}{2}$. Open the integral in (3.5) to obtain

$$\begin{aligned} M_q(k,l,\chi) = \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=3/2} &\frac{\Gamma(u+a)\Gamma(u+b)}{\Gamma(a)\Gamma(b)} \frac{G(u)}{u} \left(\frac{4\pi^2 q}{N}\right)^{-u} \\ &\times (L(2u+1,\chi) + \varepsilon(g)\overline{\chi}(q)L(2u+1,\overline{\chi}))\,du, \end{aligned}$$

where $L(s,\chi)$ is the Dirichlet L-function associated to χ. Now we move the contour to $\mathrm{Re}(u) = -1/2$ and collect the residue at $u=0$.

The integral along $\mathrm{Re}(u) = -1/2$ is $O(1/k)$ as in [2].

If $\chi = 1$, then $\varepsilon(g) = \eta_g(N)^2 = 1$ and the same calculations as in [2] give the residue

$$\frac{\Gamma'}{\Gamma}(a) + \frac{\Gamma'}{\Gamma}(b) + 2\gamma_0 - \log\frac{4\pi^2 q}{N} = 2\log k + O(1).$$

If on the other hand $\chi \neq 1$, then $L(2u+1,\chi)$ and $L(2u+1,\overline{\chi})$ are holomorphic and non-vanishing at $u=0$, hence the residue is given by $L(1,\chi) + \varepsilon(g)\overline{\chi}(q)L(1,\overline{\chi})$.

Hence

$$M_q(k,l,\chi) = \begin{cases} 2\log k + O(1), & \chi = 1, \\ L(1,\chi) + \varepsilon(g)\overline{\chi}(q)L(1,\overline{\chi}) + O(1/k), & \chi \neq 1. \end{cases} \tag{3.7}$$

3.2. The error term. Apply the Mellin inversion formula to the J-Bessel function as in [2] to obtain (here $\alpha = -1/2 - \delta$),

$$E_{g,q}(k) = \frac{i^{k-2}}{4\pi} \int\limits_{\mathrm{Re}(u)=3/2} \int\limits_{\mathrm{Re}(s)=\alpha} \frac{G(u)}{u} \frac{\Gamma(u+a)\Gamma(u+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \times T_{g,q}(u,s) q^{-s/2} (2\pi)^{-2u-s} N^u \, ds \, du, \tag{3.8}$$

where

$$T_{g,q}(u,s) = L(2u+1,\chi) S_{g,q}(u,s) + \varepsilon(g) L(2u+1,\overline{\chi}) S_{\overline{g},q}(u,s)$$

and

$$S_{g,q}(u,s) = \sum_{m=1}^{\infty} \sum_{c=1}^{\infty} \frac{\lambda_g(m) S(m,q;c)}{m^{1/2+u+s/2} c^{1-s}}. \tag{3.9}$$

Note that (3.9) is absolutely convergent by the Weil bound for the Kloosterman sum. Now let us apply the trick in [2] by opening the Kloosterman sum and applying the functional equations (2.1) to obtain

$$\begin{aligned} S_{g,q}(u,s) &= \sum_{c=1}^{\infty} \sideset{}{^*}\sum_{a(c)} e(q\overline{a}/c) c^{s-1} L(1/2+u+s/2, g, a/c) \\ &= i^l \frac{\Gamma(-u-s/2+l/2)}{\Gamma(u+s/2+l/2)} \sum_{c=1}^{\infty} \sideset{}{^*}\sum_{a(c)} \frac{e(qd/c) L(1/2-u-s/2, g_i^*, -d/c)}{(2\pi)^{-2u-s} c^{1+2u} h^{u+s-l/2}}, \end{aligned}$$

where d, g_i^* and h are fixed by first choosing $b, d \in \mathbb{Z}$ such that $ad - bc = 1$ and then taking the corresponding coset representative for

$$\Gamma_1(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Move the line of integration in s to $\mathrm{Re}(s) = \alpha = -6$ so that the L-series are all absolutely convergent. So this double sum is absolutely convergent and the inner sum equals

$$\frac{(2\pi)^{2u+s}}{c^{1+2u}} \sideset{}{^*}\sum_{a(c)} h^{-u-(s-l)/2} \sum_{m=1}^{\infty} \frac{\lambda_{g_i^*}(m)}{m^{1/2-u-s/2}} e\left(\frac{(qh-m)d}{ch}\right).$$

Hence along $s = -6 + it$ and $u = 3/2 + iv$,

$$|S_{g,q}(u,s)| \le (2\pi)^{-3}\left|\frac{\Gamma(-u-s/2+l/2)}{\Gamma(u+s/2+l/2)}\right| \sum_{c=1}^{\infty}\sum_{a(c)}{}^{*}\sum_{m=1}^{\infty}\frac{h^{(3+l)/2}|\lambda_{g_i^*}(m)|}{m^2c^4}$$
$$\le (2\pi)^{-3}\left|\frac{\Gamma(-u-s/2+l/2)}{\Gamma(u+s/2+l/2)}\right| \sum_{c=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi(c)h^{(3+l)/2}}{m^2c^4}\sum_{i=1}^{\mu}|\lambda_{g_i^*}(m)|$$
$$\ll \left|\frac{\Gamma(-u-s/2+l/2)}{\Gamma(u+s/2+l/2)}\right|,$$

since $\lambda_i^*(m) = O(\sqrt{m})$ and $1 \le h \le N$. So

$$(3.10)\qquad |E_{g,q}(k)| \ll \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{G(u)}{u}\,\frac{\Gamma(u+a)\Gamma(u+b)}{\Gamma(a)\Gamma(b)}\,\frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)}\,\frac{\Gamma(-u-s/2+l/2)}{\Gamma(u+s/2+l/2)}\right| dt\,dv \ll 1,$$

by the same calculations as in [2].

3.3. End of proof of Theorem 1. By assumption, for infinitely many k and any $f \in \mathcal{S}_k^{\mathrm{new}}(1)$, we have $L(1/2, f\otimes g_1) = L(1/2, f\otimes g_2)$. Then by (3.4), for a prime $q = p$ not dividing N_1N_2, we have

$$(3.11)\qquad \frac{\lambda_{g_1}(p)}{\sqrt{p}}M_p(k,l,\chi_1) + E_{g_1,p}(k) = \frac{\lambda_{g_2}(p)}{\sqrt{p}}M_p(k,l,\chi_2) + E_{g_2,p}(k).$$

Since $\chi_1 = 1$, χ_2 must also be trivial, because otherwise for any p with $\lambda_{g_1}(p) \ne 0$, by (3.7) and (3.10), the left-hand side is asymptotically equal to

$$\frac{2\lambda_{g_1}(p)}{\sqrt{p}}\log k,$$

while the right-hand side is only $O(1)$. Now (3.11) becomes

$$2\frac{\lambda_{g_1}(p)}{\sqrt{p}}\log k = 2\frac{\lambda_{g_2}(p)}{\sqrt{p}}\log k + O(1),$$

hence $\lambda_{g_1}(p) = \lambda_{g_2}(p)$ for almost all p. So $g_1 = g_2$, since they are newforms.

4. Proof of Theorem 2. If χ is non-trivial, we are not able to obtain $\log k$ in the main term. The idea is to raise the power of u in $I_{f\otimes g}(X,s)$ to u^2, which produces the desirable $\log k$ in the main term. However, we shall need information about the derivatives of the convolution L-functions; moreover, we cannot get rid of the gamma factors as we did in (3.3), but a proper alternative $D(1/2, f\otimes g)$ will be used to solve this problem.

Since most of the proof and calculations and arguments in this section are similar to those in Section 3, we only note down the main steps.

Let $g \in \mathcal{S}_l^{\text{new}}(N,\chi)$ with χ non-trivial, and $f \in \mathcal{S}_k^{\text{new}}(1)$.

As before, let $G(u) = e^{u^2}$. For $X > 0$, let

$$J_{f\otimes g}(X,s) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re}(u)=3/2} X^u \Lambda(s+u, f\otimes g) \frac{G(u)}{u^2}\, du.$$

Again, by moving the contour to $\operatorname{Re}(u) = -3/2$, collecting the residue at $u = 0$, applying functional equation (2.2) and changing variable $u \to -u$, we obtain

$$\Lambda'(s, f\otimes g) + \Lambda(s, f\otimes g)\log X = J_{f\otimes g}(X,s) - \varepsilon(g) J_{f\otimes \overline{g}}(X^{-1}, 1-s). \tag{4.1}$$

Evaluate both sides at $s = 1/2$ and $X = 1$ to obtain

$$\begin{aligned}\Lambda'(1/2, f\otimes g) = {}& \left(\frac{N}{4\pi^2}\right)^{1/2} \Gamma\left(\frac{k-l+1}{2}\right)\Gamma\left(\frac{k+l-1}{2}\right) \\ & \times \sum_{n=1}^{\infty} \frac{b_{f\otimes g}(n) + \varepsilon(g) b_{f\otimes g}(n)}{\sqrt{n}} U_{1/2}\left(\frac{4\pi^2 n}{N}\right),\end{aligned} \tag{4.2}$$

where

$$U_s(y) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re}(u)=3/2} y^{-u} \frac{\Gamma\left(s+u+\frac{k-l}{2}\right)\Gamma\left(s+u-1+\frac{k+l}{2}\right)}{\Gamma\left(s+\frac{k-l}{2}\right)\Gamma\left(s-1+\frac{k+l}{2}\right)} \frac{G(u)}{u^2}\, du.$$

It is easy to see that

$$\begin{aligned}&\frac{\Lambda'(1/2, f\otimes g)}{\left(\frac{N}{4\pi^2}\right)^{1/2}\Gamma\left(\frac{k-l+1}{2}\right)\Gamma\left(\frac{k+l-1}{2}\right)} \\ &= \left(\log\frac{N}{4\pi^2} + \frac{\Gamma'}{\Gamma}\left(\frac{k-l+1}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k-l+1}{2}\right)\right) L(1/2, f\otimes g) \\ &\quad + L'(1/2, f\otimes g) \\ &= (2\log k + O(1)) L(1/2, f\otimes g) + L'(1/2, f\otimes g),\end{aligned} \tag{4.3}$$

since

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1}).$$

Now denote $D(1/2, f\otimes g) = 2\log k\, L(1/2, f\otimes g) + L'(1/2, f\otimes g)$. For any positive integer q with $(q,N) = 1$, consider the sum

$$\sum_{f\in\mathcal{S}_k^{\text{new}}(1)} D(1/2, f\otimes g) \frac{\lambda_g(q)}{\omega_f}. \tag{4.4}$$

By (4.2), (4.3), the proof of Theorem 1 and the Petersson trace formula, (4.4) equals

$$\sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)}\biggl(\frac{\Lambda'(1/2,f\otimes g)}{\bigl(\frac{N}{4\pi^2}\bigr)^{1/2}\Gamma\bigl(\frac{k-l+1}{2}\bigr)\Gamma\bigl(\frac{k+l-1}{2}\bigr)}+O(1)L(1/2,f\otimes g)\biggr)\frac{\lambda_g(q)}{\omega_f} \tag{4.5}$$
$$= \sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)}\frac{\Lambda'(1/2,f\otimes g)}{\bigl(\frac{N}{4\pi^2}\bigr)^{1/2}\Gamma\bigl(\frac{k-l+1}{2}\bigr)\Gamma\bigl(\frac{k+l-1}{2}\bigr)}\frac{\lambda_g(q)}{\omega_f}+O(1)$$
$$= \sum_{f\in\mathcal{S}_k^{\mathrm{new}}(1)}\sum_{n=1}^{\infty}\frac{b_{f\otimes g}(n)-\varepsilon(g)b_{f\otimes g}(n)}{\sqrt{n}}U_{1/2}\biggl(\frac{4\pi^2 n}{N}\biggr)\frac{\lambda_g(q)}{\omega_f}+O(1)$$
$$= \frac{\lambda_g(q)}{\sqrt{q}}\widetilde{M}_q(k,l,\chi)+\widetilde{E}_{g,q}(k)+O(1),$$

where

$$\widetilde{M}_q(k,l,\chi)=\sum_{n=1}^{\infty}U_{1/2}\biggl(\frac{4\pi^2n^2q}{N}\biggr)(\chi(n)-\varepsilon(g)\overline{\chi}(qn)) \tag{4.6}$$

and

$$\widetilde{E}_{g,q}(k)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}U_{1/2}\biggl(\frac{4\pi^2 n}{N}\biggr)\sum_{n=mt^2}(\lambda_g(m)\chi(t)-\varepsilon(g)\overline{\lambda_g(m)\chi(t)}) \tag{4.7}$$
$$\times\, 2\pi i^{-k}\sum_{c=1}^{\infty}\frac{S(m,q;c)}{c}J_{k-1}\biggl(\frac{4\pi\sqrt{mq}}{c}\biggr).$$

As in Section 3.1, we obtain

$$\widetilde{M}_q(k,l,\chi)=\frac{1}{2\pi i}\int_{\mathrm{Re}(u)=3/2}\frac{\Gamma(u+a)\Gamma(u+b)}{\Gamma(a)\Gamma(b)}\frac{G(u)}{u^2}\biggl(\frac{4\pi^2q}{N}\biggr)^{-u} \tag{4.8}$$
$$\times\,(L(2u+1,\chi)-\varepsilon(g)\overline{\chi}(q)L(2u+1,\overline{\chi}))\,du.$$

As before, by moving the contour to $\mathrm{Re}(u)=-1/2$ and collecting the residue at $u=0$, we obtain

$$\widetilde{M}_q(k,l,\chi)=(L(1,\chi)-\varepsilon(g)L(1,\overline{\chi}))\log k+O(1). \tag{4.9}$$

The calculations in Section 3.2 can be carried out here with little modification and we obtain

$$\widetilde{E}_{g,q}(k)=O(1). \tag{4.10}$$

Take q such that $(q,N_1N_2)=1$ and (4.4) is independent of $g=g_1$ or $g=g_2$, under the assumption of Theorem 2. So by (4.5), (4.9) and (4.10), we have

$$\frac{\lambda_{g_1}(q)}{\sqrt{q}}(L(1,\chi_1)-\varepsilon(g_1)L(1,\overline{\chi_1}))\log k = \frac{\lambda_{g_2}(q)}{\sqrt{q}}(L(1,\chi_2)-\varepsilon(g_2)L(1,\overline{\chi_2}))\log k + O(1). \tag{4.11}$$

If $L(1,\chi_1)-\varepsilon(g_1)L(1,\overline{\chi_1})=0$, then take q to be a prime such that $\lambda_{g_2}(q)\neq 0$ and by (4.11), $L(1,\chi_2)-\varepsilon(g_2)L(1,\overline{\chi_2})=0$. Part (a) of Theorem 2 is proved.

Now assume $L(1,\chi_1)-\varepsilon(g_1)L(1,\overline{\chi_1})\neq 0$. Take any two distinct primes p_1 and p_2 such that $\lambda_{g_1}(p_1),\lambda_{g_2}(p_2)\neq 0$; this is possible since g_1 is a newform (see [7, Theorem 4.6.8], for example). Let

$$c=\frac{L(1,\chi_1)-\varepsilon(g_1)L(1,\overline{\chi_1})}{L(1,\chi_2)-\varepsilon(g_2)L(1,\overline{\chi_2})}.$$

Now by (4.11),

$$c\lambda_{g_1}(q)=\lambda_{g_2}(q), \tag{4.12}$$

and by taking $q=p_1,p_2$ and p_1p_2, we have $c^2=c$, hence $c=1$.

So for any q with $(q,N_1N_2)=1$, we have $\lambda_{g_1}(q)=\lambda_{g_2}(q)$. Since both g_1,g_2 are newforms, $g_1=g_2$ (see, for example, [7, Theorem 4.6.19]). The proof of Theorem 2 is complete.

Remark. If $L(1,\chi)-\varepsilon(g)L(1,\overline{\chi})=0$, then we are not able to distinguish two newforms of this kind in this way. For example, if χ is real and non-trivial, g has real coefficients and $\varepsilon(g)=1$, then we are not able to distinguish two newforms of this kind by this method.

Remark. Information on central values of the derivative L-functions is necessary, since for example, for newforms g with real coefficients, $\varepsilon(g)=-1$ and χ real but non-trivial, we have $L(1/2,f\otimes g)=0$ for any f, but

$$L(1,\chi)-\varepsilon(g)L(1,\overline{\chi})=2L(1,\chi)\neq 0.$$

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