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A functional relation for the Tornheim double zeta function

by

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1. Introduction

Definition 1.1. The *Tornheim double zeta function* $T(s,t,u)$, for $s,t,u \in \mathbb{C}$, $\Re(s+u)>1$, $\Re(t+u)>1$ and $\Re(s+t+u)>2$, is defined by

$$T(s,t,u) := \sum_{m,n=1}^{\infty} \frac{1}{m^s n^t (m+n)^u}. \tag{1.1}$$

This function $T(s,t,u)$ is a generalization of the Riemann zeta function $\zeta(s)$, $s \in \mathbb{C}$. Furthermore, $T(s,t,u)$ is continued meromorphically to $\mathbb{C}^3$ in [4]. By the definition, we have

$$T(s,t,u) = T(t,s,u), \qquad T(s,t,0) = \zeta(s)\zeta(t).$$

The case of $t=0$, that is $T(s,0,u)$, is called the *Euler–Zagier double zeta function* [10].

The values $T(a,b,c)$ for $a,b,c \in \mathbb{N}$ were first investigated by Tornheim [7] in 1950 and later Mordell [5] in 1958. Tornheim [7, Theorem 7] showed that $T(a,b,c)$ can be expressed as a polynomial in $\{\zeta(j) \mid 2 \le j \le a+b+c\}$ with rational coefficients when $a+b+c$ is odd, and that the same is true for $T(2r,2r,2r)$ and $T(2r-1,2r,2r+1)$ [7, Theorem 8], but he did not give the coefficients. Mordell [5, Theorem III] proved that $T(2r,2r,2r) = k_r \pi^{6r}$ for some rational number k_r. In 1985 Subbarao and Sitaramachandrarao [6, Theorem 4.1] explicitly determined $T(2p,2q,2r)+T(2q,2r,2p)+T(2r,2p,2q)$ $(p,q,r \in \mathbb{N})$. Then, by taking $p=q=r$, they gave an explicit formula for $T(2r,2r,2r)$ $(r \in \mathbb{N})$ [6, Remark 3.1]. In 1996 Huard, Williams and Zhang [3, Theorems 1–3] determined $T(r,0,N-r)$ $(r \in \mathbb{N},\ N \in 2\mathbb{N}+1,\ 1 \le r \le N-2)$, $T(p,q,N-p-q)$ $(p,q \in \mathbb{N}\cup\{0\},\ N \in 2\mathbb{N}+1,\ 1 \le p+q \le N-1,\ 0 \le p,q \le N-2)$ and $T(r,r,r)$ $(r \in \mathbb{N})$. In 2002 Tsumura [8, Theorem 1]

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proved that $T(p,q,r)+(-1)^pT(p,r,q)+(-1)^{p+r}T(r,q,p)$ is a polynomial in $\{\zeta(k) \mid 2 \le k \le p+q+r\}$ with rational coefficients for $p,q,r \in \mathbb{N}\cup\{0\}$ with $p+q \ge 2$ and $r \ge 2$. Recently, Espinosa and Moll provided an explicit formula for $T(x,y,z)$, $x,y,z \in \mathbb{R}$, in terms of integrals involving Hurwitz zeta functions (see [2, Proposition 2.1 and Theorem 2.4]). Also in 2006 Tsumura [9, Theorem 4.5] proved the following functional relation:

$$\begin{aligned}(1.2)\quad & T(a,b,s)+(-1)^bT(b,s,a)+(-1)^aT(s,a,b)\\ &= 2\sum_{\substack{j=0\\ j\equiv a\,(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{j/2}\frac{(i\pi)^{2l}}{(2l)!}\binom{b-1+j-2l}{j-2l}\zeta(b+j+s-2l)\\ &\quad -4\sum_{\substack{j=0\\ j\equiv a\,(2)}}^{a}(2^{1-a+j}-1)\zeta(a-j)\sum_{l=0}^{(j-1)/2}\frac{(i\pi)^{2l}}{(2l+1)!}\sum_{\substack{k=0\\ k\equiv b\,(2)}}^{b}\zeta(b-k)\\ &\quad \times\binom{k-1+j-2l}{j-2l-1}\zeta(k+j+s-2l)\end{aligned}$$

(where (2) means mod 2), for $a,b \in \mathbb{N}\cup\{0\}$, $b \ge 2$, $s \in \mathbb{C}$, except for the singular points of each side of this formula.

In this paper, we prove the following result.

THEOREM 1.2. *For all $a,b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned}(1.3)\quad & T(a,b,s)+(-1)^bT(b,s,a)+(-1)^aT(s,a,b)\\ &= \frac{2}{a!\,b!}\sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!\\ &\quad \times\zeta(2k)\zeta(a+b+s-2k).\end{aligned}$$

This functional relation is considerably simpler than that of Tsumura. We are not aware of a direct proof which shows that the right-hand sides of (1.2) and (1.3) are the same. “Mathematica 5.0” shows that they are equal for all $1 \le a \le b \le 100$, $a,b \in \mathbb{N}$. It therefore seems unlikely that a non-trivial functional relation can be deduced by equating (1.2) and (1.3).

In Section 3, we obtain new proofs of formulas for the special values of $T(a,b,c)$, $a,b,c \in \mathbb{N}$ mentioned in the introduction by using the functional relation (1.3).

2. Proof of Theorem 1.2. Firstly, we define $\log t$, $t \in \mathbb{C}$, and t^s, $s,t \in \mathbb{C}$, by

$$\log t := \log|t| + i\arg t,\qquad t^s := e^{s\log t},\qquad 0 \le \arg t < 2\pi.$$

And for $s, t, u \in \mathbb{C}$, $\Re(s+u) > 1$, $\Re(t+u) > 1$ and $\Re(s+t+u) > 2$, we put

$$S(s,t,u) := \sum_{\substack{m\neq 0,\, n\neq 0\\ m+n\neq 0}} \frac{1}{m^s n^t (m+n)^u}. \tag{2.1}$$

LEMMA 2.1. *For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have*

$$\begin{aligned} S(a,b,s) &= (1+e^{-\pi i(a+b+s)}) \\ &\quad \times (T(a,b,s) + (-1)^b T(b,s,a) + (-1)^a T(s,a,b)). \end{aligned} \tag{2.2}$$

Proof. Let

$$\begin{aligned}
T_1(a,b,s) &:= \sum_{m,n>0} \frac{1}{m^a n^b (m+n)^s} = T(a,b,s), \\
T_2(a,b,s) &:= \sum_{\substack{m<0,\, n>0\\ n>-m}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ n>m}} \frac{1}{(-m)^a n^b (n-m)^s} \\
&= (-1)^{-a} \sum_{m,k>0} \frac{1}{m^a (m+k)^b k^s} = (-1)^{-a} T(s,a,b), \\
T_3(a,b,s) &:= \sum_{\substack{m<0,\, n>0\\ -m>n}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ m>n}} \frac{1}{(-m)^a n^b (n-m)^s} \\
&= e^{-\pi i(a+s)} \sum_{n,k>0} \frac{1}{(n+k)^a n^b k^s} = e^{-\pi i(a+s)} T(b,s,a), \\
T_4(a,b,s) &:= \sum_{m,n<0} \frac{1}{m^a n^b (m+n)^s} = e^{-\pi i(a+b+s)} T(a,b,s), \\
T_5(a,b,s) &:= \sum_{\substack{m>0,\, n<0\\ -n>m}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ n>m}} \frac{1}{m^a (-n)^b (m-n)^s} \\
&= e^{-\pi i(b+s)} T(s,a,b), \\
T_6(a,b,s) &:= \sum_{\substack{m>0,\, n<0\\ m>-n}} \frac{1}{m^a n^b (m+n)^s} = \sum_{\substack{m,n>0\\ m>n}} \frac{1}{m^a (-n)^b (m-n)^s} \\
&= (-1)^{-b} T(b,s,a).
\end{aligned}$$

Obviously we have

$$\sum_{j=1}^{6} T_j(a,b,s) = S(a,b,s).$$

This implies (2.2). We can also see that the convergence of $S(a,b,s)$ is equivalent to the convergence of $T(a,b,s)$. ∎

LEMMA 2.2 ([11]). *For $\Re(s) > 1$, $\Re(t) > 1$ and $\Re(u) > 1$, we have*

$$S(s,t,u) = \int_0^1 \sum_{m\neq 0} \frac{e^{2\pi imx}}{m^s} \sum_{n\neq 0} \frac{e^{2\pi inx}}{n^t} \sum_{l\neq 0} \frac{e^{-2\pi ilx}}{l^u}\, dx. \tag{2.3}$$

Proof. By putting $l = m+n$, we have

$$\begin{aligned} S(s,t,u) &= \sum_{\substack{m,n,l\neq 0\\ m+n=l}} \frac{1}{m^s n^t l^u} = \sum_{m,n,l\neq 0} \int_0^1 \frac{e^{2\pi i(m+n-l)x}}{m^s n^t l^u}\, dx \\ &= \int_0^1 \sum_{m,n,l\neq 0} \frac{e^{2\pi i(m+n-l)x}}{m^s n^t l^u}\, dx = \int_0^1 \sum_{m\neq 0} \frac{e^{2\pi imx}}{m^s} \sum_{n\neq 0} \frac{e^{2\pi inx}}{n^t} \sum_{l\neq 0} \frac{e^{-2\pi ilx}}{l^u}\, dx. \end{aligned}$$

Changing the order of summation and integration is justified by absolute convergence. ■

We denote by $B_j(x)$ the Bernoulli polynomial of order j defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x)\, \frac{t^j}{j!}, \qquad |t| < 2\pi.$$

It is known (see [1, p. 266, (22), and p. 267, (24)]) that

$$B_{2j} := B_{2j}(0) = (-1)^{j+1} 2(2j)!(2\pi)^{-2j}\zeta(2j), \qquad j \in \mathbb{N}, \tag{2.4}$$

$$B_j(x) = -\frac{j!}{(2\pi i)^j} \lim_{K\to\infty} \sum_{\substack{k=-K\\ k\neq 0}}^{K} \frac{e^{2\pi ikx}}{k^j}, \qquad j \in \mathbb{N}. \tag{2.5}$$

For $k \in \mathbb{Z}$, $j \in \mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx} B_j(x)\, dx = \begin{cases} 0, & k = 0, \\ -(2\pi ik)^{-j} j!, & k \neq 0. \end{cases} \tag{2.6}$$

In fact, the case of $k = 0$ is obvious, and in the case of $k \neq 0$, we get (2.6) by using (2.5). Next we quote [1, p. 276, 19(b)], for $p + q \geq 2$, which is

$$\begin{aligned} &B_p(x)B_q(x) \\ &= \sum_{k=0}^{\max(p,q)/2} \left\{ p\binom{q}{2k} + q\binom{p}{2k} \right\} \frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k} - (-1)^p \frac{p!q!}{(p+q)!} B_{p+q}. \end{aligned} \tag{2.7}$$

Proof of Theorem 1.2. Firstly, we assume $a, b \geq 2$, $1 + e^{-\pi i(a+b+s)} \neq 0$ and $\Re(s) > 1$. By using (2.6) and (2.7), we have

$$\begin{aligned}
&-\int_0^1 B_a(x)B_b(x)\sum_{l\neq 0}\frac{e^{-2\pi i l x}}{l^s}\,dx \\
&\quad = -\int_0^1 \sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k} + b\binom{a}{2k}\right\}\frac{B_{2k}B_{a+b-2k}(x)}{a+b-2k}\sum_{l\neq 0}\frac{e^{-2\pi i l x}}{l^s}\,dx \\
&\quad = \sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k} + b\binom{a}{2k}\right\}\frac{(a+b-2k-1)!B_{2k}}{(2\pi i)^{a+b-2k}} \\
&\qquad \times \zeta(a+b+s-2k)(1+e^{-\pi i(a+b+s)}).
\end{aligned}$$

Because of (2.2), (2.4) and (2.5), we obtain (1.3) in this region. By analytic continuation, we have (1.3) for all $a, b \in \mathbb{N}$, $a, b \geq 2$ and $s \in \mathbb{C}$ except for the singular points of each side of this formula.

Next we consider the case of $a = 1$, $b \geq 2$. For $a, b \in \mathbb{N}$, $a, b \geq 2$ and $s \in \mathbb{C}$ except for the singular points, we define $K(a, b, s)$ by the right-hand side of (1.3). We quote some basic properties [3, (1.5)] proved by easy computations, for $s, t, u \in \mathbb{C}$ except for the singular points:

$$\left\{\begin{aligned}
&T(s, t-1, u+1) + T(s-1, t, u+1) = T(s, t, u), \\
&T(s, t+1, u-1) - T(s-1, t+1, u) = T(s, t, u), \\
&T(s+1, t, u-1) - T(s+1, t-1, u) = T(s, t, u).
\end{aligned}\right. \tag{2.8}$$

For $b \geq 2$, we have

$$\begin{aligned}
K(2, b, s) =\ & T(2, b, s) + (-1)^b T(b, s, 2) + (-1)^2 T(s, 2, b) \\
&= T(1, b, s+1) + (-1)^b T(b, s+1, 1) + (-1)T(s+1, 1, b) \\
&\quad + T(2, b-1, s+1) + (-1)^{b-1}T(b-1, s+1, 2) \\
&\quad + (-1)^2 T(s+1, 2, b-1)
\end{aligned}$$

by (2.8) and the result in the case $a, b \geq 2$ which we have already shown. Hence we have to show

$$K(2, b, s) = K(1, b, s+1) + K(2, b-1, s+1), \qquad b \geq 2.$$

In fact we have

$$\begin{aligned}
&\frac{2}{b!}\left\{\binom{b}{2k} + b\binom{1}{2k}\right\} + \frac{2}{2!\,(b-1)!}\left\{2\binom{b-1}{2k} + (b-1)\binom{2}{2k}\right\} \\
&\qquad = \frac{2}{2!\,b!}\left\{2\binom{b}{2k} + b\binom{2}{2k}\right\}(b+1-2k), \qquad 0 \leq k \leq b/2.
\end{aligned}$$

In the cases of $k = 0, 1, b/2$, we have this equation immediately. For $2 \leq k \leq$

$(b-1)/2$, we obtain it by

$$b\binom{b-1}{l} = \frac{b(b-1)\cdots(b-l+1)(b-l)}{l!} = (b-l)\binom{b}{l}, \qquad 0 \le l \le b.$$

We can prove (1.3) for the case of $a = b = 1$ similarly. ∎

3. New proofs of known formulas. In this section, from our theorem we deduce formulas for the special values of $T(a,b,c)$ $(a,b,c \in \mathbb{N})$ mentioned in the introduction. By taking $a = 2p$, $b = 2q$, $s = 2r$ in (1.3), we have

$$T(2p,2q,2r) + T(2q,2r,2p) + T(2r,2p,2q)$$
$$= \frac{2}{(2p)!\,(2q)!} \sum_{k=0}^{\max(p,q)} \left\{ 2p\binom{2q}{2k} + 2q\binom{2p}{2k} \right\} (2p+2q-2k-1)!(2k)!$$
$$\times \zeta(2k)\zeta(2p+2q+2r-2k).$$

This formula coincides with [6, Theorem 4.1]. (There is a misprint in [6, Theorem 4.1], "min" is to be replaced by "max".) Putting $a = b = s = r$ in (1.3) we have, after easy computations of binomial coefficients,

$$T(r,r,r) = \frac{4}{1+2(-1)^r} \sum_{k=0}^{r/2} \binom{2r-2k-1}{2k-1} \zeta(2k)\zeta(3r-2k).$$

This formula is [3, Theorem 3].

For $a,b,c \in \mathbb{N}$, we define $N(a,b,c)$ as half of the right-hand side of (1.3). We recall the harmonic product formula

$$T(a,0,b) + T(b,0,a) = \zeta(a)\zeta(b) - \zeta(a+b).$$

Putting $s = 0$ in (1.3) and multiplying by $(-1)^a$, we obtain

$$(-1)^a\zeta(a)\zeta(b) + (-1)^{a+b}T(b,0,a) + T(a,0,b) = 2(-1)^a N(a,b,0).$$

When $a + b \in 2\mathbb{N}+1$, we can remove $T(b,0,a)$ by summing the above two formulas. Hence

$$T(a,0,b) = -\frac{\zeta(a+b)}{2} + \frac{1+(-1)^b}{2}\,\zeta(a)\zeta(b) + (-1)^a N(a,b,0) \tag{3.1}$$

for all $a,b \ge 2$, $a+b \in 2\mathbb{N}+1$. Next by changing the variables in (1.3), we obtain

$$\begin{cases} (-1)^b T(a,b,c) + T(b,c,a) + (-1)^c T(c,a,b) = 2N(b,c,a), \\ (-1)^a T(a,b,c) + (-1)^c T(b,c,a) + T(c,a,b) = 2N(c,a,b). \end{cases}$$

In the case of $a+b+c \in 2\mathbb{N}+1$, we can remove $T(b,c,a)$ and $T(c,a,b)$ by multiplying the former equality by $(-1)^b$ and the latter by $(-1)^a$, and summing the resulting formulas. Hence we have

$$T(a,b,c) = (-1)^b N(b,c,a) + (-1)^a N(c,a,b), \qquad a+b+c \in 2\mathbb{N}+1. \tag{3.2}$$

By putting $s = t = 1$ in the first equation of (2.8), we obtain

$$T(1, 1, u) = 2T(1, 0, u + 1).$$

Hence we can calculate $T(1, 0, c + 1)$ if $c + 1 \in 2\mathbb{N}$. Therefore we obtain another proof of [3, Theorems 1, 2]. Moreover we get

$$T(p, q, r) + (-1)^p T(p, r, q) + (-1)^{p+r} T(r, q, p) = 2(-1)^p N(p, r, q)$$

by taking $a = p$, $b = r$ and $s = q$ in (1.3), and multiplying by $(-1)^p$. Hence we obtain another proof of [8, Theorem 1], because $N(p, q, r)$ is a polynomial in $\{\zeta(k) \mid 2 \le k \le p + q + r\}$ with rational coefficients for $p, q, r \in \mathbb{N} \cup \{0\}$ with $p + q \ge 2$ and $r \ge 2$.

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