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Generalised prime systems with periodic integer counting function

by

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Introduction. In a recent paper [7], we discussed Mellin transforms $\hat{N}(s)$ of integrators N for which $N(x)-x$ is periodic in order to study flows of holomorphic functions converging to $\zeta(s)$. Here we consider the question when such an N determines a g-prime system, i.e. $N(x)$ is the 'integer counting function' of a generalised prime system—see Section 1.3 for the definition.

An example of such a flow $\hat{N}_\lambda(s)$ was given in [7], but it was unclear whether or not they determined g-prime systems. As a consequence of our present results, we show that none of them does.

In fact, we investigate more generally when an increasing function N for which $N(x)-cx$ is periodic determines a g-prime system for a constant $c>0$. (At the outset we assume that N is right-continuous, $N(1)=1$, and $N(x)=0$ for $x<1$.) For example, $N(x)=cx+1-c$ for $x\geq 1$ determines a continuous g-prime system for $0<c\leq 2$ at least.

As for discontinuous examples, we have the prototype $N(x)=[x]$ for the usual primes and integers. For other examples, consider the g-prime system containing the usual primes except given primes $p_1,\dots,p_k$. This has integer counting function

$$N(x)=\sum_{\substack{n\leq P\\ (n,P)=1}}\left[\frac{x-n}{P}+1\right],$$

where $P=p_1\cdots p_k$. In this case $N(x+P)=N(x)+\varphi(P)$ where φ is Euler's function, and $N(x)-(\varphi(P)/P)x$ has period P.

Our results split quite naturally into continuous and discontinuous cases. In Section 2, where we consider the continuous case, the main result is that for N sufficiently 'nice' (e.g. continuously differentiable), N determines a

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g-prime system only for the trivial case where $N(x) - cx$ is constant, i.e. $N(x) = cx + 1 - c$.

For discontinuous N the picture is less straightforward. A useful tool is to consider its 'jump' function N_J, which must necessarily also have $N_J(x) - c'x$ periodic (for some $c' > 0$) and which also determines a g-prime system if N does (Theorem 1.1 below). We show that if such an N has only finitely many discontinuities in any interval but is otherwise 'smooth', then N must be a step function, the discontinuities must occur at *integer* points, and the period, say P, must be a natural number. Then, denoting the jump at n by a_n, we show that a_n is even modulo P [1] and multiplicative. This allows us to deduce our main result.

Theorem A. *Let $N \in T$ be such that $N(x) - cx$ has period P, and suppose that N determines a g-prime system. Then $P \in \mathbb{N}$ and*

$$N(x) = \sum_{\substack{n \le P \\ (n,P)=1}} \left[\frac{x-n}{P} + 1\right],$$

i.e. N is the integer counting function of the g-prime system $\mathbb{P} \setminus \{p_1, \dots, p_k\}$ where $p_1, \dots, p_k$ are the prime divisors of P.

(For the definition of T, see Section 1.2.) This actually shows that the smallest period must be squarefree and that $c = \varphi(P)/P$. Our set-up includes all the usual 'discrete' g-prime systems.

In proving Theorem A, we prove the following result on Dirichlet series with periodic coefficients, which may be of independent interest.

Theorem B. *Let $\{a_n\}_{n \in \mathbb{N}}$ be periodic, $a_1 = 1$, and suppose $a_n = \exp_* b_n$ for some $b_n \ge 0$. Then a_n is multiplicative.*

Here $*$ refers to Dirichlet convolution. Thus a_n and b_n are related by $\sum_{n=1}^{\infty} a_n/n^s = \exp\{\sum_{n=1}^{\infty} b_n/n^s\}$.

1. Preliminaries

1.1. Riemann–Stieltjes convolution. Let S denote the space of functions $f : \mathbb{R} \to \mathbb{C}$ which are zero on $(-\infty, 1)$, right-continuous, and of local bounded variation. (See e.g. [3, pp. 50–70].) This is a vector space over addition. Let S^+ denote the subspace of S consisting of increasing functions. Also, for $\alpha \in \mathbb{R}$, let $S_\alpha = \{f \in S : f(1) = \alpha\}$, while $S_\alpha^+ = S^+ \cap S_\alpha$.

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[1] That is, $a_n = a_{(n,P)}$.

For functions $f, g \in S$, define the *convolution* (or *Mellin–Stieltjes convolution*) by [2]

$$(f * g)(x) = \int_{1-}^{x} f\left(\frac{x}{t}\right) dg(t).$$

We note that S is closed under $*$ and that $*$ is commutative and associative. The identity (with respect to $*$) is $i(x) = 1$ for $x \geq 1$ and zero otherwise.

(a) If f or g is continuous (on $\mathbb{R}$), then $f * g$ is continuous.

(b) *Exponentials*. For $f \in S_1$, there exists $g \in S_0$ such that $f = \exp_* g$, i.e.

$$f = \sum_{n=0}^{\infty} \frac{g^{*n}}{n!},$$

where $g^{*n} = g * g^{*(n-1)}$ and $g^{*0} = i$. Also $f = \exp_* g$ if and only if $f * g_L = f_L$ (see [5]), where $f_L \in S$ is the function defined for $x \geq 1$ by $f_L(x) = \int_1^x \log t \, df(t)$.

(c) For $f \in S$, define the *Mellin transform* of f by $\hat{f}(s) = \int_{1-}^{\infty} x^{-s} \, df(x)$. This exists if $f(x) = O(x^A)$ for some A. Note that $\widehat{f * g} = \hat{f}\hat{g}$ and $\widehat{\exp_* f} = \exp \hat{f}$.

(d) Let $f, g \in S$ be continuously differentiable on $(1, \infty)$. Let $g_1(x) = \int_{1-}^{x} (1/t) \, dg(t)$. Then $f * g$ is also continuously differentiable on $(1, \infty)$ with

$$(f * g)' = f' * g_1 + f(1)g'.$$

Proof. Let $x > 1$ and consider $(f * g)(x + h) - (f * g)(x)$ for h small. First suppose that $h > 0$. We have

$$\begin{aligned} (1.1) \qquad & \frac{(f * g)(x + h) - (f * g)(x)}{h} \\ &= \int_{1-}^{x} \frac{f((x + h)/t) - f(x/t)}{h} \, dg(t) + \frac{1}{h} \int_{x}^{x+h} f\left(\frac{x + h}{t}\right) dg(t). \end{aligned}$$

The integrand in the first integral tends pointwise to $(1/t)f'(x/t)$, so by the continuity of f' this integral tends to (see [1], p. 218)

$$\int_{1-}^{x} \frac{f'(x/t)}{t} \, dg(t) = (f' * g_1)(x) \quad \text{ as } h \to 0.$$

The second term equals

$$f(1)\frac{g(x + h) - g(x)}{h} + \frac{1}{h} \int_{x}^{x+h} \left(f\left(\frac{x + h}{t}\right) - f(1)\right) dg(t).$$

([2]) All limits of integration are understood to be + (i.e. from the right) except where they are explicitly stated to be −.

The first summand tends to $f(1)g'(x)$ while the integrand tends to 0 by right-continuity of f at 1. Hence so does the integral.

If $h < 0$, write $h = -k$ and split up the integral as $(1/k)\int_1^{x-k}$ and $(1/k)\int_{x-k}^{x}$ and argue as before. ∎

For the proofs of (a)–(c) see [3] and [5].

1.2. The 'jump' function

Definition 1.1.

(i) For $f \in S$ and each $x \in \mathbb{R}$, we denote by $\Delta f(x)$ the left-hand jump of f at x; i.e.

$$\Delta f(x) = f(x) - f(x-) = \lim_{h\to 0^+}(f(x) - f(x-h)).$$

This is well-defined for monotone f and hence for $f \in S$. Note also that Δf is non-zero on a countable set only [1, p. 162].

(ii) For $f \in S^+$, let f_J denote the *jump function* of f, i.e.

$$f_J(x) = \sum_{x_r \le x} \Delta f(x_r),$$

where the x_r denote the discontinuities of f.

The function f_J is increasing and $f = f_J + f_C$, where f_C is continuous and increasing ([1, p. 186]).

Let δ_a denote the function which is 1 on $[a,\infty)$ and zero otherwise. Note that $\delta_a * \delta_b = \delta_{ab}$. Letting D_f denote the (countable) set of discontinuities of f, we may write

$$f_J = \sum_{\alpha\in D_f} \Delta f(\alpha)\delta_\alpha. \tag{1.2}$$

The series has only non-negative terms and converges absolutely.

Properties. Let $f, g \in S^+$.

(a) $(f * g)_J = f_J * g_J$.

Write $f = f_J + f_C$ and similarly for g. Then

$$f * g = (f_J + f_C) * (g_J + g_C) = f_J * g_J + f_J * g_C + f_C * g_J + f_C * g_C. \tag{1.3}$$

The last three terms are all continuous, and so their jump functions are identically zero. Therefore we need to show $(f_J * g_J)_J = f_J * g_J$.

To see this, use (1.2) for f_J and g_J. Hence

$$f_J * g_J = \sum_{\alpha\in D_f}\sum_{\beta\in D_g} \Delta f(\alpha)\Delta g(\beta)\delta_\alpha * \delta_\beta = \sum_{\alpha\in D_f}\sum_{\beta\in D_g} \Delta f(\alpha)\Delta g(\beta)\delta_{\alpha\beta},$$

which is a sum of the form $\sum_\gamma c_\gamma\delta_\gamma$, i.e. a jump function. Thus $(f_J * g_J)_J = f_J * g_J$ as required.

(b) *For $x \geq 1$, we have*

$$\Delta(f * g)(x) = \sum_{\substack{\alpha\beta = x \\ \alpha \in D_f,\, \beta \in D_g}} \Delta f(\alpha)\Delta g(\beta). \tag{1.4}$$

Take Δ of both sides of (1.3). As the last three terms are all continuous, $\Delta = 0$ for these functions. For the remaining term

$$\Delta(f_J * g_J)(x) = \sum_{\alpha \in D_f,\, \beta \in D_g} \Delta f(\alpha)\Delta g(\beta)\Delta\delta_{\alpha\beta}(x) = \sum_{\substack{\alpha\beta = x \\ \alpha \in D_f,\, \beta \in D_g}} \Delta f(\alpha)\Delta g(\beta),$$

since $\Delta\delta_a(x) = 1$ for $x = a$ and zero otherwise.

(c) $D_{f*g} = D_f D_g = \{\alpha\beta : \alpha \in D_f,\ \beta \in D_g\}$.

For, if $x \notin D_f D_g$ (i.e. $x \neq \alpha\beta$ for any $\alpha \in D_f$ and $\beta \in D_g$), then there is no contribution to the sum in (1.4). Hence $\Delta(f * g)(x) = 0$ and $x \notin D_{f*g}$. Thus $D_{f*g} \subset D_f D_g$.

For the converse, if $x \in D_f D_g$ then $x = \alpha\beta$ for some $\alpha \in D_f$ and $\beta \in D_g$, so that

$$\Delta(f * g)(x) = \Delta(f * g)(\alpha\beta) \geq \Delta f(\alpha)\Delta g(\beta) > 0,$$

as all the other terms in (1.4) are non-negative. Hence $x \in D_{f*g}$ and $D_{f*g} = D_f D_g$ follows.

(d) For $f \in S$, let f_L denote the function $f_L(x) = \int_1^x \log t \, df(t)$. Then $\Delta f_L(x) = \Delta f(x) \log x$ (see [3, p. 341]) and hence $(f_J)_L = (f_L)_J$. (Both sides equal $\sum_{\alpha \in D_f} \Delta f(\alpha) \log \alpha \, \delta_\alpha$.)

The subspace T. Consider those functions in S whose right-hand derivative exists and is continuous in $(1, \infty)$, i.e.

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

exists for each $x > 1$ and f'_+ is continuous here. Let T denote the subspace of such functions which have a finite number of discontinuities per bounded interval. For example, all step functions in S lie in T with $f'_+ \equiv 0$. Further, for $f \in T$, $f'_+ \equiv 0$ if and only if f is a step function. This follows from the fact that if f is continuous on an interval, and f has a continuous one-sided derivative, then in fact f' exists (and of course equals the one-sided derivative)—see [9, p. 355]. Thus on each interval where f is continuous and $f'_+ \equiv 0$, we must have $f' \equiv 0$ so that f is constant here.

Part (d) of 1.1 generalises to functions in T: *if $f, g \in T$ then $f * g \in T$ and*

$$(f * g)'_+ = f'_+ * g_1 + f_{J,1} * g'_+,$$

where g_1 is as before and $f_{J,1} = (f_J)_1$.

Proof. By 1.2(c), $D_{f*g} \subset D_f D_g$, so $f * g$ has at most finitely many discontinuities per bounded interval.

We have, on $(1,\infty)$,

$$(f*g)'_+ = (f_J * g_J)'_+ + (f_J * g_C)'_+ + (f_C * g_J)'_+ + (f_C * g_C)'_+.$$

Now $f_J * g_J$ is again a step function, so $(f_J * g_J)'_+ = 0$. Also, $f'_+ = (f_C)'_+$ hence f_C is continuously differentiable, and similarly for g_C. By 1.1(d), $(f_C * g_C)'_+ = f'_C * g_{C,1}$. For the remaining terms

$$(f_J * g_C)'_+(x) = \bigg(\sum_{\alpha\in D_f} \Delta f(\alpha) g_C\Big(\frac{x}{\alpha}\Big)\bigg)'_+ = \sum_{\alpha\in D_f} \frac{\Delta f(\alpha)}{\alpha} g'_C\Big(\frac{x}{\alpha}\Big).$$

This is clear for $x \notin D_f$ (since then $\alpha \neq x$), but also true if $x \in D_f$ since $g_C(\frac{x}{\alpha}) = 0$ for $x \le \alpha$. Thus $(f_J * g_C)'_+ = f_{J,1} * g'_C$ and similarly $(f_C * g_J)'_+ = f'_C * g_{J,1}$. Putting these together gives

$$(f*g)'_+ = f_{J,1} * g'_C + f'_C * g_{J,1} + f'_C * g_{C,1} = f_{J,1} * g'_+ + f'_+ * g_1.$$

Thus $(f*g)'_+$ is continuous and $f * g \in T$. ∎

1.3. Generalised prime systems. We distinguish between two different types of g-prime system.

DEFINITION 1.2. An *outer g-prime system* is a pair of functions Π, N with $\Pi \in S_0^+$ and $N \in S_1^+$ such that $N = \exp_* \Pi$.

Of course, if $\Pi \in S_0^+$, then $\exp_* \Pi \in S_1^+$, so (Π, N) is an outer g-prime system (with $N = \exp_* \Pi$). On the other hand, if $N \in S_1^+$, then $N = \exp_* \Pi$ for some $\Pi \in S_0$ by 1.1(b), but Π need not be increasing. If Π is increasing, then we say N *determines an outer g-prime system*. The above definition is somewhat more general than the usual 'generalised primes', since we have not mentioned the equivalent of the prime counting function $\pi(x)$.

DEFINITION 1.3. A *g-prime system* is an outer g-prime system for which there exists $\pi \in S_0^+$ such that

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi(x^{1/k}).$$

We say N *determines a g-prime system* if there exists such an increasing $\pi \in S_0$.

REMARKS. (a) As such, $\pi(x)$ is given by

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \Pi(x^{1/k}).$$

In fact this sum always converges for $\Pi \in S^+$ (since $\Pi(x^{1/k})$ decreases with k and $\sum_{k=1}^{\infty} \mu(k)/k$ converges). But of course π need not be increasing.

(b) A g-prime system is *discrete* if π is a step function with integer jumps. In this case the g-primes are the discontinuities of π and the step is the multiplicity.

(c) An outer g-prime system is *continuous* if N (and hence Π, see below) is continuous in $(1, \infty)$.

(d) For an outer g-prime system (Π, N), let $\psi = \Pi_L$ (i.e. $\psi(x) = \int_1^x \log t \, d\Pi(t)$) denote the *generalised Chebyshev function*.

Note that $\psi \in S_0^+$, and that $N = \exp_* \Pi$ is equivalent to $\psi * N = N_L$ (see [3] and [5]).

If N determines a g-prime system and $N(x) = cx + O(x(\log x)^{-\gamma})$ for some $\gamma > 3/2$, then by Beurling's prime number theorem (3) (see [2] or [4]), $\psi(x) \sim x$. Also $\psi_1(x) = \log x + \kappa + o(1)$ for some constant κ, where $\psi_1(x) = \int_1^x (1/t) \, d\psi(t)$.

(e) Applying 1.2(c) to outer g-primes shows that $D_{N_L} = D_N D_\psi$. But $D_{N_L} = D_N \setminus \{1\}$, so $D_N \setminus \{1\} = D_N D_\psi$.

Theorem 1.1. *Let (Π, N) be an outer g-prime system. Then*

(a) *$\Delta\Pi \le \Delta N$. In particular, Π is continuous at the points of continuity of N.*
(b) *(Π_J, N_J) is an outer g-prime system.*

Proof. (a) Apply Δ to both sides of $\psi * N = N_L$ and use $\Delta N_L(x) = \Delta N(x) \log x$. Thus

$$\Delta N(x) \log x = \Delta(\psi * (N_J + N_C))(x) = \Delta(\psi * N_J)(x) \ge \Delta\psi(x),$$

since N has a jump of 1 at 1. But $\Delta\psi(x) = \Delta\Pi(x) \log x$, so $\Delta\Pi \le \Delta N$ and (a) follows.

(b) Take the jump function of both sides of the equation $\psi * N = N_L$. Thus $(\psi * N)_J = (N_L)_J$. By 1.2(a) and (d) this is $\psi_J * N_J = (N_J)_L$. Since N_J and ψ_J are increasing, this implies (Π_J, N_J) forms a g-prime system. ∎

Theorem 1.1 gives a useful necessary condition for $N \in S_1^+$ to determine a g-prime system, namely that N_J must determine a g-prime system. Of course, this is of no use if N is continuous, in which case $N_J = i$, the identity with respect to $*$.

Finally, we remark that if N is continuously differentiable on $(1, \infty)$, then so is ψ and $\psi' = N_L' - N' * \psi_1$. The proof follows 1.1(d) with $f = N$ and $g = \psi$, so that $(f * g)' = N_L'$. The first integral on the RHS of (1.1) then

(3) This is usually formulated for g-prime systems, but actually proved for outer g-prime systems. No use of $\pi(x)$ being increasing is made, only that of $\Pi(x)$.

tends to $f' * g_1 = N' * \psi_1$, while the second integral lies between

$$\frac{N(1)}{h} \int_x^{x+h} d\psi(t) \quad \text{and} \quad \frac{N(1+h)}{h} \int_x^{x+h} d\psi(t).$$

Since N is right-continuous at 1, it follows that $(\psi(x+h) - \psi(x))/h$ must therefore tend to a limit as $h \to 0^+$. Similarly for $h \to 0^-$.

In the same way, $N \in T$ implies $\psi \in T$.

2. Continuous g-prime systems with $N(x) - cx$ periodic. Suppose now that $N \in S_1$ and $N(x) = cx - R(x)$ where $R(x)$ is periodic for some $c > 0$. Extend R to the whole real line by periodicity. Thus R is right continuous, locally of bounded variation, and $R(1) = c - 1$.

In what follows we shall always write $N = \exp_* \Pi$ where $\Pi \in S_0$.

Theorem 2.1. *Let $N(x) = cx - R(x) \in S_1^+$, where R is continuously differentiable and periodic, and $c > 0$. Then Π is increasing if and only if R is constant; i.e. $N(x) = cx + 1 - c$ for $x \geq 1$.*

Proof. If R is constant, then $N(x) = cx + 1 - c$ $(x \geq 1)$ and $\hat{N}(s) = 1 + c/(s-1)$. Thus

$$\hat{\psi}(s) = \frac{-\hat{N}'(s)}{\hat{N}(s)} = \frac{1}{s-1} - \frac{1}{s+c-1},$$

which implies $\psi'(x) = 1 - x^{-c} \geq 0$. Hence Π is increasing.

For the converse, let R be non-constant and suppose, for a contradiction, that Π is increasing. Equivalently, suppose that $\psi' \geq 0$. Differentiate the relation $N_L = \psi * N$, using 1.1(d). Thus, for $x > 1$,

$$N'(x) \log x = (N' * \psi_1)(x) + \psi'(x) \geq (N' * \psi_1)(x), \tag{2.1}$$

where $\psi_1(x) = \int_1^x (1/t)\, d\psi(t)$. Since $N' = c - R'$, this becomes

$$R'(x) \log x - (R' * \psi_1)(x) \leq c \log x - c\psi_1(x).$$

By Beurling's PNT, the right-hand side tends to a limit as $x \to \infty$, so for some constant A and all $x > 1$,

$$R'(x) \log x - (R' * \psi_1)(x) \leq A. \tag{2.2}$$

Let P be a period of R. Extend R to $\mathbb{R}$ by periodicity. By continuity and periodicity of R' there exists $x_0 \in [0, P]$ such that

$$R'(x_0) = \max_{x \in \mathbb{R}} R'(x).$$

Furthermore, for $\delta > 0$ sufficiently small, the set of points x in $[0, P]$ for which $R'(x) \leq R'(x_0) - \delta$ contains an interval, say $[\alpha, \beta]$ with $0 < \alpha < \beta < P$. (If not then R' is constant, which forces R constant.) Let $x = nP + x_0$ in (2.2) where $n \in \mathbb{N}$. Since $\log(nP+x_0) = \psi_1(nP+x_0)+O(1)$ and R' has period P, (2.2) can be written as

$$\int_{1-}^{nP+x_0} R'(x_0) - R'\left(P\left\{\frac{nP+x_0}{tP}\right\}\right) d\psi_1(t) \le A. \tag{2.3}$$

(A different constant A.) Note that the integrand is non-negative. Furthermore, the integrand is at least δ for $t \in \left[\frac{nP+x_0}{kP+\beta}, \frac{nP+x_0}{kP+\alpha}\right]$ for each positive integer $k \le n$.

Let K be a fixed positive integer less than n. Thus the LHS of (2.3) is at least

$$\sum_{k=1}^{K} \int_{\frac{nP+x_0}{kP+\beta}}^{\frac{nP+x_0}{kP+\alpha}} \delta\, d\psi_1(t) = \delta \sum_{k=1}^{K}\left(\psi_1\left(\frac{nP+x_0}{kP+\alpha}\right) - \psi_1\left(\frac{nP+x_0}{kP+\beta}\right)\right).$$

As $n \to \infty$, the kth term in the sum tends to

$$\log\left(\frac{kP+\beta}{kP+\alpha}\right) = -\log\left(1 - \frac{\beta-\alpha}{kP+\beta}\right) \ge \frac{\beta-\alpha}{kP+\beta}.$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} \int_{1-}^{nP+x_0} \left(R'(x_0) - R'\left(P\left\{\frac{nP+x_0}{tP}\right\}\right)\right) d\psi_1(t) &\ge \delta(\beta-\alpha)\sum_{k=1}^{K}\frac{1}{kP+\beta}\\ &\ge \delta' \log K\end{aligned}$$

for some $\delta' > 0$. This is true for every $K \ge 1$ so the left-hand side of (2.3) cannot be bounded. This contradiction proves the theorem. ∎

Remark. (i) We see that $N(x) = cx+1-c$ determines an outer g-prime system for every $c > 0$. What about g-prime systems, i.e. for which values of c is π increasing? We show in the appendix that this happens for $0 < c \le \lambda$ and fails for $c > \lambda$ for some $\lambda > 2$.

(ii) The proof of Theorem 2.1 can be readily extended to the case where R is absolutely continuous and $R'(x)$ has a maximum value, say at $x = x_0$, and the set

$$\{x \in [0,P] : R'(x) \le R'(x_0) - \delta\}$$

contains an interval, for some $\delta > 0$.

In particular this shows that none of the functions N_λ with $\lambda > 1$ (as defined in [7, Section 3]) forms part of a g-prime system, except of course when $\rho_\lambda = 0$. (To recall: $N_\lambda(x) = x - R_\lambda(x)$ for $x \ge 1$ and zero otherwise, where $R_\lambda(x)$ is periodic with period 1 and defined for $0 \le x < 1$ by $R_\lambda(x) = \rho_\lambda(\zeta(1-\lambda, 1-x) - \zeta(1-\lambda))$. Here ρ_λ is a continuous function of λ with $\rho_1 = 1$.)

For $\lambda > 2$, this follows from Theorem 2.1 since R_λ is continuously differentiable and non-constant. For $1 < \lambda \le 2$, this follows on noting that R_λ is absolutely continuous and R'_λ is maximum at $0+$.

3. G-prime systems with $N(x) - cx$ periodic and finitely many discontinuities. Suppose now that N has discontinuities (other than at 1). To check whether N comes from a g-prime system we consider its jump function N_J. By Theorem 1.1, a necessary condition that N determines a g-prime system is that N_J does.

Our strategy for determining the possible N will be as follows. Writing $N = N_J + N_C$, we first show by extending Theorem 2.1 that we must have $N_C(x) = a(x-1)$ for some $a \ge 0$. Then we show that the discontinuities must occur at the (rational) integers and that the period, say P, is an integer. Writing a_n for the jump at n we therefore have $a_{n+P} = a_n$ for $n \ge 2$. Next we show that $a_{1+P} = a_1$ is forced, so a_n is truly periodic. Using a result of Saias and Weingartner [8] on Dirichlet series with periodic coefficients, we deduce that (i) a_n must be even $\pmod P$, and (ii) a_n is multiplicative. We are then in a position to deduce $N_C \equiv 0$ (i.e. N is a step function) and determine exactly which N arise from g-prime systems.

First we extend Theorem 2.1 to members of T.

Theorem 3.1. *Let $N(x) = cx - R(x) \in T$, where R is periodic and such that Π is increasing. Then $N(x) = N_J(x) + a(x-1)$ for some $a \ge 0$.*

Proof. We proceed as in the proof of Theorem 2.1 but with R'_+ in place of R'. Now (2.1) becomes

$$N'_+(x)\log x = (N'_+ * \psi_1)(x) + (N_{J,1} * \psi'_+)(x) \ge (N'_+ * \psi_1)(x),$$

and (2.2) still holds with R' replaced by R'_+. If R'_+ is not constant, then as before, we can find an $x_0 \in [0, P]$ which maximises R'_+ and for which $R'_+(x) \le R'_+(x_0) - \delta$ holds throughout some interval for some (sufficiently small) $\delta > 0$. We obtain a contradiction as before and hence N'_+ is constant.

But N has finitely many discontinuities in bounded intervals, so $N'_+ = (N_C)'_+$. So $N'_+ \equiv a$ implies (since N_C is continuous) that $N_C(x) = a(x-1)$, using $N_C(1) = 0$. Since N_C is increasing, we must have $a \ge 0$. ∎

Later on, we shall see that the only possible value of a is 0.

Notation. Let λ denote the total jump of N per interval of length P, i.e. $N_J(x+P) - N_J(x) = \lambda$ for $x \ge 1$. Thus $N_J(x) = (\lambda/P)x + O(1)$ and, by integration by parts, $(N_J)_L(x) = (\lambda/P)x\log x + O(x)$. Note that $\lambda = 0$ implies N is continuous, while $\lambda = cP$ implies $N = N_J$.

For the following, D_N denotes the set of discontinuities of N in $(0, \infty)$ and $D_N^* = D_N \cap (1, P+1]$. We suppose that D_N^* is a finite, but non-empty, set.

PROPOSITION 3.2. *Let D_N^* have k elements. Suppose $\alpha \in D_N$ is such that α is irrational. Then there are at most k^2 numbers $\beta \in D_N$ such that $\alpha\beta \in D_N$.*

Proof. Suppose, for a contradiction, that there are $l > k^2$ numbers $\beta \in D_N$ such that $\alpha\beta \in D_N$. Let $D_N^* = \{c_1, \dots, c_k\}$. Each β is of the form $nP + c_i$. There are k choices for c_i so some c_{i_0} will appear at least $k+1$ times. (If not and all appear at most k times, then there can be at most k^2 such numbers β.)

Thus we have (at least) $k+1$ equations

$$\alpha(nP + c_{i_0}) = mP + c_j,$$

with (possibly different) $m, n \in \mathbb{N}$ and some $c_j \in D_N^*$. As D_N^* has only k elements, at least one c_j must occur twice, i.e. there exist positive integers n_1, n_2, m_1, m_2 such that

$$\alpha(n_1 P + c_{i_0}) = m_1 P + c_{j_0} \quad \text{and} \quad \alpha(n_2 P + c_{i_0}) = m_2 P + c_{j_0}.$$

Note that $n_1 \neq n_2$ and $m_1 \neq m_2$, otherwise they are not genuinely different equations. Subtracting these two gives

$$\alpha(n_2 - n_1) = m_2 - m_1,$$

and α is rational—a contradiction. ∎

PROPOSITION 3.3. *The set D_N contains only rational numbers and P is rational.*

Proof. By 1.2(a) and Theorem 1.1,

$$(N_J)_L(x) = (N_J * \psi_J)(x) = \sum_{\substack{\alpha\beta \le x \\ \alpha,\beta \in D_N}} \varDelta N(\alpha)\varDelta\psi(\beta). \tag{3.1}$$

Since $(N_J)_L(x) = (\lambda/P)x\log x + O(x)$ and $D_\psi D_N = D_{N_L} = D_N \setminus \{1\}$, we may rewrite (3.1) as

$$\sum_{\alpha \le x} \varDelta N(\alpha) \sum_{\substack{\beta \le x/\alpha \\ \alpha\beta \in D_N}} \varDelta\psi(\beta) = \frac{\lambda}{P} x \log x + O(x). \tag{3.2}$$

For α irrational, by Proposition 3.2 there are at most k^2 possible βs for which $\alpha\beta \in D_N$, where $k = |D_N^*|$. For each such β, $\varDelta\psi(\beta) \le \varDelta N(\beta)\log\beta \le C\log\beta$ for some C. Hence the inner sum on the left of (3.2) is at most $Ck^2\log(x/\alpha)$. Thus the contribution of irrational α to the LHS of (3.2) is less than

$$Ck^2 \sum_{\alpha \le x} \varDelta N(\alpha) \log\frac{x}{\alpha} = Ck^2 \int_{1-}^{x} \log\frac{x}{t}\, dN_J(t) = Ck^2 \int_1^x \frac{N_J(t)}{t}\, dt = O(x).$$

Hence

$$\sum_{\substack{\alpha\le x\\ \alpha\,\text{rational}}} \Delta N(\alpha) \sum_{\substack{\beta\le x/\alpha\\ \alpha\beta\in D_N}} \Delta\psi(\beta) = \frac{\lambda}{P} x\log x + O(x). \tag{3.3}$$

But the LHS of (3.3) is (using Beurling's PNT for $\psi_J(x)$)

$$\sum_{\substack{\alpha\le x\\ \alpha\,\text{rational}}} \Delta N(\alpha)\psi_J\left(\frac{x}{\alpha}\right) \sim x \sum_{\substack{\alpha\le x\\ \alpha\,\text{rational}}} \frac{\Delta N(\alpha)}{\alpha}. \tag{3.4}$$

Now

$$N_{J,\mathbb{Q}}(x) := \sum_{\substack{\alpha\le x\\ \alpha\,\text{rational}}} \Delta N(\alpha) = \frac{\mu}{P} x + O(1)$$

for some $\mu \le \lambda$ by periodicity. (Precisely, μ is the jump per interval of length P from the rational discontinuities.) The RHS of (3.4) is therefore

$$x\int_1^x \frac{1}{t}\, dN_{J,\mathbb{Q}}(t) = x\int_1^x \frac{N_{J,\mathbb{Q}}(t)}{t^2}\, dt + O(x) = \frac{\mu}{P} x\log x + O(x).$$

It follows that $\mu = \lambda$ and there are no irrational numbers in D_N.

Finally, $\alpha \in D_N$ with $\alpha > 1$ implies $\alpha + P \in D_N$ by periodicity. As D_N contains only rationals, this forces P rational. ∎

Proposition 3.4. *$D_N \subset \mathbb{N}$ and $P \in \mathbb{N}$.*

Proof. Since $D_N \setminus \{1\} = D_{\psi * N} = D_\psi D_N$, if $\alpha \in D_\psi$ then $\alpha\beta \in D_N$ for every $\beta \in D_N$. In particular (using $D_\psi \subset D_N$), $\alpha \in D_\psi$ implies $\alpha^n \in D_N$ for every $n \in \mathbb{N}$. By periodicity, $\alpha^n - kP \in D_N$ for every integer k provided $\alpha^n - kP \ge 1$.

Now write $\alpha = r/s$ and $P = t/u$ where $r, s, t, u \in \mathbb{N}$ and $(r, s) = (t, u) = 1$. For D_N^* to be finite, the numbers $1 + P\{(\alpha^n - 1)/P\}$ $(n = 1, 2, \ldots)$ (take $k = [(\alpha^n - 1)/P]$ above) must repeat themselves infinitely often. Here $\{x\}$ is the fractional part of x. Thus for infinitely many values of n,

$$\alpha^n - kP = \alpha^{n_0} - k_0 P$$

for some integers k, k_0, and n_0. As such,

$$P = \frac{\alpha^n - \alpha^{n_0}}{k - k_0} = \frac{(r/s)^n - (r/s)^{n_0}}{k - k_0} = \frac{t}{u}.$$

Multiplying through by $(k - k_0)us^{n_0}$ shows that $s^{n-n_0} \mid ur^n$ for infinitely many n. But $(r, s) = 1$, so $s^{n-n_0} \mid u$ for infinitely many n. This is only possible if $s = 1$, i.e. $\alpha \in \mathbb{N}$. Hence $D_\psi \subset \mathbb{N}$.

Consequently, $D_\Pi \subset \mathbb{N}$ also, and $D_{\Pi^{*k}} \subset \mathbb{N}$ for every positive integer k. Since $N = \sum_{k=0}^{\infty} \Pi^{*k}/k!$, it follows that $D_N \subset \mathbb{N}$ as well.

Finally, $m \in D_N$ with $m > 1$ implies $m + P \in D_N$ by periodicity. Since $D_N \subset \mathbb{N}$, this shows that $P \in \mathbb{N}$. ∎

4. Determining the jumps. Now that we have established the discontinuities are at the integers, it remains to determine the possible jumps. Write $a_n = \Delta N(n)$ and $c_n = \Delta\psi(n)$. Thus $a_1 = 1$ and $a_{n+P} = a_n$ for $n > 1$. The equation $\Delta N_L = (\Delta N) * \psi_J$ translates as

$$a_n \log n = \sum_{d|n} c_d a_{n/d}. \tag{4.1}$$

Thus $c_1 = 0$; for a prime p, $c_p = a_p \log p$; and for distinct primes p and q, we have (after some calculation) $c_{pq} = (a_{pq} - a_p a_q) \log pq$.

Next we show that a_n is truly periodic ($a_{n+P} = a_n$ for $n \geq 1$). For the proof, let $\langle \mathbb{P}_{r,P} \rangle$ denote the set of numbers of the form $p_1 \ldots p_k$ where the p_i are distinct primes, all congruent to r (mod P). Here r is coprime to P. Each such set is infinite by Dirichlet's theorem on primes in arithmetic progressions.

Proposition 4.1. $a_{P+1} = 1$.

Proof. First we prove that $a_{P+1} = 0$ or 1.

Let $p_1, \ldots, p_k$ be distinct primes all of the form 1 (mod P), with $k \geq 3$. Let $n = p_1 \cdots p_k$, which is also 1 (mod P). Note that for every $d \,|\, n$, $d = 1$ (mod P), so that $a_d = a_{P+1}$ if $d > 1$. In particular we have $c_{p_i p_j} = a_{P+1}(1 - a_{P+1}) \log p_i p_j$ for any $1 \leq i, j \leq k$ with $i \neq j$. As $c_n \geq 0$, (4.1) implies

$$\begin{aligned} a_{P+1} \log n &\geq \sum_{1 \leq i < j \leq k} c_{p_i p_j} a_{n/p_i p_j} = a_{P+1}^2 (1 - a_{P+1}) \sum_{1 \leq i < j \leq k} \log p_i p_j \\ &= a_{P+1}^2 (1 - a_{P+1})(k-1) \log n. \end{aligned}$$

This is impossible for k sufficiently large unless a_{P+1} equals 0 or 1.

Next we show that $a_{P+1} = 0$ implies $a_n = 0$ for all $n > 1$, and hence that $N_J(x) = 1$ for $x \geq 1$—i.e. the continuous case.

We proceed by induction. Suppose $a_{P+1} = 0$ and that $a_n = 0$ for all $n > 1$ such that [4] $\Omega(n) < k$, some $k \geq 1$. (It is vacuously true for $k = 1$.) Then $a_{nr} = 0$ for all such n and all $r \equiv 1$ (mod P), by periodicity. In particular, we can take $r \in \langle \mathbb{P}_{1,P} \rangle$. Note that this implies $c_{nr} = 0$ also for such n and r.

[4] As usual, $\Omega(n)$ denotes the total number of prime factors of n, while $\omega(n)$ denotes the number of distinct prime factors of n.

Now let n be such that $\Omega(n) = k$. Then, with $r \in \langle \mathbb{P}_{1,P} \rangle$ such that $(n,r) = 1$,

$$a_{nr} \log nr = \sum_{d|nr} c_d a_{nr/d} = \sum_{d_1|n} \sum_{d_2|r} c_{d_1 d_2} a_{nr/d_1 d_2}.$$

Now $d_2 \in \langle \mathbb{P}_{1,P} \rangle$ also, so by assumption, $c_{d_1 d_2} = 0$ if $\Omega(d_1) < k$. Hence only the terms with $\Omega(d_1) = k$ give a contribution, i.e. only if $d_1 = n$. Also $a_{nr} = a_n$ by periodicity. Thus

$$a_n \log nr = \sum_{d_2|r} c_{nd_2} a_{r/d_2} = c_{nr}, \tag{4.2}$$

since only the term with $d_2 = r$ makes a_{r/d_2} non-zero.

Now consider $a_{n^2 r}$ with n and r as above. Then

$$a_{n^2 r} \log n^2 r \geq \sum_{d|r} c_{nd} a_{nr/d}.$$

Using (4.2) and noting that $a_{n^2 r} = a_{n^2}$, we therefore have (5)

$$a_{n^2} \log n^2 r \geq a_n^2 \sum_{d|r} \log nd = \frac{a_n^2}{2} d(r) \log n^2 r,$$

i.e. $2a_{n^2} \geq a_n^2 d(r)$ for all $r \in \langle \mathbb{P}_{1,P} \rangle$ such that $(n,r) = 1$. But r can be chosen such that $d(r)$ is arbitrarily large, and we have a contradiction if $a_n > 0$. Thus $a_n = 0$ is forced.

Hence by induction, $a_n = 0$ for all $n > 1$. ∎

Thus, for the discontinuous case, $\hat{N}_J(s)$ is a Dirichlet series with purely periodic coefficients. Further, if N_J determines a g-prime system, then $\hat{N}_J$ has no zeros in H_1 (6). Now we use the main result of Saias and Weingartner ([8, Corollary]: *Let F be a Dirichlet series with periodic coefficients. Then F does not vanish in H_1 if and only if $F = PL_\chi$, where P is a Dirichlet polynomial with no zeros in H_1 and χ is a Dirichlet charcter.*

Thus $\hat{N}_J = PL_\chi$ for some Dirichlet polynomial P and Dirichlet character χ. We shall see below that the positivity of the coefficients of $\hat{N}_J$ implies that χ must be a principal character, showing that we actually have $\hat{N}_J = Q\zeta$ for some Dirichlet polynomial Q.

PROPOSITION 4.2. *$\hat{N}_J(s) = Q(s)\zeta(s)$ where Q is a Dirichlet polynomial with no zeros in H_1. Furthermore, a_n is even modulo P, i.e. $a_n = a_{(n,P)}$, and $Q(s) = \sum_{d|P} q(d)/d^s$ for some $q(d)$.*

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(5) Using $2\sum_{d|n} \log kd = d(n) \log k^2 n$.

(6) For $\theta \in \mathbb{R}$, H_θ denotes the half-plane $\{s \in \mathbb{C} : \Re s > \theta\}$.

Proof. From above, $\hat{N}_J(s) = P(s)L_\chi(s)$, where $P(s) = \sum_{n=1}^{N} b_n n^{-s}$ say. Extend b_n so that $b_n = 0$ for $n > N$. By inversion,

$$b_n = \sum_{d|n} \mu(d)\chi(d)a_{n/d} = 0 \quad \text{ for } n > N.$$

In particular, for every prime $p > N$, $a_p = \chi(p)$. A simple induction on $\Omega(n)$ shows that, more generally, $a_n = \chi(n)$ whenever all the prime factors of n are greater than N. Consequently, for all such n, $a_n = 0$ or 1 (since $a_n \geq 0$ while $\chi(n) = 0$ or a root of unity).

Now let $p > \max\{N, P\}$ be prime. Then $p \equiv r \pmod{P}$ for some r with $(r, P) = 1$. Let $n = p^{\phi(P)}$. Then $n \equiv r^{\phi(P)} \equiv 1 \pmod{P}$ and hence

$$1 = a_1 = a_n = \chi(n) = \chi(p^{\phi(P)}) = \chi(p)^{\phi(P)}.$$

But $\chi(p) = 0$ or 1, so $\chi(p) = 1$ for all sufficiently large p.

This implies χ must be a principal character. For suppose χ is a character modulo m. Let $(r, m) = 1$. For a sufficiently large prime p in each residue class $r \pmod{m}$, $1 = \chi(p) = \chi(r)$ by periodicity. Thus $\chi(r) = 1$ whenever $(r, m) = 1$, i.e. χ is principal. Thus

$$\hat{N}_J(s) = P(s)L_{\chi_0}(s) = P(s)\zeta(s)\prod_{p|m}\left(1 - \frac{1}{p^s}\right) = Q(s)\zeta(s),$$

where Q is again a Dirichlet polynomial, non-zero in H_1. Denoting the coefficients of Q by $q(n)$, we see that $q(1) = 1$, $q(n) = 0$ for n sufficiently large, and

$$a_n = \sum_{d|n} q(d).$$

To show a_n is even modulo P, we first show that for $d \mid P$, $a_{pd} = a_d$ for all sufficiently large primes p. It is true for $d = 1$, so suppose it is true if $\Omega(d) < k$ for some $k \geq 1$.

Let $d \mid P$ be such that $\Omega(d) = k$. Let p be prime and sufficiently large so that $(p, d) = 1$ and $q(pd) = 0$. Then

$$\begin{aligned} 0 = q(pd) &= \sum_{c|pd} \mu(c)a_{pd/c} = \sum_{c|d} \mu(c)a_{pd/c} + \sum_{c|d} \mu(pc)a_{d/c} \\ &= a_{pd} + \sum_{\substack{c|d \\ c>1}} \mu(c)a_{pd/c} - \sum_{c|d} \mu(c)a_{d/c} = a_{pd} - a_d \end{aligned}$$

since $a_{pd/c} = a_{d/c}$ as $\Omega(d/c) < k$ in the first sum.

Let $d = (n, P)$. Then $(n/d, P/d) = 1$ and there exist arbitrarily large primes p congruent to $n/d \pmod{P/d}$. For such primes p, $pd \equiv n \pmod{P}$, and by periodicity $a_n = a_{pd} = a_d$ for p sufficiently large. Thus $a_n = a_{(n,P)}$.

As a result, we can write

$$\hat{N}_J(s) = \sum_{d|P} \sum_{\substack{n=1\\(n,P)=d}}^{\infty} \frac{a_n}{n^s} = \sum_{d|P} \frac{a_d}{d^s} \sum_{\substack{m=1\\(m,P/d)=1}}^{\infty} \frac{1}{m^s} = \sum_{d|P} \frac{a_d}{d^s} \prod_{p|P/d}\Big(1-\frac{1}{p^s}\Big)\zeta(s)$$
$$= Q(s)\zeta(s),$$

which shows that $q(n)$ is supported on the divisors of P. ∎

THEOREM 4.3. *a_n is multiplicative.*

Proof. Equivalently, we show $q(n)$ is multiplicative. Let the period be $P = p_1^{m_1}\cdots p_k^{m_k}$. Write

$$Q(s) = \sum_{d|P} \frac{q(d)}{d^s} = \exp\Big\{\sum_{n=1}^{\infty} \frac{t(n)}{n^s}\Big\}$$

for some $t(n)$, where $t(1) = 0$. Since $\hat{N}_J(s) = \exp\{\sum_{n=1}^{\infty} b_n/n^s\}$ for some $b_n \ge 0$, Proposition 4.2 implies that $t(n) = b_n \ge 0$ for n not a prime power. The aim is to show that $t(n) = 0$ for such n.

Since the $q(n)$ are supported on the divisors of P, $t(n)$ is supported on the set $\{p_1^{n_1}\cdots p_k^{n_k} : n_1,\dots,n_k \in \mathbb{N}_0\}$.

For each $p \mid P$ let

$$Q_p(s) = \sum_{r=0}^{\infty} \frac{q(p^r)}{p^{rs}}.$$

(This is a polynomial in p^{-s}.) Then

$$\prod_{p|P} Q_p(s) = \exp\Big\{\sum_{n \text{ prime power}} \frac{t(n)}{n^s}\Big\}.$$

Now define $T_1(s)$ and $t_1(n)$ by

$$\text{(4.3)} \qquad \frac{Q(s)}{\prod_{p|P} Q_p(s)} = \exp\{T_1(s)\} = \exp\Big\{\sum_{n=1}^{\infty} \frac{t_1(n)}{n^s}\Big\},$$

i.e. $t_1(n) = t(n)$ for n not a prime power and zero otherwise.

If the Dirichlet series for $T_1(s)$ converges everywhere, then the result follows. Indeed, the LHS of (4.3) is then entire and of order 1, while if $t_1(n_0) > 0$ for some $n_0 > 1$, then the RHS of (4.3) is, for negative s, at least $e^{t_1(n_0)n_0^{-s}}$, which has infinite order. The contradiction implies T_1 is identically zero and $Q = \prod_p Q_p$.

Suppose then that the series for T_1 has a finite abscissa of convergence, say $-\beta$. Since the coefficients are non-negative, $-\beta$ must be a singularity of the function, i.e. $-\beta$ must be a zero of one of the $Q_p(s)$. (As we shall see later, $Q_p(s) \neq 0$ in H_0, so $\beta \geq 0$, but we do not require to know this at this stage.)

We can write down the 'spatial extension' of (4.3). We can think of this as substituting $z_i = p_i^{-s}$. For p prime, let $\tilde{Q}_p(z) = \sum_{r=0}^{\infty} q(p^r)z^r$, so that $\tilde{Q}_p(p^{-s}) = Q_p(s)$. Now define

$$\tilde{Q}(z_1, \ldots, z_k) = \sum_{b_1,\ldots,b_k \geq 0} q(p_1^{b_1} \cdots p_k^{b_k}) z_1^{b_1} \cdots z_k^{b_k}$$

(the series is of course finite) and similarly for $\tilde{T}_1$. Then (4.3) becomes

$$\tag{4.4} \begin{aligned} \frac{\tilde{Q}(z_1, \ldots, z_k)}{\tilde{Q}_{p_1}(z_1) \cdots \tilde{Q}_{p_k}(z_k)} &= \exp\{\tilde{T}_1(z_1, \ldots, z_k)\} \\ &= \exp\Big\{ \sum_{n_1,\ldots,n_k \geq 0} t_1(p_1^{n_1} \cdots p_k^{n_k}) z_1^{n_1} \cdots z_k^{n_k} \Big\}. \end{aligned}$$

Since (4.3) holds for $\sigma > -\beta$, (4.4) holds in the domain $\{(z_1, \ldots, z_k) : |z_1| < p_1^{\beta}, \ldots, |z_k| < p_k^{\beta}\}$.

Let r be the smallest positive integer such that $t_1(n) = 0$ whenever $\omega(n) < r$. (Thus $2 \leq r \leq k$.) Put $z_{r+1}, \ldots, z_k = 0$. Then (4.4) becomes

$$\tag{4.5} \frac{\tilde{Q}(z_1, \ldots, z_r)}{\tilde{Q}_{p_1}(z_1) \cdots \tilde{Q}_{p_r}(z_r)} = \exp\Big\{ \sum_{n_1,\ldots,n_r \geq 0} t_1(p_1^{n_1} \cdots p_r^{n_r}) z_1^{n_1} \cdots z_r^{n_r} \Big\}$$

where we identified $\tilde{Q}(z_1, \ldots, z_r)$ with $\tilde{Q}(z_1, \ldots, z_r, 0, \ldots, 0)$. Without loss of generality, we may assume that the numerator and denominator of the left-hand side of (4.5) have no common factors. (If there are any, cancel them, and apply the argument to what remains.)

Let $z_i = x_i$ $(i = 1, \ldots, r)$ be real and positive. Take logs of (4.5) and differentiate with respect to each of the variables $x_1, \ldots, x_r$. This gives

$$\tag{4.6} \begin{aligned} &\sum_{n_1,\ldots,n_r \geq 0} n_1 \cdots n_r t_1(p_1^{n_1} \cdots p_r^{n_r}) x_1^{n_1} \cdots x_r^{n_r} \\ &\qquad = \frac{\partial^r}{\partial x_1 \cdots \partial x_r} \log \tilde{Q}(x_1, \ldots, x_r) = \frac{P(x_1, \ldots, x_r)}{\tilde{Q}(x_1, \ldots, x_r)^r} \end{aligned}$$

for some polynomial P. The crucial point here is that the polynomials $\tilde{Q}_p$ have all disappeared.

Now, $\tilde{Q}_p(p^{\beta}) = 0$ for some $p \,|\, P$, say $p = p_1$. Fix $x_2, \ldots, x_r$ and let $x_1 \to p_1^{\beta}$ through real values from below. If $\tilde{Q}(p_1^{\beta}, x_2, \ldots, x_r) \neq 0$, then the RHS of (4.6) remains bounded, and hence (since $t_1(n) \geq 0$) the series

$$\tag{4.7} \sum_{n_1,\ldots,n_k \geq 1} n_1 \cdots n_r t_1(p_1^{n_1} \cdots p_r^{n_r}) p_1^{n_1 \beta} x_2^{n_2} \cdots x_r^{n_r} \quad \text{converges,}$$

while the LHS of (4.5) tends to infinity, so

$$\sum_{n_1,\dots,n_r\ge 0} t_1(p_1^{n_1}\cdots p_r^{n_r})p_1^{n_1\beta}x_2^{n_2}\cdots x_r^{n_r} \quad \text{diverges.} \tag{4.8}$$

But (4.7) and (4.8) are in contradiction since in (4.8) we actually require $n_1,\dots,n_r \ge 1$ (if any $n_j = 0$, there is no contribution to the sum as $\omega(p_1^{n_1}\cdots p_r^{n_r}) < r$).

Thus this forces $\tilde{Q}(p_1^{\beta}, x_2,\dots,x_r) = 0$ for every x_i $(i = 2,\dots,r)$ in some interval, and hence for all such x_i, since $\tilde{Q}$ is a polynomial. But this implies $x_1 - p_1^{\beta}$ is a factor of both $\tilde{Q}(x_1,\dots,x_r)$ and $\tilde{Q}_{p_1}(x_1)$—a contradiction. Hence T_1 is identically zero and the result follows. ∎

Remark. This proves Theorem B of the introduction.

Determining a for which $N_J(x) + a(x-1)$ is a g-prime system. The problem thus reduces to determining $Q_p(s)$. We shall see in Theorem 4.4 that the zeros of $Q_p(s)$ all have real part less than or equal to zero. We use this fact to deduce that the only permissible value of a is 0.

For, using this fact, the zeros of Q then all lie in $\mathbb{C}\setminus H_0$. In particular, in H_0, the zeros of $\hat{N}_J$ are precisely the zeros of ζ and hence $\hat{N}_J$ has no real positive zeros. Indeed, $Q(\sigma) > 0$ for $\sigma > 0$ since $Q(\sigma)$ is real and non-zero here and as $\sigma\to\infty$, $Q(\sigma)\to 1$. Thus $\hat{N}_J(\sigma) < 0$ for $0 < \sigma < 1$. Also $\hat{N}(\sigma) = \hat{N}_J(\sigma) - a/(1-\sigma) < 0$ for $\sigma\in(0,1)$.

Now $N = N_J + N_C$ and $\psi = \psi_J + \psi_C$ and by assumption ψ_C is increasing. (Here $N_C(x) = a(x-1)$, so that $\hat{N}_C(s) = a/(s-1)$.) Thus

$$\hat{\psi}_C(s) = \hat{\psi}(s) - \hat{\psi}_J(s) = \frac{\hat{N}_J'(s)}{\hat{N}_J(s)} - \frac{\hat{N}'(s)}{\hat{N}(s)},$$

since (Π_J, N_J) and (Π, N) are g-prime systems. Note that $\hat{\psi}_C \ne -\hat{N}_C'/\hat{N}_C$ as (Π_C, N_C) is not a g-prime system (indeed $N_C(1) = 0$).

Both $\psi(s)$ and $\psi_J(s)$ are meromorphic functions, holomorphic in $\overline{H}_1\setminus\{1\}$, with simple poles at $s = 1$ and residue 1. Thus $\psi_C(s)$ has a removable singularity at 1 and poles at the zeros of $\hat{N}$ and $\hat{N}_J$.

Landau's oscillation theorem (cf. [3, p. 137]) applied to $\hat{\psi}_C$ implies that $\hat{\psi}_C$ has a singularity at its abscissa of convergence, say θ. Of course $\theta < 1$ must be a zero of $\hat{N}$ or $\hat{N}_J$. But neither $\hat{N}$ nor $\hat{N}_J$ has real positive zeros, so $\theta \le 0$. But then $\hat{\psi}_C$ must be holomorphic in H_0, implying that $\hat{N}$ and $\hat{N}_J$ have the same zeros here, i.e. all the non-trivial Riemann zeros. But at each such zero, say ρ, also $\hat{N}_C(\rho) = 0$. This is impossible as $\hat{N}_C$ has no zeros, except if $a = 0$.

Hence $a = 0$ is forced and $N = N_J$.

Criteria for g-primes. We have $\hat{N}(s) = Q(s)\zeta(s) = \exp\{T(s) + \log\zeta(s)\} = \exp\{\hat{\varPi}(s)\}$. Thus

$$\hat{\varPi}(s) = \sum_{n=1}^{\infty} \frac{\varLambda_1(n) + t(n)}{n^s}.$$

For $\varPi$ to be increasing, the coefficients of $\hat{\varPi}$ must be non-negative, that is, $\varLambda_1(n) + t(n) \geq 0$ for all $n \in \mathbb{N}$. As $t(n)$ is supported on the powers of the prime divisors of P, we have

$$(*) \qquad \varPi \text{ is increasing } \Leftrightarrow t(p^k) \geq -\frac{1}{k} \text{ for } p \mid P \text{ and } k \in \mathbb{N}.$$

Note that $t(p) = q(p) = a_p - 1 \geq -1$ for p prime, so $(*)$ is satisfied for $k = 1$.

Turning now to $\pi(x)$, we observe that N determines g-primes if π is increasing, where $\pi(x) = \sum_{k=1}^{\infty} (\mu(k)/k)\varPi(x^{1/k})$. But

$$\hat{\pi}(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k}\hat{\varPi}(ks) = \sum_{p} \frac{1}{p^s} + \sum_{k,n\geq 1} \frac{\mu(k)t(n)}{kn^{ks}} = \sum_{n=1}^{\infty} \frac{\pi_n}{n^s},$$

say, for some coefficients π_n. Thus π is increasing if and only if $\pi_n \geq 0$ for all n. Now $\pi_1 = 0$ and $\pi_p = 1 + t(p) \geq 0$ for p prime, while $\pi_n = 0$ for n not a prime power. Hence

$$(**) \qquad \pi \text{ is increasing } \Leftrightarrow \sum_{d\mid n} \frac{\mu(d)}{d} t(p^{n/d}) \geq 0 \text{ for } n \geq 2 \text{ and } p \mid P.$$

To deal with these criteria, it is useful to write them in terms of the zeros of $\tilde{Q}_p$.

The zeros of $\tilde{Q}_p$. Let $p \mid P$ and let k be the degree of $\tilde{Q}_p$. Then $\tilde{Q}_p$ has k zeros $\lambda_1, \ldots, \lambda_k$. Letting $\mu_r = 1/\lambda_r$ gives $\tilde{Q}_p(z) = (1 - \mu_1 z)\cdots(1 - \mu_k z)$ and

$$\log \tilde{Q}_p(z) = \sum_{r=1}^{k} \log(1 - \mu_r z) = -\sum_{n=1}^{\infty} \biggl(\frac{1}{n}\sum_{r=1}^{k} \mu_r^n\biggr) z^n.$$

Since $\log \tilde{Q}_p(z) = \sum_{r=1}^{\infty} t(p^r) z^r$, equating coefficients gives

$$t(p^n) = -\frac{1}{n}\sum_{r=1}^{k} \mu_r^n.$$

Hence $(*)$ is satisfied for a prime $p \mid P$ if and only if

$$(\dagger) \qquad \tau_n := \sum_{r=1}^{k} \mu_r^n \leq 1 \quad \text{for } n \in \mathbb{N}.$$

Turning to $(**)$, let $s_n(w) = \sum_{d|n} \mu(d) w^{n/d}$ for $w \in \mathbb{C}$. Then

$$\sum_{d|n} \frac{\mu(d)}{d} t(p^{n/d}) = -\frac{1}{n} \sum_{r=1}^{k} s_n(\mu_r),$$

and $(**)$ is satisfied for a prime $p \mid P$ if and only if

$$(\dagger\dagger) \qquad \sum_{r=1}^{k} s_n(\mu_r) \le 0 \quad \text{for } n \ge 2.$$

THEOREM 4.4. *Let $\tilde{Q}_p$, k and $\mu_1, \ldots, \mu_k$ be as above. For $k = 1$, $(\dagger)$ is satisfied if and only if $|\mu_1| \le 1$. For $k > 1$, if $(\dagger)$ is satisfied, then $|\mu_r| < 1$ for all r.*

Proof. For $k = 1$ this is trivial so assume $k > 1$ and that $(\dagger)$ is satisfied. The numbers $\mu_1, \ldots, \mu_k$ are either real or occur in complex conjugate pairs. Denote the real ones by $\mu_1, \ldots, \mu_l$ and the complex ones by $\nu_1 e^{\pm i\theta_1}, \ldots, \nu_m e^{\pm i\theta_m}$ where $\nu_r > 0$ and $0 < \theta_r < \pi$. Thus $(\dagger)$ becomes

$$(4.9) \qquad \tau_n = \mu_1^n + \cdots + \mu_l^n + 2(\nu_1^n \cos n\theta_1 + \cdots + \nu_m^n \cos n\theta_m) \le 1.$$

Assume without loss of generality that $|\mu_1| \ge \cdots \ge |\mu_l|$ and $\nu_1 \ge \cdots \ge \nu_m$. If $|\mu_1| \ge 1$, then $\mu_1^{2n} \ge 1$ and (4.9) implies

$$\nu_1^{2n} \cos 2n\theta_1 + \cdots + \nu_m^{2n} \cos 2n\theta_m \le 0 \quad \text{for all } n \in \mathbb{N}.$$

Suppose $\nu_1 = \cdots = \nu_q > \nu_{q+1}$ for some $q \le m$; then this involves

$$(4.10) \qquad \cos 2n\theta_1 + \cdots + \cos 2n\theta_q \le \frac{a}{A^n} \quad (n \in \mathbb{N})$$

for some a and $A > 1$. But this is impossible as we show below.

Thus if any μ_r is real, then $|\mu_r| < 1$. Now suppose $\nu_1 = \cdots = \nu_q > \nu_{q+1}$ and $\nu_1 \ge 1$. Then (4.9) implies

$$(4.11) \qquad \cos 2n\theta_1 + \cdots + \cos 2n\theta_q \le \frac{1}{2} + \frac{a}{A^n} \quad (n \in \mathbb{N})$$

for some a and $A > 1$. We show this is impossible, which in turn implies (4.10) is impossible.

Let $\phi_r = \theta_r/\pi$. By Dirichlet's theorem (see [6, p. 170]), the numbers $n\phi_1, \ldots, n\phi_q$ can be made arbitrarily close to q integers simultaneously, i.e. given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|n\phi_r - K_r| < \varepsilon$ for $r = 1, \ldots, q$ and integers K_r. Thus, for some $|\delta_r| < \varepsilon$,

$$\cos 2n\theta_r = \cos 2\pi n\phi_r = \cos 2\pi(K_r + \delta_r) = \cos 2\pi\delta_r > \cos 2\pi\varepsilon,$$

which can be made as close to 1 as we please. The inequalities (4.10) and (4.11) are impossible and hence $\nu_r < 1$ for all r. ∎

To deal with $(\dagger\dagger)$ we require the following.

LEMMA 4.5.

(a) *Let* $w \in \mathbb{R}$. *Then* $s_n(w) \le 0$ *for all* $n > 1$ *if and only if* $w = 0$ *or* 1.
(b) *Let* $w_1, \dots, w_k$ *be non-zero complex numbers of modulus less than one, and symmetric about* $\mathbb{R}$, *i.e.* $\overline{w}_i = w_j$ *for some* j. *Then* $s_n(w_1) + \dots + s_n(w_k)$ *changes sign infinitely often.*

Proof. (a) For p prime, $s_p(w) = w^p - w > 0$ for $w > 1$, while for p an odd prime, $s_{2p}(w) = w^{2p} - w^p - w^2 + w > 0$ whenever $w < -1$ for p sufficiently large. This leaves $-1 \le w \le 1$. For $w = 1$, $s_n(w) = 0$ for all $n > 1$ and the condition $s_n(w) \le 0$ is satisfied, while for $w = -1$, $s_n(w) = 0$ for $n > 2$ and $s_2(-1) = 2$, so the condition (narrowly) fails in this case. For $w = 0$ the result holds trivially.

Now suppose $-1 < w < 1$, $w \ne 0$. Consider the entire function defined by the Dirichlet series

$$H_w(s) = \sum_{n=1}^{\infty} \frac{w^n}{n^s}.$$

Note that

$$\frac{H_w(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{s_n(w)}{n^s}.$$

Now if $s_n(w)$ is ultimately of one sign, then the abscissa of convergence of this series must be a singularity of H_w/ζ. This singularity must be real, and there can be no others further to the right. But the first real singularity (furthest to the right) is at -2, so H_w must be zero at all the complex zeros of ζ. This is a contradiction as H_w, being bounded in any strip, has at most $O(T)$ zeros up to height T here.

(b) This time

$$\frac{H_{w_1}(s) + \dots + H_{w_k}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{s_n(w_1) + \dots + s_n(w_k)}{n^s}.$$

If $s_n(w_1) + \dots + s_n(w_k)$ is ultimately of one sign, then the abscissa of convergence is a singularity of the LHS. Each H_{w_i} is entire, so the first real singularity occurs at -2. As in (a), this gives a contradiction. ∎

Proof of Theorem A. By Lemma 4.5(b), if $k > 1$, (††) cannot be satisfied (for then $|\mu_r| < 1$ for all r). So, for π to be increasing, we require $k = 1$, i.e. $\tilde{Q}_p(z) = 1 + q(p)z$. Hence $\mu_1 = -q(p)$ and (††) holds if and only if $s_n(\mu_1) = s_n(-q(p)) \le 0$ for $n \ge 2$. By (a) of Lemma 4.5, this only happens if $q(p) = 0$ or -1. Thus

$$\hat{N}(s) = \zeta(s) \prod_{p|P} \left(1 + \frac{q(p)}{p^s}\right) = \zeta(s) \prod_{i=1}^{l} \left(1 - \frac{1}{p_i^s}\right)$$

for some prime divisors $p_1, \dots, p_l$ of P. ∎

Outer g-prime systems with $N(x)-cx$ periodic. The condition in Theorem 4.4 does not allow us to determine which coefficients a_n will lead to outer g-prime systems, as they are only necessary and not sufficient. Instead we use the relation

$$kq(p^k)=\sum_{r=1}^{k} rt(p^r)q(p^{k-r}), \tag{4.12}$$

which follows directly from $Q=e^T$. This allows us to calculate $t(p^k)$ explicitly in special cases. Suppose $\tilde{Q}_p$ has degree 1. Then $q(p^r)=0$ for $r>1$ and (4.12) gives $kt(p^k)=-(k-1)t(p^{k-1})q(p)$ for $k\geq 2$. Thus

$$t(p^k)=\frac{(-1)^{k-1}q(p)^k}{k}.$$

As a result, $(*)$ holds if and only if $(-q(p))^k\leq 1$ for all k, which is easily seen to be equivalent to $-1\leq q(p)\leq 1$ for all $p\,|\,P$ (i.e. $0\leq a_p\leq 2$). In particular, we have proven:

Theorem C. *Let $N\in T$ be such that $N(x)-cx$ has squarefree period P. Then N determines an outer g-prime system if and only if*

$$N(x)=\sum_{d|P} q(d)\left[\frac{x}{d}\right],$$

where $q(\cdot)$ is multiplicative, $q(p)\in[-1,1]$, and $c=\prod_{p|P}(1+q(p)/p)$.

For example, the outer g-prime systems for which $N(x)-cx$ has period 6 are given by

$$N(x)=[x]+\lambda\left[\frac{x}{2}\right]+\mu\left[\frac{x}{3}\right]+\lambda\mu\left[\frac{x}{6}\right],$$

where $(\lambda,\mu\in[-1,1])$ and $(1+\lambda/2)(1+\mu/3)=c$.

Appendix. When does $N(x)=cx+1-c$ determine a g-prime system? From the proof of Theorem 2.1 we saw that $\psi'(x)=1-x^{-c}$ for $x\geq 1$. Thus ψ (equivalently Π) is increasing for every $c\geq 0$. What about π? Let $\theta=\pi_L$ be the generalisation of Chebyshev's θ-function. Then $\theta(x)=\sum_{n=1}^{\infty}\mu(n)\psi(x^{1/n})$ so that

$$\theta'(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}x^{1/n-1}\psi'(x^{1/n})=\frac{1}{x}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}(x^{1/n}-x^{(1-c)/n}).$$

Let f be the entire function

$$f(z)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}(e^{z/n}-1)=\sum_{k=1}^{\infty}\frac{z^k}{k!\zeta(k+1)}.$$

Then $e^x\theta'(e^x) = f(x) - f((1-c)x)$ and θ is increasing if and only if

$$(\mathrm{A}_c) \qquad f(x) \ge f((1-c)x) \quad \forall x \ge 0.$$

For $0 \le c \le 2$ this is easily seen to hold as

$$f(x) - f((1-c)x) = \sum_{k=1}^{\infty} \frac{(1-(1-c)^k)x^k}{k!\zeta(k+1)}$$

and the coefficients are all non-negative if (and only if) $0 \le c \le 2$.

Now consider $c > 2$. It is clear that (A_c) holds for all $c > 2$ (actually for $c \ge 1$) if and only if

$$(\mathrm{B}) \qquad f(-x) \le 0 \quad \text{for } x \ge 0.$$

For if (B) is true, then since $(1-c)x \le 0$, we have

$$f((1-c)x) \le 0 \le f(x)$$

and (A_c) holds. Conversely, assume (A_c) holds for all $c > 2$. Suppose, for a contradiction, that $f(-x_0) > 0$ for some $x_0 > 0$. Then

$$0 < f(-x_0) = f\left((1-c)\cdot\frac{x_0}{c-1}\right) \le f\left(\frac{x_0}{c-1}\right)$$

for every $c > 2$. This is false for c sufficiently large as the RHS can be arbitrarily close to zero. Thus (B) is true.

However, we show that (B) is false, and hence that (A_c) fails for some $c > 2$.

THEOREM A1. *There exists $\lambda > 2$ such that for $c \le \lambda$, π is increasing, while for $c > \lambda$, π is not increasing.*

Proof. Clearly, if (A_c) holds for some $c = c_0 > 1$, then it holds for all smaller c, since (A_c) is equivalent to

$$(\mathrm{A}'_c) \qquad f(-y) \le f\left(\frac{y}{c-1}\right) \quad \forall y \ge 0$$

and f is increasing on $(0,\infty)$. Also, if (A'_c) holds for all $c < c_1$, then by continuity of f, it holds for $c = c_1$. Now we show (B) is false.

Starting from the formula [7] $\frac{1}{2\pi i}\int_{(-1,0)} \Gamma(s)x^{-s}\,ds = e^{-x} - 1$ $(x > 0)$ we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{(-1,0)} \frac{\Gamma(s)}{\zeta(1-s)}x^{-s}\,ds &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\cdot\frac{1}{2\pi i}\int_{(-1,0)}\Gamma(s)\left(\frac{x}{n}\right)^{-s}ds\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}(e^{-x/n}-1) = f(-x),\end{aligned}$$

---

[7] Here $\int_{(\alpha,\beta)}$ means $\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}$ for any $\sigma \in (\alpha,\beta)$.

using the absolute and uniform convergence of the Dirichlet series for $1/\zeta(1-s)$. Changing the variable gives

$$f(-x) = \frac{1}{2\pi i} \int_{(1,2)} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1}\, ds.$$

By Mellin inversion

$$\frac{\Gamma(1-s)}{\zeta(s)} = \int_0^\infty \frac{f(-x)}{x^s}\, dx \qquad (1 < \sigma < 2).$$

Hence

$$\begin{aligned}\int_1^\infty \frac{f(-x)}{x^s}\, dx &= \frac{\Gamma(1-s)}{\zeta(s)} - \int_0^1 \frac{f(-x)}{x^s}\, dx \\ &= \frac{\Gamma(1-s)}{\zeta(s)} + \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k!\zeta(k+1)(k+1-s)}.\end{aligned}$$

Since the LHS converges and is holomorphic in H_1, the singularities at $2, 3, 4, \ldots$ on the RHS are all removable, as is the singularity at $s = 1$.

Suppose now that $f(-x)$ is ultimately of one sign. Then the abscissa of convergence of the LHS Mellin transform must be a (real) singularity of the function. But the first real singularity occurs at -2 (zero of ζ). This is a contradiction as there are singularities at the non-trivial zeros of ζ to the right of this. Thus $f(-x)$ cannot be ultimately of one sign, i.e. f changes sign infinitely often in $(-\infty, 0)$ and has infinitely many zeros here.

Thus (A'_c) fails for some $c \geq 2$ and hence all larger c. Let λ denote the supremum of those c for which (A'_c) holds. Thus (A'_c) holds for $c \leq \lambda$ and fails for $c > \lambda$.

Finally, $\lambda > 2$ since $f(y/(\lambda-1)) \geq f(-y)$ for all $y \geq 0$ with equality for some $y > 0$ (or λ would not be optimal) and this is false for $\lambda = 2$. ∎

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