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Repelling periodic points and logarithmic equidistribution in non-archimedean dynamics

by

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Dedicated to Professor David Drasin on his seventieth birthday

1. Introduction. Let K be an algebraically closed field complete with respect to a non-trivial absolute value (or valuation) $|\cdot|$. Then K is said to be *non-archimedean* if $|z-w| \le \max\{|z|,|w|\}$ $(z,w \in K)$ (e.g. p-adic $\mathbb{C}_p$). Otherwise, K is said to be *archimedean*, and then $K \cong \mathbb{C}$. It is always assumed that K is of characteristic 0 in this article. For non-archimedean K, the projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$ is not compact. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is a compact augmentation of $\mathbb{P}^1$ and contains $\mathbb{P}^1$ as its dense subset. For the details on P^1, see [1, §2], [11, §2.1]. For archimedean K, P^1 reduces to $\mathbb{P}^1$.

Let f be a rational function on $\mathbb{P}^1$ of algebraic degree $d > 1$. The action of f on $\mathbb{P}^1$ continuously extends to an open and (fiber-)discrete map on P^1. The (Berkovich) *Julia set* $\mathcal{J}(f)$ is the set of all $z_0 \in \mathsf{P}^1$ at which

$$\bigcap_{U:\,\text{open in }\mathsf{P}^1,\, z_0\in U} \Big(\bigcup_{k\in\mathbb{N}} f^k(U)\Big) = \mathsf{P}^1 \setminus E(f)$$

(cf. [11, Definition 2.8]). Here the exceptional set $E(f)$ of f consists of at most two points in $\mathbb{P}^1$. The (Berkovich) *Fatou set* $\mathcal{F}(f)$ is $\mathsf{P}^1 \setminus \mathcal{J}(f)$.

Let $f^\#$ denote the chordal derivative of f. A periodic point $p \in \mathbb{P}^1$ of f such that $f^k(p) = p$ is said to be *superattracting*, *attracting*, *indifferent* or *repelling* if the absolute value of the multiplier $(f^k)^\#(p) = |(f^k)'(p)|$ is $= 0$, < 1, $= 1$ or > 1, respectively. Let us denote the sets of superattracting, attracting and repelling periodic points of f in $\mathbb{P}^1$ by $SAT(f)$, $AT(f)$, $R(f)$, respectively. For non-archimedean K, the following is an open problem: if

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the classical Julia set $\mathcal{J}(f)\cap\mathbb{P}^1$ is non-empty, is it true that

$$\overline{R(f)}=\mathcal{J}(f)\cap\mathbb{P}^1? \tag{1.1}$$

The closure of $R(f)$ is taken in $\mathbb{P}^1$ with respect to the chordal distance.

The Dirac measure at $w\in\mathsf{P}^1$ is denoted by δ_w. For a (possibly constant) rational function a on $\mathbb{P}^1$, there are exactly $d^k+\deg a$ roots of the equation $f^k=a$ in $\mathbb{P}^1$ counting their multiplicity, unless $f^k\not\equiv a$. Let us consider the sequence of the averaged distributions

$$\nu_k^a:=\frac{1}{d^k+\deg a}\sum_{w\in\mathbb{P}^1:\, f^k(w)=a(w)}\delta_w$$

of roots of $f^k=a$ in $\mathbb{P}^1$, where the sum takes into account the multiplicity of each root. Let μ_f be the equilibrium measure of f on P^1. The function $f^\#$ extends continuously to P^1. We define the *Lyapunov exponent* of μ_f as

$$L(f):=\int_{\mathsf{P}^1}\log f^\#\,d\mu_f.$$

We first show a logarithmic equidistribution of periodic points with respect to μ_f:

THEOREM 1. *Let f be a rational function on $\mathbb{P}^1$ of degree $d>1$. Then*

$$\lim_{k\to\infty}\frac{1}{kd^k}\sum_z\log(f^k)^\#(z)=L(f),$$

where the sum is over all $z\in(AT(f)\setminus SAT(f))\cup R(f)$ such that $f^k(z)=z$.

As an application of Theorem 1, we give a partial positive answer to the question (1.1).

THEOREM 2. *Let f be a rational function on $\mathbb{P}^1$ of degree >1. If $L(f)>0$, then $\overline{R(f)}=\mathcal{J}(f)\cap\mathbb{P}^1$.*

REMARK 1.1. For archimedean K, $L(f)>0$ always holds, and Theorem 2 can give yet another proof of the repelling density in the archimedean case. But $L(f)>0$ is not always the case for non-archimedean K.

In Section 5, we also show the formula

$$L(f)=-\log|d|-\frac{2}{d}\log|\operatorname{Res}F|+\sum_{j=1}^{2d-2}G^F(C_j^F) \tag{1.2}$$

(due to DeMarco [9] for archimedean K; the notation will be explained in Section 5). Theorem 2 may be stated without invoking the Berkovich space ($L(f)$ uses it).

THEOREM 3. *Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. If*

$$-\log|d| - \frac{2}{d}\log|\mathrm{Res}\, F| + \sum_{j=1}^{2d-2} G^F(C_j^F) > 0,$$

then $\overline{R(f)} = \mathcal{J}(f) \cap \mathbb{P}^1$.

This improves Bézivin [7, Théorème], where f was a polynomial and some additional conditions were assumed.

2. Background. For the foundations of potential theory and dynamics on P^1, see [1], [11]. For archimedean K, see also [21, III, §11], [5, VII].

Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$.

NOTATION 2.1. Let us also denote by $|\cdot|$ both the maximal norm (used for non-archimedean K) and the Euclidean norm (used for archimedean K) on K^2. The origin of K^2 is denoted by 0, and π is the canonical projection $K^2 \setminus \{0\} \to \mathbb{P}^1$. The (normalized) *chordal distance* $[z, w]$ on $\mathbb{P}^1$ is defined as

$$[z, w] := |p \wedge q|/(|p| \cdot |q|) \ (\leq 1)$$

if $z = \pi(p)$ and $w = \pi(q)$. Here we put $(z_0, z_1) \wedge (w_0, w_1) := z_0w_1 - z_1w_0$ on $K^2 \times K^2$. The *chordal derivative* $f^\#$ is

$$f^\#(z) := \lim_{\mathbb{P}^1 \ni w \to z} [f(w), f(z)]/[w, z],$$

and extends continuously to P^1. The critical set $C(f)$ of f is defined as $C(f) := \{c \in \mathbb{P}^1;\ f^\#(c) = 0\}$.

A *non-degenerate homogeneous lift* F of f, which is unique up to multiplication in $K \setminus \{0\}$, is a homogeneous polynomial endomorphism of algebraic degree d on K^2 such that $\pi \circ F = f \circ \pi$ on $K^2 \setminus \{0\}$ and $F^{-1}(0) = \{0\}$.

The extension of f on P^1 induces the push-forward f_* and pullback f^* on both spaces of continuous functions and of Radon measures on P^1 ([1, §9], [11, §2.2]).

Let us denote by Ω both the Dirac measure at the canonical (Gauss) point $\mathcal{S}_{\mathrm{can}} \in \mathsf{P}^1$ (defined for non-archimedean K [1, §1.2], [11, §2.1]) and the normalized Fubini–Study area element $\omega = |dz|/(\pi(1 + |z|^2))$ on $\mathbb{P}^1$ (defined for archimedean K). For non-archimedean K, the chordal distance $[z, w]$ canonically extends to the generalized Hsia kernel $\delta_{\mathcal{S}_{\mathrm{can}}}(z, w)$ on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ ([1, §4.4], [11, §2.1]). For simplicity, it is also denoted by $[z, w]$. Let us denote the Laplacian on P^1 by Δ ([1, §5], [10, §7.7], [20, §3] for non-archimedean K), which is normalized as

$$\Delta \log[\cdot, w] = \delta_w - \Omega$$

for each $w \in \mathsf{P}^1$ ([1, Example 5.19], [11, §2.4]; in [1] the opposite sign convention on Δ is adopted).

The function $(\log|F|)/d - \log|\cdot|$ on K^2 descends to a continuous function T_F on $\mathbb{P}^1$, and continuously extends to P^1. Then

$$\Delta T_F = \frac{1}{d} f^* \Omega - \Omega$$

([1, §10.1], [11, §3.1]). The *dynamical Green function* g_F is the uniform limit

$$g_F := \sum_{j=0}^{\infty} \frac{1}{d^j} (f^j)^* T_F$$

on P^1 ([1, §10.1], [11, §3.1]). The *equilibrium measure* μ_f of f is defined as

$$\mu_f := \Omega + \Delta g_F,$$

which is indeed independent of the choice of F. This is an f-balanced (and f-invariant) probability measure on P^1 ([1, §10], [8, §2], [11, §3.1] for non-archimedean K).

The *exceptional set* $E(f)$ of f is the maximal f-backward invariant finite subset of $\mathbb{P}^1$, which is possibly empty and consists of at most two points. A rational function a on $\mathbb{P}^1$ is said to be *exceptional* (with respect to f) if it identically equals a point in $E(f)$; otherwise it is *non-exceptional* (with respect to f). The equidistribution theorem for moving targets in complex dynamics due to Lyubich [13, Theorem 3] and its non-archimedean counterpart due to Favre and Rivera-Letelier [11, Theorème B] is

THEOREM 2.2. *Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. Then for every non-exceptional rational function a on $\mathbb{P}^1$, $\nu_k^a \to \mu_f$ weakly as $k \to \infty$.*

3. A logarithmic equidistribution of roots of $f^k = a$. Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and F be a non-degenerate homogeneous lift of f.

For a Radon measure μ on P^1, the *chordal potential* is

$$U_\mu(z) := \int_{\mathsf{P}^1} \log\,[z, w]\, d\mu(w)$$

for $z \in \mathsf{P}^1$. Then U_μ is a quasipotential of μ in the sense that

$$\Delta U_\mu = \mu - \mu(\mathsf{P}^1)\Omega$$

([1, Example 5.12]). For the details on U_μ, see [1, Proposition 6.12], [11, §2.4], [21, III, §11].

LEMMA 3.1. *Suppose that a sequence of positive measures ν_k on P^1 tends to μ_f weakly as $k \to \infty$. Then the convergence*

$$\lim_{k\to\infty} \int_{\mathsf{P}^1} \log f^{\#}\, d\nu_k = L(f)$$

holds if for each $c \in C(f)$, $\lim_{k\to\infty} U_{\nu_k}(c) = U_{\mu_f}(c)$.

Proof. By a direct computation involving Euler's identity,

$$f^{\#}(z) = \frac{1}{|d|}\frac{|p|^2}{|F(p)|^2}|\det DF(p)| \tag{3.1}$$

if $z = \pi(p)$ (cf. [12, Theorem 4.3]). The Jacobian determinant $\det DF$ of F, which is a homogeneous polynomial on K^2 of degree $2d-2$, factors as

$$\det DF(p) = \prod_{j=1}^{2d-2}(p \wedge C_j^F)$$

($C_j^F \in K^2 \setminus \{0\}$, $j = 1, \ldots, 2d-2$). Then the equality (3.1) descends to

$$\log f^{\#}(z) = -\log|d| + \sum_{j=1}^{2d-2}(\log[z, c_j] + \log|C_j^F|) - 2dT_F(z) \tag{3.2}$$

on $\mathbb{P}^1$, and extends to P^1. Here the $c_j := \pi(C_j^F)$ ($j = 1, \ldots, 2d-2$) range over $C(f)$. Let us integrate (3.2) with respect to $d\nu_k(z)$ and $d\mu_f(z)$, and take the difference of the integrals. Then

$$\int_{\mathsf{P}^1} \log f^{\#}\, d\nu_k - L(f) = \sum_{i=1}^{2d-2}(U_{\nu_k}(c_j) - U_{\mu_f}(c_j)) - 2d\int_{\mathsf{P}^1} T_F\, d(\nu_k - \mu_f).$$

Since T_F is continuous on P^1, the assumption $\lim_{k\to\infty}\nu_k = \mu_f$ implies that $\int_{\mathsf{P}^1} T_F\, d(\nu_k - \mu_f) \to 0$ as $k \to \infty$. Now the proof is complete. ∎

Lemma 3.2. *The chordal potential $U_{f^*\Omega}$ is continuous on P^1. Moreover, uniformly on P^1,*

$$\lim_{k\to\infty} U_{(f^k)^*\Omega/d^k} = U_{\mu_f}. \tag{3.3}$$

Proof. Since $\Delta T_F = f^*\Omega/d - \Omega = \Delta U_{f^*\Omega/d}$, the function $U_{f^*\Omega/d} - T_F$ is constant on P^1: this is immediate if K is archimedean, and for non-archimedean K, it follows from the continuity of T_F and a continuity property of the chordal potential ([1, Proposition 6.12]) and a property of Δ ([1, Lemma 5.24], [11, §2.4]). Hence $U_{f^*\Omega}$ is continuous on P^1. By the same argument as above or a direct computation, U_Ω is constant on P^1. From the definition of μ_f, we have $\mu_f - (f^k)^*\Omega/d^k = \Delta\sum_{j=k}^{\infty}(f^j)^*T_F/d^j$. Hence by the same argument as above, the function $U_{\mu_f} - U_{(f^k)^*\Omega/d^k} - \sum_{j=k}^{\infty}(f^j)^*T_F/d^j$ is constant on P^1. Integrating this in $d\Omega$, by the Fubini theorem, we have

$$U_{\mu_f}(z) - U_{(f^k)^*\Omega/d^k}(z) = \int_{\mathsf{P}^1}\sum_{j=k}^{\infty}\frac{(f^j)^*T_F}{d^j}\, d(\delta_z - \Omega),$$

which tends to 0 uniformly in $z \in \mathsf{P}^1$ as $k \to \infty$. ∎

For rational functions f, a on $\mathbb{P}^1$, the function $z \mapsto [f(z), a(z)]$ on $\mathbb{P}^1$ continuously extends to P^1. Let us denote this extension by $[f, a]_{\mathsf{P}^1}(z)$.

LEMMA 3.3. *Let a be a non-exceptional rational function on $\mathbb{P}^1$, and let (S_k) be a sequence of subsets of P^1. Then for every $z \in \mathsf{P}^1$,*

$$U_{\nu_k^a|(\mathsf{P}^1\setminus S_k)}(z) - U_{\mu_f}(z) = \lim_{w\to z}\left(\frac{1}{d^k+\deg a}\log\,[f^k,a]_{\mathsf{P}^1}(w) - U_{\nu_k^a|S_k}(w)\right) + o(1)$$

as $k \to \infty$.

Proof. Let a be any rational function on $\mathbb{P}^1$, and put $d_k := d^k + \deg a$. By convention, put $a^*\Omega := 0$ when a is constant. Recall that U_Ω is constant on P^1, and observe that

$$\frac{1}{d_k}\Delta\log\,[f^k,a]_{\mathsf{P}^1} = \nu_k^a - \frac{(f^k)^*\Omega + a^*\Omega}{d_k}$$

([11, §3.4]). Hence by the argument used in the proof of Lemma 3.2, the function $\log\,[f^k,a]_{\mathsf{P}^1}(\cdot)/d_k - U_{\nu_k^a} + (U_{(f^k)^*\Omega} + U_{a^*\Omega})/d_k$ is constant on P^1. Integrating it in $d\Omega$, by the Fubini theorem, we obtain

$$\frac{1}{d_k}\log\,[f^k,a]_{\mathsf{P}^1}(\cdot) = U_{\nu_k^a} - \frac{U_{(f^k)^*\Omega} + U_{a^*\Omega}}{d_k} + \frac{1}{d_k}\int_{\mathsf{P}^1}\log\,[f^k,a]_{\mathsf{P}^1}\,d\Omega \tag{3.4}$$

([18, (1.5)]), and for every $z \in \mathsf{P}^1$, by a continuity property of the chordal potential [1, Proposition 6.12],

$$\lim_{w\to z}\left(\frac{1}{d_k}\log\,[f^k,a]_{\mathsf{P}^1}(w) - U_{\nu_k^a|S_k}(w)\right) = U_{\nu_k^a|(\mathsf{P}^1\setminus S_k)}(z) - \frac{U_{(f^k)^*\Omega}(z) + U_{a^*\Omega}(z)}{d_k} + \frac{1}{d_k}\int_{\mathsf{P}^1}\log\,[f^k,a]_{\mathsf{P}^1}\,d\Omega. \tag{3.5}$$

Suppose in addition that a is non-exceptional. From (3.5) and Lemma 3.2, it remains to show that

$$\lim_{k\to\infty}\frac{1}{d_k}\int_{\mathsf{P}^1}\log\,[f^k,a]_{\mathsf{P}^1}\,d\Omega = 0 \tag{3.6}$$

(cf. [14]). Fix $(k_j) \subset \mathbb{N}$. By Theorem 2.2, $\lim_{j\to\infty}\nu_{k_j}^a = \mu_f$ weakly as $j \to \infty$, so by a standard cut-off argument,

$$\limsup_{j\to\infty} U_{\nu_{k_j}^a} \le U_{\mu_f}.$$

For every $z \in \mathsf{P}^1$, taking $\limsup_{j\to\infty}$ in (3.4 for $k = k_j$), we have

$$\limsup_{j\to\infty}\frac{1}{d_{k_j}}\log\,[f^{k_j},a]_{\mathsf{P}^1}(z)$$

$$=\limsup_{j\to\infty}\bigg(U_{\nu^a_{k_j}}(z)-\frac{U_{(f^{k_j})^*\Omega}(z)+U_{a^*\Omega}(z)}{d_{k_j}}+\frac{1}{d_{k_j}}\int\limits_{\mathsf{P}^1}\log\,[f^{k_j},a]_{\mathsf{P}^1}\,d\Omega\bigg)$$

$$\le\limsup_{j\to\infty}U_{\nu^a_{k_j}}(z)-U_{\mu_f}(z)+\limsup_{j\to\infty}\frac{1}{d_{k_j}}\int\limits_{\mathsf{P}^1}\log\,[f^{k_j},a]_{\mathsf{P}^1}\,d\Omega,$$

so

$$\limsup_{j\to\infty}\frac{1}{d_{k_j}}\log\,[f^{k_j},a]_{\mathsf{P}^1}(z)\le\limsup_{j\to\infty}\frac{1}{d_{k_j}}\int\limits_{\mathsf{P}^1}\log\,[f^{k_j},a]_{\mathsf{P}^1}\,d\Omega\le 0. \tag{3.7}$$

Observe that there is a fixed point z_0 of f in $\mathbb{P}^1\setminus E(f)$: for, if there is a multiple root of $f=\mathrm{Id}_{\mathbb{P}^1}$ in $\mathbb{P}^1$, then this root is not in $E(f)$. Otherwise, all $d+1>2$ roots of $f=\mathrm{Id}_{\mathbb{P}^1}$ in $\mathbb{P}^1$ are simple, so distinct. Since $\#E(f)\le 2$, at least one root of $f=\mathrm{Id}_{\mathbb{P}^1}$ in $\mathbb{P}^1$ is not in $E(f)$.

In particular, $\#\bigcup_{k\in\mathbb{N}}f^{-k}(z_0)=\infty$, so there is $N\in\mathbb{N}$ such that $f^{-N}(z_0)\setminus a^{-1}(z_0)\neq\emptyset$. Take $z_1\in f^{-N}(z_0)\setminus a^{-1}(z_0)$. For every $j\in\mathbb{N}$ large enough, $[f^{k_j}(z_1),a(z_1)]=[f^{k_j-N}(z_0),a(z_1)]=[z_0,a(z_1)]>0$, so

$$\limsup_{j\to\infty}\frac{1}{d_{k_j}}\log\,[f^{k_j}(z_1),a(z_1)]=0.$$

This with (3.7) for $z=z_1$ completes the proof of (3.6). ■

Let a be a non-exceptional rational function on $\mathbb{P}^1$, and let (S_k) be a sequence of subsets of P^1 such that $\lim_{k\to\infty}\nu^a_k(S_k)=0$. Then from Theorem 2.2, $\lim_{k\to\infty}\nu^a_k|(\mathsf{P}^1\setminus S_k)=\mu_f$ weakly. Lemmas 3.1 and 3.3 yield

Theorem 4. *Let f be a rational function on $\mathbb{P}^1$ of degree $d>1$, and let a be a non-exceptional rational function on $\mathbb{P}^1$. Let (S_k) be a sequence of subsets of P^1 satisfying $\lim_{k\to\infty}\nu^a_k(S_k)=0$. Then the logarithmic equidistribution*

$$\lim_{k\to\infty}\int\limits_{\mathsf{P}^1\setminus S_k}\log f^{\#}\,d\nu^a_k=L(f)$$

holds if for each $c\in C(f)$,

$$\lim_{k\to\infty}\lim_{\mathbb{P}^1\ni z\to c}\bigg(\frac{1}{d^k+\deg a}\log\,[f^k(z),a(z)]-\int\limits_{S_k}\log\,[z,w]\nu^a_k(w)\bigg)=0. \tag{3.8}$$

4. A proof of Theorems 1 and 2. Theorem 1 is a principal application of Theorem 4.

Take $a=\mathrm{Id}_{\mathbb{P}^1}$. For each $k\in\mathbb{N}$, we take $S_k=SAT(f)\cap\{w\in\mathbb{P}^1;\ f^k(w)=w\}$. Since $\#SAT(f)<\infty$ (from $\#C(f)<\infty$) and each $p\in S_k$ is simple as a root of $f^k=\mathrm{Id}_{\mathbb{P}^1}$, we have $\lim_{k\to\infty}\nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(S_k)=0$. Observe also that from $\#SAT(f) < \infty$, for every $c \in C(f)$ and every $k \in \mathbb{N}$,

$$\inf_{w \in S_k \setminus \{c\}} [c,w] \geq \inf_{w \in SAT(f) \setminus \{c\}} [c,w] > 0,$$

so that

$$\lim_{k\to\infty} \int_{S_k \setminus \{c\}} \log [c,w] \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) = 0. \tag{4.1}$$

The equality (4.1) will be used repeatedly in the rest of this section. Let $c \in C(f)$, and let us check the condition (3.8).

If $c \in C(f) \cap \mathcal{F}(f) \setminus SAT(f)$, then the Fatou component of f containing c is either an immediate attractive or parabolic basin of f, or is non-cyclic under f (by the Denjoy–Wolff classification of cyclic Fatou components and its non-archimedean counterpart due to Rivera-Letelier [17, Théorème de Classification]). Hence $\inf_{k\in\mathbb{N}} [f^k(c), c] > 0$, and noting that $c \notin S_k$ (so $S_k = S_k \setminus \{c\}$), by (4.1) we have

$$\lim_{k\to\infty} \left(\frac{1}{d^k+1} \log [f^k(c), c] - \int_{S_k} \log [c,w]\, \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) \right) = 0.$$

If $c \in C(f) \cap SAT(f)$, then putting $p := \min\{k \in \mathbb{N};\ f^k(c) = c\}$, by (4.1) we have

$$\begin{aligned}
&\lim_{p\mathbb{N}\ni k\to\infty} \lim_{\mathbb{P}^1 \ni z \to c} \left(\frac{1}{d^k+1} \log [f^k(z), z] - \int_{S_k} \log [z,w] \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) \right) \\
&= \lim_{p\mathbb{N}\ni k\to\infty} \left(\frac{1}{d^k+1} \lim_{\mathbb{P}^1 \ni z \to c} \log \frac{[f^k(z), z]}{[z,c]} - \int_{S_k \setminus \{c\}} \log [c,w] \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) \right) = 0,
\end{aligned}$$

since

$$\begin{aligned}
\lim_{\mathbb{P}^1 \ni z \to c} \log \frac{[f^k(z), z]}{[z,c]} &= \lim_{\mathbb{P}^1 \ni z \to c} \log \frac{|f^k(z) - c - (z-c)|}{|z-c|} \\
&= \log |(f^k)^{\#}(c) - 1| = 0.
\end{aligned}$$

Noting that $\inf_{k \in (\mathbb{N} \setminus p\mathbb{N})} [f^k(c), c] > 0$ and $c \notin S_k$ (so $S_k = S_k \setminus \{c\}$) for every $k \in \mathbb{N} \setminus p\mathbb{N}$, by (4.1) we also have

$$\lim_{(\mathbb{N}\setminus p\mathbb{N})\ni k\to\infty} \left(\frac{1}{d^k+1} \log [f^k(c), c] - \int_{S_k} \log [c,w]\, \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) \right) = 0.$$

Hence if $c \in C(f) \cap SAT(f)$ $(\subset \mathcal{F}(f))$, then

$$\lim_{k\to\infty} \lim_{\mathbb{P}^1 \ni z \to c} \left(\frac{1}{d^k+1} \log [f^k(z), z] - \int_{S_k} \log [z,w] \nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w) \right) = 0.$$

Recall Przytycki [16, Lemma 1] (the original proof for archimedean K works for non-archimedean K): if $c \in C(f) \cap \mathcal{J}(f)$, then there is $L \geq 1$ such that for every $k \in \mathbb{N}$,

$$[f^k(c), c] \geq L^{-k}.$$

Hence if $c \in C(f) \cap \mathcal{J}(f)$, then noting that $c \notin S_k$ (so $S_k = S_k \setminus \{c\}$), by (4.1) we have

$$\lim_{k\to\infty}\left(\frac{1}{d^k+1}\log\,[f^k(c), c] - \int_{S_k}\log\,[c,w]\,\nu_k^{\mathrm{Id}_{\mathbb{P}^1}}(w)\right) = 0.$$

Now Theorem 4 (and the chain rule) implies that

$$\frac{1}{k}\Big(\int_{AT(f)\setminus SAT(f)}\log\,(f^k)^{\#}\,d\nu_k^{\mathrm{Id}_{\mathbb{P}^1}} + \int_{R(f)}\log\,(f^k)^{\#}\,d\nu_k^{\mathrm{Id}_{\mathbb{P}^1}}\Big)$$
$$= \frac{1}{k}\int_{\mathsf{P}^1\setminus S_k}\log\,(f^k)^{\#}\,d\nu_k^a = \int_{\mathsf{P}^1\setminus S_k}\log f^{\#}\,d\nu_k^a \to L(f)$$

as $k \to \infty$. The proof of Theorem 1 is complete.

Remark 4.1. In the arithmetic setting where $K = \mathbb{C}_v$ for a number field k with a non-trivial absolute value (or place) v and where f has its coefficients in k, Theorem 1 is obtained in [19] using Roth's theorem from Diophantine approximation theory. For archimedean K, a version of Theorem 1 is shown in [4] (see also [3]) using $L(f) > 0$.

Let us complete the proof of Theorem 2. Let f be a rational function on $\mathbb{P}^1$ of degree > 1. Since

$$\int_{AT(f)\setminus SAT(f)}\log\,(f^k)^{\#}\,d\nu_k^{\mathrm{Id}_{\mathbb{P}^1}} \leq 0,$$

if $L(f) > 0$, then by Theorem 1, $R(f) \neq \emptyset$. By Bézivin [6, Théorème 3], then $\overline{R(f)} = \mathcal{J}(f) \cap \mathbb{P}^1$ (the original proof for p-adic K works for both non-archimedean and archimedean K). ∎

5. Proof of (1.2). Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and F be a non-degenerate homogeneous lift of f. Let us consider the weighted F-kernel (Arakelov–Green function of μ_f [1, §10.2]) on P^1 defined as

$$\Phi_F(z,w) := \log\,[z,w] - g_F(z) - g_F(w),$$

and the F-potential of the equilibrium measure μ_f defined as

$$U_{F,\mu_f}(z) := \int_{\mathsf{P}^1}\Phi_F(z,w)\,d\mu_f(w)$$

on P^1. For the details on $U_{F,\mu}$, see [1, Proposition 8.68].

Since $\Delta U_{F,\mu_f} = \mu_f - \mu_f = 0$, by the argument used in the proof of Lemma 3.2, U_{F,μ_f} identically equals a constant V_F on P^1. For the definition of the homogeneous resultant $\operatorname{Res} F$ of F, see [9, §6]. By [9, Theorem 1.5] (for archimedean K) and [1, §10.2] (for non-archimedean K),

$$V_F = -\frac{1}{d(d-1)} \log |\operatorname{Res} F|$$

(for a simple computation, see [15, Appendix]). The escaping rate function (homogeneous dynamical height [1, §10.2]) of F on $K^2 \setminus \{0\}$ is

$$G^F := g_F \circ \pi + \log |\cdot| = \lim_{k\to\infty} \frac{1}{d^k} \log |F^k|.$$

The equality (3.2) in Section 3 is rewritten as

$$\log f^{\#}(z) = -\log |d| + \sum_{j=1}^{2d-2} (\Phi_F(z, c_j) + G^F(C_j^F)) + 2(g_F(f(z)) - g_F(z))$$

on P^1, where $\{C_j^F \in K^2 \setminus \{0\}; j = 1, \ldots, 2d-2\}$ satisfies $\det DF(p) = \prod_{j=1}^{2d-2} (p \wedge C_j^F)$. Integrating this unintegrated version of (1.2) in $d\mu_f(z)$ yields

$$\begin{aligned} L(f) &= -\log |d| + (2d-2)V_F + \sum_{j=1}^{2d-2} G^F(C_j^F) + 2 \int_{\mathsf{P}^1} g_F \, d(f_*\mu_f - \mu_f) \\ &= -\log |d| - \frac{2}{d} \log |\operatorname{Res} F| + \sum_{i=1}^{2d-2} G^F(C_i^F) \end{aligned}$$

from $f_*\mu_f = \mu_f$. ∎

Remark. For another simple computation of $L(f)$ in the archimedean K case, see Bassanelli–Berteloot [2, Theorem 3.1, Propositions 4.8, 4.10].

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