A functional relation for the
Tornheim double zeta function

by

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1. Introduction

DEFINITION 1.1. The Tornheim double zeta function $T(s, t, u)$, for $s, t, u \in \mathbb{C}$, $\Re(s + u) > 1$, $\Re(t + u) > 1$ and $\Re(s + t + u) > 2$, is defined by

\begin{equation}
T(s, t, u) := \sum_{m,n=1}^{\infty} \frac{1}{m^{s}n^{t}(m+n)^{u}}.
\end{equation}

This function $T(s, t, u)$ is a generalization of the Riemann zeta function $\zeta(s)$, $s \in \mathbb{C}$. Furthermore, $T(s, t, u)$ is continued meromorphically to $\mathbb{C}^3$ in [4]. By the definition, we have

$T(s, t, u) = T(t, s, u)$, $T(s, t, 0) = \zeta(s)\zeta(t)$.

The case of $t = 0$, that is $T(s, 0, u)$, is called the Euler–Zagier double zeta function [10].

The values $T(a, b, c)$ for $a, b, c \in \mathbb{N}$ were first investigated by Tornheim [7] in 1950 and later Mordell [5] in 1958. Tornheim [7, Theorem 7] showed that $T(a, b, c)$ can be expressed as a polynomial in $\{\zeta(j) \mid 2 \leq j \leq a+b+c\}$ with rational coefficients when $a+b+c$ is odd, and that the same is true for $T(2r, 2r, 2r)$ and $T(2r-1, 2r, 2r+1)$ [7, Theorem 8], but he did not give the coefficients. Mordell [5, Theorem III] proved that $T(2r, 2r, 2r) = k_r \pi^{6r}$ for some rational number $k_r$. In 1985 Subbarao and Sitaramachandrarao [6, Theorem 4.1] explicitly determined $T(2p, 2q, 2r)+T(2q, 2r, 2p)+T(2r, 2p, 2q)$ ($p, q, r \in \mathbb{N}$). Then, by taking $p = q = r$, they gave an explicit formula for $T(2r, 2r, 2r)$ ($r \in \mathbb{N}$) [6, Remark 3.1]. In 1996 Huard, Williams and Zhang [3, Theorems 1–3] determined $T(r, 0, N-r)$ ($r \in \mathbb{N}$, $N \in 2\mathbb{N}+1$, $1 \leq r \leq N-2$), $T(p, q, N-p-q)$ ($p, q \in \mathbb{N} \cup \{0\}$, $N \in 2\mathbb{N}+1$, $1 \leq p+q \leq N-1$, $0 \leq p, q \leq N-2$) and $T(r, r, r)$ ($r \in \mathbb{N}$). In 2002 Tsumura [8, Theorem 1]...
proved that \( T(p, q, r) + (-1)^p T(p, r, q) + (-1)^{p+r} T(r, q, p) \) is a polynomial in \( \{ \zeta(k) \mid 2 \leq k \leq p + q + r \} \) with rational coefficients for \( p, q, r \in \mathbb{N} \cup \{0\} \) with \( p + q \geq 2 \) and \( r \geq 2 \). Recently, Espinosa and Moll provided an explicit formula for \( T(x, y, z) \), \( x, y, z \in \mathbb{R} \), in terms of integrals involving Hurwitz zeta functions (see [2, Proposition 2.1 and Theorem 2.4]). Also in 2006 Tsumura [9, Theorem 4.5] proved the following functional relation:

\[
(1.2) \quad T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b)
\]

\[
= 2 \sum_{j=0}^{a} (21-a+j-1) \zeta(a-j) \sum_{l=0}^{j/2} \frac{(i\pi)^{2l}}{(2l)!} \binom{b-1+j-2l}{j-2l} \zeta(b+j+s-2l)
\]

\[
- 4 \sum_{j=0}^{a} (21-a+j-1) \zeta(a-j) \sum_{l=0}^{(j-1)/2} \frac{(i\pi)^{2l}}{(2l+1)!} \sum_{k=0}^{b} \zeta(b-k)
\]

\[
\times \binom{k-1+j-2l}{j-2l-1} \zeta(k+j+s-2l)
\]

(1.2) is considerably simpler than that of Tsumura. We are not aware of a direct proof which shows that the right-hand sides of (1.2) and (1.3) are the same. “Mathematica 5.0” shows that they are equal for all \( 1 \leq a \leq b \leq 100 \), \( a, b \in \mathbb{N} \). It therefore seems unlikely that a non-trivial functional relation can be deduced by equating (1.2) and (1.3).

In Section 3, we obtain new proofs of formulas for the special values of \( T(a, b, c) \), \( a, b, c \in \mathbb{N} \) mentioned in the introduction by using the functional relation (1.3).

2. Proof of Theorem 1.2. Firstly, we define \( \log t, t \in \mathbb{C} \), and \( t^s \), \( s, t \in \mathbb{C} \), by

\[
\log t := \log |t| + i \arg t, \quad t^s := e^{s \log t}, \quad 0 \leq \arg t < 2\pi.
\]
And for $s, t, u \in \mathbb{C}$, $\Re(s + u) > 1$, $\Re(t + u) > 1$ and $\Re(s + t + u) > 2$, we put

$$S(s, t, u) := \sum_{m \neq 0, n \neq 0 \atop m + n \neq 0} \frac{1}{m^s n^t (m + n)^u}.$$  \hfill (2.1)

**Lemma 2.1.** For all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points, we have

$$S(a, b, s) = (1 + e^{-\pi i (a + b + s)}) \times (T(a, b, s) + (-1)^b T(b, s, a) + (-1)^a T(s, a, b)).$$  \hfill (2.2)

**Proof.** Let

$$T_1(a, b, s) := \sum_{m, n > 0} \frac{1}{m^a n^b (m + n)^s} = T(a, b, s),$$

$$T_2(a, b, s) := \sum_{m < 0, n > 0 \atop n > -m} \frac{1}{m^a n^b (m + n)^s} = \sum_{m, n > 0} \frac{1}{m^a n^b (n - m)^s} = (-1)^{-a} \sum_{m, k > 0} \frac{1}{m^a (m + k)^b k^s} = (-1)^{-a} T(s, a, b),$$

$$T_3(a, b, s) := \sum_{m < 0, n > 0 \atop -m > n} \frac{1}{m^a n^b (m + n)^s} = \sum_{m, n > 0} \frac{1}{m^a n^b (n - m)^s} = e^{-\pi i (a + s)} \sum_{n, k > 0} \frac{1}{(n + k)^a n^b k^s} = e^{-\pi i (a + s)} T(b, s, a),$$

$$T_4(a, b, s) := \sum_{m, n < 0} \frac{1}{m^a n^b (m + n)^s} = e^{-\pi i (a + b + s)} T(a, b, s),$$

$$T_5(a, b, s) := \sum_{m > 0, n < 0 \atop -n > m} \frac{1}{m^a n^b (m + n)^s} = \sum_{m, n > 0} \frac{1}{m^a (-n)^b (m - n)^s} = e^{-\pi i (b + s)} T(s, a, b),$$

$$T_6(a, b, s) := \sum_{m > 0, n < 0 \atop m > -n} \frac{1}{m^a n^b (m + n)^s} = \sum_{m, n > 0} \frac{1}{m^a (-n)^b (m - n)^s} = (-1)^{-b} T(b, s, a).$$

Obviously we have

$$\sum_{j=1}^{6} T_j(a, b, s) = S(a, b, s).$$

This implies (2.2). We can also see that the convergence of $S(a, b, s)$ is equivalent to the convergence of $T(a, b, s)$. \hfill \blacksquare
Lemma 2.2 ([11]). For $\Re(s) > 1$, $\Re(t) > 1$ and $\Re(u) > 1$, we have

\begin{equation}
S(s, t, u) = \sum_{m, n, l \neq 0} \frac{e^{2\pi imx}}{m^s n^t l^u} \sum_{m, n, l \neq 0} e^{2\pi inx} \sum_{m, n, l \neq 0} e^{-2\pi ilx} dx.
\end{equation}

Proof. By putting $l = m + n$, we have

\begin{equation}
S(s, t, u) = \sum_{m, n, l \neq 0} \frac{1}{m^s n^t l^u} = \sum_{m, n, l \neq 0} \sum_{m, n, l \neq 0} e^{2\pi i(m+n-l)x} dx = \sum_{m, n, l \neq 0} \sum_{m, n, l \neq 0} e^{2\pi imx} \sum_{m, n, l \neq 0} e^{2\pi inx} \sum_{m, n, l \neq 0} e^{-2\pi ilx} dx.
\end{equation}

Changing the order of summation and integration is justified by absolute convergence.

We denote by $B_j(x)$ the Bernoulli polynomial of order $j$ defined by

\begin{equation}
t e^x - 1 = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}, \quad |t| < 2\pi.
\end{equation}

It is known (see [1, p. 266, (22), and p. 267, (24)]) that

\begin{align}
B_{2j} &= (-1)^{j+1} 2(2j)! (2\pi)^{-2j} \zeta(2j), \quad j \in \mathbb{N}, \\
B_j(x) &= -\frac{j!}{(2\pi i)^j} \lim_{K \to \infty} \sum_{k=-K}^{K} \frac{e^{2\pi ikx}}{k^j}, \quad j \in \mathbb{N}.
\end{align}

For $k \in \mathbb{Z}$, $j \in \mathbb{N}$ we have

\begin{equation}
\int_0^1 e^{-2\pi ikx} B_j(x) dx = \begin{cases} 0, & k = 0, \\
-(2\pi ik)^{-j} j!, & k \neq 0.
\end{cases}
\end{equation}

In fact, the case of $k = 0$ is obvious, and in the case of $k \neq 0$, we get (2.6) by using (2.5). Next we quote [1, p. 276, 19(b)], for $p + q \geq 2$, which is

\begin{equation}
B_p(x)B_q(x) = \sum_{k=0}^{\max(p, q)/2} \left\{ p\left(\frac{q}{2k}\right) + q\left(\frac{p}{2k}\right) \right\} B_{2k} B_{p + q - 2k}(x) \frac{p^k q^k}{p + q - 2k} - (-1)^{p} \frac{p! q!}{(p + q)!} B_{p + q}.
\end{equation}
Proof of Theorem 1.2. Firstly, we assume \( a, b \geq 2 \), \( 1 + e^{-\pi i(a+b+s)} \neq 0 \) and \( \Re(s) > 1 \). By using (2.6) and (2.7), we have

\[
- \int_0^1 B_a(x)B_b(x) \sum_{l \neq 0} \frac{e^{-2\pi ilx}}{l^s} \, dx
\]

\[
= - \int_0^1 \sum_{k=0}^{\max(a,b)/2} \left\{ a \left( \frac{b}{2k} \right) + b \left( \frac{a}{2k} \right) \right\} \frac{B_{2k}B_{a+b-2k}(x)}{a+b-2k} \sum_{l \neq 0} \frac{e^{-2\pi ilx}}{l^s} \, dx
\]

\[
= \sum_{k=0}^{\max(a,b)/2} \left\{ a \left( \frac{b}{2k} \right) + b \left( \frac{a}{2k} \right) \right\} \frac{(a+b-2k-1)!B_{2k}}{(2\pi i)^{a+b-2k}}
\]

\[
\times \zeta(a+b+s-2k)(1 + e^{-\pi i(a+b+s)}).
\]

Because of (2.2), (2.4) and (2.5), we obtain (1.3) in this region. By analytic continuation, we have (1.3) for all \( a, b \in \mathbb{N}, a, b \geq 2 \) and \( s \in \mathbb{C} \) except for the singular points of each side of this formula.

Next we consider the case of \( a = 1, b \geq 2 \). For \( a, b \in \mathbb{N}, a, b \geq 2 \) and \( s \in \mathbb{C} \) except for the singular points, we define \( K(a,b,s) \) by the right-hand side of (1.3). We quote some basic properties \([3, (1.5)]\) proved by easy computations, for \( s, t, u \in \mathbb{C} \) except for the singular points:

\[
T(s,t-1,u+1) + T(s-1,t,u+1) = T(s,t,u),
\]

\[
T(s,t+1,u-1) - T(s-1,t+1,u) = T(s,t,u),
\]

\[
T(s+1,t,u-1) - T(s+1,t-1,u) = T(s,t,u).
\]

(2.8)

For \( b \geq 2 \), we have

\[
K(2,b,s) = T(2,b,s) + (-1)^b T(b,s,2) + (-1)^2 T(s,2,b)
\]

\[
= T(1,b,s+1) + (-1)^b T(b,s+1,1) + (-1) T(s+1,1,b)
\]

\[
+ T(2,b-1,s+1) + (-1)^{b-1} T(b-1,s+1,2)
\]

\[
+ (-1)^2 T(s+1,2,b-1)
\]

by (2.8) and the result in the case \( a,b \geq 2 \) which we have already shown. Hence we have to show

\[
K(2,b,s) = K(1,b,s+1) + K(2,b-1,s+1), \quad b \geq 2.
\]

In fact we have

\[
\frac{2}{b!} \left\{ \left( \frac{b}{2k} \right) + b \left( \frac{1}{2k} \right) \right\} = \frac{2}{2! (b-1)!} \left\{ 2 \left( \frac{b-1}{2k} \right) + (b-1) \left( \frac{2}{2k} \right) \right\}
\]

\[
= \frac{2}{2! b!} \left\{ 2 \left( \frac{b}{2k} \right) + b \left( \frac{2}{2k} \right) \right\} (b+1-2k), \quad 0 \leq k \leq b/2.
\]

In the cases of \( k = 0, 1, b/2 \), we have this equation immediately. For \( 2 \leq k \leq b/2 \), we have
(b − 1)/2, we obtain it by
\[ b \binom{b - 1}{l} = \frac{b(b - 1) \cdots (b - l + 1)(b - l)}{l!} = (b - l) \binom{b}{l}, \quad 0 \leq l \leq b. \]

We can prove (1.3) for the case of a = b = 1 similarly. ■

3. New proofs of known formulas. In this section, from our theorem we deduce formulas for the special values of \( T(a, b, c) \) \((a, b, c \in \mathbb{N})\) mentioned in the introduction. By taking \( a = 2p, b = 2q, s = 2r \) in (1.3), we have
\[
T(2p, 2q, 2r) + T(2q, 2r, 2p) + T(2r, 2p, 2q)
\]
\[
= \frac{2}{(2p)! (2q)!} \sum_{k=0}^{\max(p,q)} \left\{ 2p \binom{2q}{2k} + 2q \binom{2p}{2k} \right\} (2p + 2q - 2k - 1)! (2k)!
\]
\[
\times \zeta(2k) \zeta(2p + 2q + 2r - 2k).
\]

This formula coincides with [6, Theorem 4.1]. (There is a misprint in [6, Theorem 4.1], “min” is to be replaced by “max”.) Putting \( a = b = s = r \) in (1.3) we have, after easy computations of binomial coefficients,
\[
T(r, r, r) = \frac{4}{1 + 2(-1)^r} \sum_{k=0}^{r/2} \left( \binom{2r - 2k - 1}{2k - 1} \right) \zeta(2k) \zeta(3r - 2k).
\]

This formula is [3, Theorem 3].

For \( a, b, c \in \mathbb{N} \), we define \( N(a, b, c) \) as half of the right-hand side of (1.3). We recall the harmonic product formula
\[
T(a, 0, b) + T(b, 0, a) = \zeta(a) \zeta(b) - \zeta(a + b).
\]

Putting \( s = 0 \) in (1.3) and multiplying by \((-1)^a\), we obtain
\[
(-1)^a \zeta(a) \zeta(b) + (-1)^{a+b} T(b, 0, a) + T(a, 0, b) = 2(-1)^a N(a, b, 0).
\]

When \( a + b \in 2\mathbb{N} + 1 \), we can remove \( T(b, 0, a) \) by summing the above two formulas. Hence
\[
T(a, 0, b) = -\frac{\zeta(a + b)}{2} + \frac{1 - (-1)^b}{2} \zeta(a) \zeta(b) + (-1)^a N(a, b, 0)
\]
for all \( a, b \geq 2, a + b \in 2\mathbb{N} + 1 \). Next by changing the variables in (1.3), we obtain
\[
\begin{cases}
(\zeta(a) + b) T(a, b, c) + T(b, c, a) + (-1)^c T(c, a, b) = 2N(b, c, a), \\
(\zeta(a) - c) T(a, b, c) + (-1)^c T(b, c, a) + T(c, a, b) = 2N(c, a, b).
\end{cases}
\]

In the case of \( a + b + c \in 2\mathbb{N} + 1 \), we can remove \( T(b, c, a) \) and \( T(c, a, b) \) by multiplying the former equality by \((-1)^b\) and the latter by \((-1)^a\), and summing the resulting formulas. Hence we have
\[
T(a, b, c) = (-1)^b N(b, c, a) + (-1)^a N(c, a, b), \quad a + b + c \in 2\mathbb{N} + 1.
\]
By putting \( s = t = 1 \) in the first equation of (2.8), we obtain

\[
T(1, 1, u) = 2T(1, 0, u + 1).
\]

Hence we can calculate \( T(1, 0, c + 1) \) if \( c + 1 \in 2\mathbb{N} \). Therefore we obtain another proof of [3, Theorems 1, 2]. Moreover we get

\[
T(p, q, r) + (-1)^p T(p, r, q) + (-1)^p + r T(r, q, p) = 2(-1)^p N(p, r, q)
\]

by taking \( a = p, b = r \) and \( s = q \) in (1.3), and multiplying by \((-1)^p\). Hence we obtain another proof of [8, Theorem 1], because \( N(p, q, r) \) is a polynomial in \( \{\zeta(k) \mid 2 \leq k \leq p + q + r\} \) with rational coefficients for \( p, q, r \in \mathbb{N} \cup \{0\} \) with \( p + q \geq 2 \) and \( r \geq 2 \).

Acknowledgments. I thank Professors Kohji Matsumoto and Hiroyuki Tsumura for useful advice.

References


Received on 2.2.2006
and in revised form on 30.7.2006

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