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Lattice points in bodies with algebraic boundary

by

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1. Introduction. Let F be a polynomial of even degree d in s variables with integer coefficients. Assume that the leading homogeneous part $F^{(d)}$ in the decomposition $F = F^{(d)} + G$ with $\deg(G) < d$ is positive definite. Then $D_F(R) = \{x \in \mathbb{R}^d \mid F(x) \le R\}$ is compact. Denote by $A_F(R)$ the number of lattice points of the standard lattice $\mathbb{Z}^s$ which are contained in $D_F(R)$. Then $A_F(R)$ is approximately equal to $\mathrm{vol}(D_F(R))$. It is easy to see that the discrepancy $P_F(R) = A_F(R) - \mathrm{vol}(D_F(R))$ satisfies

$$P_F(R) = \Omega(R^{s/d-1}). \tag{1}$$

One only has to observe that $A_F(R+\varepsilon) = A_F(R)$ for $R \in \mathbb{N}$ and $0 < \varepsilon < 1$, but $\mathrm{vol}(D_F(R+\varepsilon)) - \mathrm{vol}(D_F(R)) \gg R^{s/d-1}$. Our aim is to give a sharp upper bound for $P_F(R)$. To formulate the main result we introduce the invariant $h(F)$ of F, defined as the smallest integer h such that $F^{(d)}$ has a representation

$$F^{(d)} = \sum_{i=1}^{h} A_i B_i$$

with homogeneous polynomials $A_i, B_i \in \mathbb{Q}[X_1, \ldots, X_s]$ of positive degree.

THEOREM 1. *Assume that $h(F) > \varrho(d)$ where $\varrho(2) = 4$, $\varrho(4) = 288$ and $\varrho(d) = d(d-1)2^{d-1}(\log 2)^{-d} d!$ for $d > 4$. Then for $R \ge 1$,*

$$P_F(R) = O(R^{s/d-1}). \tag{2}$$

In the case $d = 2$ it is easy to see that $h(F) = s$. Thus Theorem 1 contains as a special case the well known theorem of Walfisz [10] and Landau [4] who proved (2) for *rational* quadratic forms of dimension $s > 4$. If $F^{(d)}$ is non-singular, i.e. the only solution of $\frac{\partial}{\partial x_i}(F^{(d)}(x)) = 0$, $1 \le i \le s$, in $\mathbb{C}^s$ is $x = 0$, then $h(F) \ge s/2$ (cf. [7, p. 282]). In this case the theorem gives the exact order of $P_F(R)$ if $s > 2\varrho(d)$. The proof of Theorem 1 uses a variant of the Hardy–Littlewood method. For general F this method was first used by

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Schmidt in his famous work on diophantine equations [6], [8]. For special F the estimate (2) can be true for much smaller s. As an example we prove

THEOREM 2. *Let* $F_0(X) = \sum_{i=1}^{s} \lambda_i X_i^d$ *with* $d \geq 2$ *even and integer coefficients* $\lambda_i > 0$. *Then* $P_{F_0}(R) = O(R^{s/d-1})$, *provided that* $s \geq \min(d2^{d-1}, \varrho_0(d))$. *Here* ϱ_0 *denotes an explicitly computable function which satisfies* $\varrho_0(d) \sim 2d^3 \log d$ *for* $d \to \infty$.

As noted by Randol [5] Theorem 2 cannot be true if $s < d^2 - d + 1$. See Krätzel [3] for a detailed study of $P_{F_0}(R)$ for small s. With some obvious modifications our proof shows that Theorem 2 remains true for real coefficients $\lambda_i > 0$.

Recently, Bentkus and Götze [1] studied $P_F(R)$ for polynomials F with real coefficients and leading homogeneous part

$$F^{(d)}(X) = \sum_{i=1}^{s_0} \lambda_i X_i^d + P(X) \quad (\lambda_i > 0). \tag{3}$$

Here P denotes a homogeneous polynomial of degree d such that the degree of P viewed as a polynomial in $(X_1, \ldots, X_{s_0})$ is strictly smaller than d. They proved (2) under the assumptions that $s_0 = s$ and $s > \alpha(d)$ or $s_0 < s$ and $s_0 > 2^d \alpha(d)$, where $\alpha(2) = 8$, $\alpha(4) = 1512$ and $\alpha(d) = d2^{d-1}e^{3d \log d}$ for $d > 4$. The condition (3) on the leading homogeneous part of F is rather restrictive. Bentkus and Götze already remarked that one should expect that (2) is true for general F if $h(F)$ is sufficiently large. The main advantage of their method is that it applies to polynomials with real coefficients, whereas we have to assume that F has integer coefficients.

2. The Hardy–Littlewood method. Let $B = (-1, 1]^s$. Assume that $R \in \mathbb{N}$ and $D_F(R) \subseteq R^{1/d} B$ for $R \geq c(F)$ sufficiently large. Otherwise consider cF instead of F, where $c \in \mathbb{N}$ is sufficiently large, and use $A_F(R) = A_{cF}(cR)$. To count the number of lattice points in $D_F(R)$ we introduce the auxiliary function $\chi = I_{(-R-1/2, R+1/2)} * \delta$ which is the convolution of the indicator function with a symmetric probability density $\delta \in \mathbb{C}^\infty(\mathbb{R})$ satisfying $\operatorname{supp}(\delta) \subseteq [-1/2, 1/2]$. Then $\chi(u) = 1$ if $|u| \leq R$, $\chi(u) = 0$ if $|u| \geq R + 1$ and $0 \leq \chi(u) \leq 1$ if $R < |u| < R + 1$. By Fourier inversion one obtains

$$\chi(u) = \int_{\mathbb{R}} \widehat{\chi}(t) e(-tu)\, dt = \int_{\mathbb{R}} \widehat{\chi}(t) e(tu)\, dt, \tag{4}$$

where

$$\widehat{\chi}(t) = \int_{\mathbb{R}} \chi(u) e(tu)\, du = \widehat{I}_{(-R-1/2, R+1/2)}(t) \widehat{\delta}(t).$$

Here $e(x) = e^{2\pi i x}$ as usual. Furthermore,

$$\widehat{I}_{(-R-1/2,R+1/2)}(t) = \frac{1}{\pi t}\sin(2\pi t(R+1/2)).$$

Applying j-fold partial integration one obtains $\widehat{\delta}(t) \ll_j (|t|+1)^{-j}$ for $j \geq 0$. Hence

$$\widehat{\chi}(t) \ll \frac{1}{|t|}(1+|t|)^{-j} \qquad (j \geq 0). \tag{5}$$

Set $N = \lceil (R+1)^{1/d}\rceil + 1/2$. Then $F(k) \leq R$ implies $k \in NB$ and (4) yields

$$A_F(R) = \sum_{n \in NB \cap \mathbb{Z}^s} \chi(F(n)) = \int_{\mathbb{R}} S_N(t)\widehat{\chi}(t)\,dt \tag{6}$$

with

$$S_N(t) = \sum_{n \in NB \cap \mathbb{Z}^s} e(tF(n)).$$

This should be compared with the following integral which counts the number of lattice points on the boundary of $D_F(R)$:

$$\int_0^1 S_N(t)e(-tR)\,dt.$$

It is not surprising that the properties of $S_N(t)$ known from the Hardy–Littlewood method can be used to analyse $A_F(R)$. The main difference comes from the behaviour of $\widehat{\chi}(t)$ for small t. Note that $S_N(t)$ is one-periodic if F has integer coefficients. The following proposition deals with these small values of t.

Proposition. *Assume that for $N \geq 1$:*

(A) $\displaystyle\int_{(0,1]} |S_N(t)|\,dt \ll N^{s-d}$.

(B) $\displaystyle\int_{(N^{1-d},1]} |S_N(t)|\,\frac{dt}{t} \ll N^{s-d}$.

(C) *There exists an $\omega > d$ such that for $|t| \leq N^{1-d}$*

$$\sum_{n \in NB' \cap \mathbb{Z}^s} e(tF(n+u)) \ll N^{s-\omega d}|t|^{-\omega} \tag{7}$$

uniformly in $u \in B$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes.

(D) *There exists an $\omega > d$ such that for $|t| \geq N^{-d}$,*

$$\int_{NB'} e(tF(x))\,dx \ll N^{s-\omega d}|t|^{-\omega} \tag{8}$$

uniformly in all boxes $B' \subseteq B$ with sides parallel to the coordinate axes.

Then $P_F(R) \ll R^{s/d-1}$.

The proof of this Proposition is given in Section 3. Here we describe the "axiomatic" form of the Hardy–Littlewood method given by Schmidt [6]. If F is a polynomial with integer coefficients, $S_N(t)$ can be evaluated asymptotically in a neighbourhood of a rational number with small denominator. The union of these neighbourhoods is called the *major arcs*. To be precise let $0 < \Delta \le 1$ and set, for $1 \le a \le q \le N^{\Delta}$ with $(a,q)=1$,

$$\mathfrak{M}_{\Delta}(q,a) = \left\{ t \in \mathbb{R}/\mathbb{Z} \;\middle|\; \left| t - \frac{a}{q} \right| < \frac{1}{q} N^{\Delta - d} \right\}.$$

Then the major arcs and minor arcs are defined by

$$\mathfrak{M}_{\Delta} = \bigcup_{\substack{1 \le a \le q \le N^{\Delta} \\ (a,q)=1}} \mathfrak{M}_{\Delta}(q,a) \quad \text{and} \quad \mathfrak{m}_{\Delta} = (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_{\Delta}.$$

Note that $\mathfrak{M}_{\Delta}$ is the union of disjoint intervals if N is sufficiently large.

If F is homogeneous, i.e. $F = F^{(d)}$, we define $\Omega(F)$ as the supremum of all $\omega > 0$ such that for all $\Delta \in (0,1]$ and $t \in \mathfrak{m}_{\Delta}$,

$$\sum_{n \in NB' \cap \mathbb{Z}^s} e(tF(n+u)) \ll_{F,\omega} N^{s-\omega\Delta} \tag{9}$$

uniformly for all $u \in B$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes. If F is an arbitrary polynomial with leading form $F^{(d)}$ we define $\Omega(F)$ as the supremum of all $\omega > 0$ such that for all $\Delta \in (0,1]$ and $t \in \mathfrak{m}_{\Delta}$,

$$\sum_{n \in NB' \cap \mathbb{Z}^s} e(tF^{(d)}(n) + P(n)) \ll_{F,\omega} N^{s-\omega\Delta} \tag{10}$$

uniformly for all polynomials $P \in \mathbb{R}[X_1, \ldots, X_s]$ with $\deg(P) < d$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes.

$\Omega(F)$ is similar to the invariant $\omega(F)$ introduced by Schmidt [6]. The latter is defined as the supremum of all $\omega > 0$ such that for all $\Delta \in (0,1]$ and $t \in \mathfrak{m}_{\Delta}$, (9) is true with $u = 0$ uniformly for all boxes $B' \subseteq B$. We prove that the assumption $\Omega(F) > d$ implies (A)–(D) of the above Proposition.

THEOREM 3. *If* $\Omega(F) > d$ *then* $P_F(R) \ll R^{s/d-1}$.

Theorem 1 follows immediately from Theorem 3 and the following inequality:

$$\Omega(F) \ge \frac{h(F)}{\tau(d)}\,. \tag{11}$$

Here $\tau(2) = 2$, $\tau(4) = 72$ and $\tau(d) < (d-1)2^{d-1}(\log 2)^{-d} d!$ in general. With $\Omega(F)$ replaced by $\omega(F)$ this is Theorem 6.A in [6, p. 86]. We have to verify that Schmidt's inequality remains true with our modified invariant $\Omega(F)$. To see this note that Schmidt's proof starts with a d-fold application of Weyl's inequality. This transforms the exponential sum in the definition of $\Omega(F)$ into an exponential sum of the form $\sum e(G_d(n_1,\dots,n_d))$, where $G(X) = tF^{(d)}(X) + P(X)$ and G_d is the unique symmetric multilinear form which satisfies $G^{(d)}(X) = \frac{(-1)^d}{d!} G_d(X,\dots,X)$. If P is a polynomial of degree strictly less than d, then $P_d = 0$. It follows that $G_d = tF_d^{(d)}$. Hence the new exponential sum does not depend on P. From this moment on, one proceeds as in [6]. Note that $\Omega(F)$ and the above lower bound on $\Omega(F)$ depend only on the leading form of F.

3. Proof of the Proposition. Assume that conditions (A)–(D) of the Proposition are satisfied. The representation (6), together with (5), (A) and (B), yields

$$\begin{aligned} A_F(R) &= \int\limits_{|t|\le N^{1-d}} S_N(t)\widehat{\chi}(t)\,dt \\ &\quad + O\Big(\int\limits_{(N^{1-d},1]} |S_N(t)|\,\frac{dt}{t} + \sum_{j=1}^{\infty}\frac{1}{j^2}\int\limits_{(j,j+1]} |S_N(t)|\,dt\Big) \\ &= \int\limits_{|t|\le N^{1-d}} S_N(t)\widehat{\chi}(t)\,dt + O(N^{s-d}). \end{aligned} \tag{12}$$

If $|t| \le N^{1-d}$ we use an asymptotic expansion of $S_N(t)$. There are several ways to obtain it. We use the following expansion of a sufficiently smooth complex-valued function $g : \mathbb{R}^s \to \mathbb{C}$ due to Bentkus and Götze [1]. Let $J \in \mathbb{N}$, and $x, u_1, \dots, u_J \in \mathbb{R}^s$. Then

$$g(x) = g(x+u_1) + \sum_{j=1}^{J-1} g_j + r_J, \tag{13}$$

where for $1 \le j < J$,

$$g_j = \sum_{|\alpha|=j} c(\alpha) g^{(j)}(x+u_{m+1})[u_1^{\alpha_1}\dots u_m^{\alpha_m}]$$

and

$$r_J = \sum_{|\alpha|=J} c'(\alpha)\int_0^1 (1-\tau)^{\alpha_m-1} g^{(J)}(x+\tau u_m)[u_1^{\alpha_1}\dots u_m^{\alpha_m}]\,d\tau.$$

The summation extends over all $\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}^m$ with $1 \le m \le j$ and $|\alpha| = \sum_{i=1}^m \alpha_i = j$. Furthermore, $g^{(j)}(x)[u_1^{\alpha_1}\dots u_m^{\alpha_m}]$ denotes the j-fold directional derivative

$$g^{(j)}(x)[u_1^{\alpha_1}\dots u_m^{\alpha_m}] = \frac{\partial^j}{\partial\lambda_1^{\alpha_1}\dots\partial\lambda_m^{\alpha_m}}\, g(x+\lambda_1 u_1 + \dots + \lambda_m u_m)\Big|_{\lambda_1=\dots=\lambda_m=0}$$

and

$$c(\alpha) = \frac{(-1)^m}{\alpha_1! \dots \alpha_m!}, \qquad c'(\alpha) = \frac{(-1)^m}{\alpha_1! \dots \alpha_{m-1}!(\alpha_m - 1)!}.$$

This expansion can be obtained by iteratively applying Taylor expansions, first to $\lambda \mapsto g(x + \lambda u_1)$ and then for every summand $g^{(\alpha_1)}(x)[u_1^{\alpha_1}]$ in the resulting expansion to $\lambda \mapsto g^{(\alpha_1)}(x + \lambda u_2)[u_1^{\alpha_1}]$. After J such steps one obtains (13).

We use (13) with $g(x) = e(tF(x))$. Summing over $x \in NB \cap \mathbb{Z}^s$ and integrating over $(u_1, \dots, u_J) \in T^J$ with $T = (-1/2, 1/2]^s$, yields

$$S_N(t) = G_0(t) + \sum_{j=1}^{J-1} G_j(t) + R_J(t)\,, \tag{14}$$

where

$$G_0(t) = \int_T \sum_{x \in NB \cap \mathbb{Z}^s} g(x + u_1)\, du_1 = \int_{NB} g(x)\, dx,$$

$$G_j(t) = \sum_{|\alpha| = j} c(\alpha) \int_{T^m} \Big(\int_{NB} g^{(j)}(x)[u_1^{\alpha_1} \dots u_m^{\alpha_m}]\, dx \Big)\, d(u_1, \dots, u_m),$$

$$R_J(t) = \sum_{|\alpha| = J} c'(\alpha) \int_0^1 (1 - \tau)^{\alpha_m - 1}$$

$$\times \int_{T^m} \sum_{x \in NB \cap \mathbb{Z}^s} g^{(J)}(x + \tau u_m)[u_1^{\alpha_1} \dots u_m^{\alpha_m}]\, d(u_1, \dots, u_m)\, d\tau.$$

With the choice $J = d$ we prove that

$$\int_{|t| \le N^{1-d}} R_d(t) \widehat{\chi}(t)\, dt \ll N^{s-d} \tag{15}$$

and for $0 \le j < d$,

$$\int_{|t| > N^{1-d}} G_j(t) \widehat{\chi}(t)\, dt \ll N^{s-d}. \tag{16}$$

From this it follows that

$$\begin{aligned} \int_{|t| \le N^{1-d}} S_N(t) \widehat{\chi}(t)\, dt &= \sum_{j=0}^{d-1} \int_{|t| \le N^{1-d}} G_j(t) \widehat{\chi}(t)\, dt + O(N^{s-d}) \\ &= \sum_{j=0}^{d-1} H_j + O(N^{s-d}), \end{aligned}$$

where

$$H_j = \int_{\mathbb{R}} G_j(t) \widehat{\chi}(t)\, dt.$$

Together with (12) and the definition of N we obtain

$$A_F(R) = \sum_{j=0}^{d-1} H_j + O(R^{s/d-1}).$$

H_0 yields the main term since

$$\begin{aligned} H_0 &= \int_{\mathbb{R}} G_0(t)\widehat{\chi}(t)\,dt = \int_{NB}\int_{\mathbb{R}} e(tF(x))\widehat{\chi}(t)\,dt\,dx = \int_{NB} \chi(F(x))\,dx \\ &= \int_{F(x)\le R} dx + O\Big(\int_{R<F(x)\le R+1} dx\Big) = \mathrm{vol}(D_F(R)) + O(R^{s/d-1}). \end{aligned}$$

In the remaining part of this section we prove (15), (16) and $H_j = 0$ for $j \ge 1$. This will complete the proof of the Proposition. We begin with the following lemma which can be proved by induction.

LEMMA 3.1. *Let $g(x) = e(tF(x))$ and $x, u_1, \ldots, u_j \in \mathbb{R}^s$. Then*

$$g^{(j)}(x)[u_1,\ldots,u_j] = g(x)\sum_{l=1}^{j}(2\pi it)^l P_{j,l}(x), \tag{17}$$

where $P_{j,l}$, $1 \le l \le j$, are polynomials with $\deg(P_{j,l}) \le ld - j$ *whose coefficients are linear in $u_1,\ldots,u_j$. They can be determined recursively by*

$$P_{j+1,1}(x) = \sum_{i=1}^{s} \frac{\partial}{\partial x_i}(P_{j,1}(x))u_{j+1}^{(i)},$$

$$P_{j+1,l}(x) = \sum_{i=1}^{s} \frac{\partial}{\partial x_i}(P_{j,l}(x))u_{j+1}^{(i)} + P_{j,l-1}(x)\sum_{i=1}^{s}\frac{\partial F}{\partial x_i}(x)u_{j+1}^{(i)} \quad (2 \le l \le j),$$

$$P_{j+1,j+1}(x) = P_{j,j}(x)\sum_{i=1}^{s}\frac{\partial F}{\partial x_i}(x)u_{j+1}^{(i)},$$

and

$$P_{1,1}(x) = \sum_{i=1}^{s}\frac{\partial F}{\partial x_i}(x)u_1^{(i)}.$$

Here $u_j^{(i)}$ denotes the ith component of u_j.

To prove (15) we consider the cases $|t| \le N^{-d}$ and $N^{-d} < |t| \le N^{1-d}$ separately. If $|t| \le N^{-d}$ we estimate $g^{(d)}$ trivially. Since $P_{j,l}(x) \ll N^{ld-j}$ uniformly in $u_1,\ldots,u_j \in T$ and $x \in 2NB$, (17) and $|t|N^d \le 1$ imply $g^{(j)}(x)[u_1,\ldots,u_j] \ll |t|N^{d-j}$. Hence $R_d(t) \ll |t|N^s$. Together with $\widehat{\chi}(t) \ll |t|^{-1}$ this yields

$$\int_{|t|\le N^{-d}} R_J(t)\widehat{\chi}(t)\,dt \ll \int_{|t|\le N^{-d}} N^s\,dt \ll N^{s-d}. \tag{18}$$

In the case $N^{-d} < |t| \le N^{1-d}$ we use assumption (C). Since the estimate in (C) is uniform in all boxes $B' \subseteq B$ with sides parallel to the coordinate axes we can apply partial summation. This yields, for an arbitrary polynomial P,

$$\sum_{n \in NB} e(tF(n+u))P(n+u) \ll N^{\deg(P)+s-\omega d}|t|^{-\omega}$$

uniformly in $u \in T$. Together with (17) we obtain

$$\begin{aligned}\sum_{n \in NB\cap\mathbb{Z}^s} g^{(d)}(n+\tau u_m)[u_1^{\alpha_1}\ldots u_m^{\alpha_m}] \\ &= \sum_{l=1}^{d}(2\pi it)^l \sum_{n\in NB\cap\mathbb{Z}^s} P_{d,l}(n+\tau u_m)e(tF(n+\tau u_m)) \\ &\ll N^{-d+s-\omega d}|t|^{-\omega}\sum_{l=1}^{d}(|t|N^d)^l \ll N^{d^2-d+s-\omega d}|t|^{d-\omega}.\end{aligned}$$

Since $\omega > d$ it follows that

$$\int_{(N^{-d},N^{1-d}]} R_d(t)\widehat{\chi}(t)\,dt \ll N^{d^2-d+s-\omega d} \int_{(N^{-d},N^{1-d}]} t^{d-\omega-1}\,dt \ll N^{s-d}.$$

This together with (18) implies (15).

To prove (16) we use (D). Since the estimate in (D) is uniform in all boxes $B' \subseteq B$ we can apply partial integration. This gives, for an arbitrary polynomial P and $|t| \ge N^{-d}$,

$$\int_{NB} P(x)e(tF(x))\,dx \ll N^{\deg(P)+s-\omega d}|t|^{-\omega}.$$

Hence Lemma 3.1 implies, for $|t| \ge N^{-d}$ (uniformly in $u_1,\ldots,u_m \in T$),

$$\begin{aligned}\int_{NB} g^{(j)}(x)[u_1^{\alpha_1}\ldots u_m^{\alpha_m}]\,dx &= \sum_{l=1}^{j}\int_{NB}(2\pi it)^l P_{j,l}(x)e(tF(x))\,dx \\ &\ll N^{s-j-\omega d}|t|^{-\omega}\sum_{l=1}^{j}(|t|N^d)^l \ll N^{s+j(d-1)-\omega d}|t|^{j-\omega}.\end{aligned}$$

For $0 \le j < d$ this together with (5) yields

$$\int_{|t|>N^{1-d}} G_j(t)\widehat{\chi}(t)\,dt \ll N^{s+j(d-1)-\omega d}\Big(\int_{(N^{1-d},1]} t^{j-\omega-1}\,dt + \int_{(1,\infty)} t^{-2}\,dt\Big) \ll N^{s-\omega}.$$

Since $\omega > d$ this implies (16).

Finally, we prove

Lemma 3.2. *$H_j = 0$ for $j \ge 1$.*

Proof. By Lemma 3.1 and the definition of H_j we obtain, for $j \geq 1$,

$$\begin{aligned} H_j &= \int_{\mathbb{R}} G_j(t)\widehat{\chi}(t)\,dt \\ &= \sum_{|\alpha|=j} c(\alpha) \int_{\mathbb{R}} \int_{T^m} \int_{NB} g^{(j)}(x)[u_1^{\alpha_1} \dots u_m^{\alpha_m}]\widehat{\chi}(t)\,dx\,d(u_1 \dots u_m)\,dt \\ &= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \sum_{l=1}^{j} \int_{NB} P_{j,l}(x) \int_{\mathbb{R}} e(tF(x))\widehat{\chi^{(l)}}(t)\,dt\,dx\,d(u_1 \dots u_m) \\ &= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \sum_{l=1}^{j} \int_{NB} P_{j,l}(x)\chi^{(l)}(F(x))\,dx\,d(u_1 \dots u_m) \\ &= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \sum_{l=1}^{j} \int_{\mathbb{R}^s} P_{j,l}(x)\chi^{(l)}(F(x))\,dx\,d(u_1 \dots u_m). \end{aligned}$$

Here we used $\widehat{\chi^{(l)}}(t) = (2\pi i t)^l \widehat{\chi}(t)$ and the fact that $\chi^{(l)}(F(x)) = 0$ if $x \notin NB$. In the case $j = 1$ Lemma 3.1 yields

$$\begin{aligned} H_1 &= -\int_{T} \int_{\mathbb{R}^s} P_{1,1}(x)\chi^{(1)}(F(x))\,dx\,du_1 \\ &= -\int_{\mathbb{R}^s} \sum_{i=1}^{s} \frac{\partial F}{\partial x_i}(x)\chi^{(1)}(F(x))\,dx \int_{T} u_1^{(i)}\,du_1 = 0. \end{aligned}$$

Remember that $T = (-1/2, 1/2]^s$. For $j \geq 1$ we prove that

$$\sum_{l=1}^{j+1} \int_{\mathbb{R}^s} P_{j+1,l}(x)\chi^{(l)}(F(x))\,dx = 0. \tag{19}$$

This implies $H_j = 0$ for $j \geq 2$. To prove (19) set

$$H_{j,l} = \int_{\mathbb{R}^s} \sum_{i=1}^{s} \frac{\partial}{\partial x_i}(P_{j,l}(x))u_{j+1}^{(i)}\chi^{(l)}(F(x))\,dx.$$

Using partial integration one obtains, for $2 \leq l \leq j+1$,

$$\begin{aligned} &\int_{\mathbb{R}^s} P_{j,l-1}(x) \sum_{i=1}^{s} \frac{\partial F}{\partial x_i}(x)u_{j+1}^{(i)}\chi^{(l)}(F(x))\,dx \\ &\qquad = \sum_{i=1}^{s} u_{j+1}^{(i)} \int_{\mathbb{R}^s} P_{j,l-1}(x)\frac{\partial}{\partial x_i}(\chi^{(l-1)}(F(x)))\,dx \\ &\qquad = -\sum_{i=1}^{s} u_{j+1}^{(i)} \int_{\mathbb{R}^s} \frac{\partial}{\partial x_i}(P_{j,l-1}(x))\chi^{(l-1)}(F(x))\,dx = -H_{j,l-1}. \end{aligned}$$

This together with the representation of $P_{j+1,l}$ in Lemma 3.1 implies

$$\int_{\mathbb{R}^s} P_{j+1,1}(x)\chi^{(1)}(F(x))\,dx = H_{j,1},$$
$$\int_{\mathbb{R}^s} P_{j+1,l}(x)\chi^{(l)}(F(x))\,dx = H_{j,l} - H_{j,l-1} \quad (2 \le l \le j),$$
$$\int_{\mathbb{R}^s} P_{j+1,j+1}(x)\chi^{(j+1)}(F(x))\,dx = -H_{j,j}.$$

Adding these $j+1$ equations yields (19). This completes the proof of Lemma 3.2 and the proof of the Proposition.

4. Proof of Theorem 3. We have to prove that $\Omega(F) > d$ implies (A)–(D) of the Proposition. We start with (D). It is only here that we use, for inhomogeneous F, the more sophisticated definition (10) instead of (9).

Lemma 4.1. *If* $0 < \omega < \Omega(F)$ *then*

$$\int_{NB'} e(tF(u))\,du \ll N^s \min(1, (|t|N^d)^{-\omega})$$

uniformly for all boxes $B' \subseteq B$ *with sides parallel to the coordinate axes.*

Proof. The estimate is trivial for $|t| \le N^{-d}$. If $|t| > N^{-d}$ the substitution $u = Q^{-1}x$ with $QN \ge 1$ yields

$$\text{(20)} \quad \int_{NB'} e(tF(u))\,du = Q^{-s} \int_{QNB'} e(tF(Q^{-1}x))\,dx$$
$$= Q^{-s}\Big(\sum_{n\in QNB'\cap\mathbb{Z}^s} e(tF(Q^{-1}n)) + O(|t|N^d(QN)^{s-1})\Big).$$

To prove (20) cover QNB' by cubes of the form $n+T$, $T = [-1/2, 1/2)^s$. There are at most $O((QN)^{s-1})$ cubes which intersect the boundary of QNB'. Furthermore, for $x \in n+T$ with $n \in QNB'$, one finds

$$e(tF(Q^{-1}x)) = e(tF(Q^{-1}n)) + O(|t|Q^{-1}N^{d-1})$$

since $\frac{\partial}{\partial x_i}(tF(Q^{-1}x)) \ll |t|Q^{-1}\frac{\partial F}{\partial x_i}(Q^{-1}x) \ll |t|Q^{-1}N^{d-1}$. This proves (20). The exponential sum in (20) has the form

$$\sum_{n\in QNB'\cap\mathbb{Z}^s} e(tQ^{-d}F^{(d)}(n) + P(n))$$

with a polynomial $P \in \mathbb{R}[X_1, \ldots, X_s]$ of degree strictly smaller than d. For $0 < \Delta \le 1$ choose Q such that $|t|Q^{-d} = (QN)^{\Delta-d}$. Then $QN \ge 1$ and $|t|Q^{-d}$ lies on the boundary of $\mathfrak{M}_\Delta(1,1)$. By the definition (10) of $\Omega(F)$ the exponential sum is $\ll (QN)^{s-\omega\Delta}$. If F is homogeneous the same follows from the alternative definition (9). Now (20) implies

$$\int\limits_{NB'} e(tF(u))\,du \ll Q^{-s}(QN)^{s-\omega\Delta} + |t|Q^{-1}N^{s+d-1}$$
$$\ll N^{s-\omega d}|t|^{-\omega} + N^{s}(|t|N^{d})^{1-1/\Delta}.$$

Both terms on the right hand side are equal if we set $\Delta = (1+\omega)^{-1} \in (0,1]$.

LEMMA 4.2. *$\Omega(F) > d$ implies that condition* (C) *of the Proposition is satisfied.*

Proof. Condition (C) is trivially satisfied if $|t| \le N^{-d}$. If $N^{-d} < |t| \le N^{1-d}$ choose $\Delta(t)$ such that $|t| = N^{\Delta(t)-d}$, i.e. $\Delta(t) = d + \log|t|/\log N$. The condition $N^{-d} < |t| \le N^{1-d}$ ensures $\Delta(t) \in (0,1]$. With this choice t lies on the boundary of $\mathfrak{M}_{\Delta(t)}(1,1)$. Hence $t \in \mathfrak{m}_{\Delta(t)}$ and the definition (10) or (9) implies, for every $\Omega(F) > \omega > d$,

$$\sum_{n\in NB'\cap\mathbb{Z}^s} e(tF(n+u)) \ll N^{s-\omega\Delta(t)} \ll N^{s-\omega d}|t|^{-\omega}$$

uniformly for all $u \in B$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes. This proves (C).

To verify conditions (A) and (B) of the Proposition, we split the domain of integration into a part covered by minor arcs and a second part covered by major arcs.

LEMMA 4.3 (minor arcs). *If $\Omega(F) > d$ and $0 < \Delta < 1$ then*

$$\int\limits_{\mathfrak{m}_\Delta} |S_N(t)|\,dt \ll N^{s-d}, \tag{21}$$

$$\int\limits_{(N^{1-d},1]\cap\mathfrak{m}_\Delta} |S_N(t)|\,\frac{dt}{t} \ll N^{s-d}. \tag{22}$$

Proof. We prove (22). The proof of (21) is analogous; see [6, p. 24, Lemma 4.B], for an even sharper estimate. Choose ω such that $\Omega(F) > \omega > d$. If $\Delta = 1$ the definition of $\Omega(F)$ implies $S_N(t) \ll N^{s-\omega}$ for all $t \in \mathfrak{m}_1$. Hence

$$\int\limits_{(N^{1-d},1]\cap\mathfrak{m}_1} |S_N(t)|\,\frac{dt}{t} \ll N^{s-\omega} \int\limits_{(N^{1-d},1]} \frac{dt}{t} \ll N^{s-\omega}\log N \ll N^{s-d}.$$

If $0 < \Delta < 1$ we split $(\Delta,1]$ into subintervals $(\Delta_{i-1},\Delta_i]$, where $\Delta = \Delta_0 < \Delta_1 < \ldots < \Delta_n = 1$. Then

$$\mathfrak{m}_\Delta = ((\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_1) \cup \bigcup_{i=1}^{n} \mathfrak{M}_{\Delta_i} \setminus \mathfrak{M}_{\Delta_{i-1}} = \mathfrak{m}_1 \cup \bigcup_{i=1}^{n} \mathfrak{r}_i,$$

where $\mathfrak{r}_i = \mathfrak{M}_{\Delta_i} \setminus \mathfrak{M}_{\Delta_{i-1}} \subseteq \mathfrak{M}_{\Delta_i}$. Since $\mathfrak{M}_\Delta$ has Lebesgue measure

$$\lambda(\mathfrak{M}_\Delta) \ll \sum_{1\le a\le q\le N^\Delta} q^{-1}N^{\Delta-d} \ll N^{2\Delta-d},$$

it follows that $\lambda(\mathfrak{r}_i) \ll N^{2\Delta_i-d}$. Furthermore, the definition of $\Omega(F)$ yields for $t \in \mathfrak{r}_i \subseteq \mathfrak{m}_{\Delta_{i-1}}$ the estimate $S_N(t) \ll N^{s-\omega\Delta_{i-1}}$. Hence we obtain

$$\begin{aligned}\int_{(N^{d-1},1]\cap\mathfrak{m}_\Delta} |S_N(t)|\,\frac{dt}{t} &\ll \int_{(N^{d-1},1]\cap\mathfrak{m}_1} |S_N(t)|\,\frac{dt}{t} + \sum_{i=1}^{n}\int_{(N^{d-1},1]\cap\mathfrak{r}_i} |S_N(t)|\,\frac{dt}{t}\\ &\ll N^{s-d} + \sum_{i=1}^{n} N^{s-\omega\Delta_{i-1}} \int_{(N^{d-1},1]\cap\mathfrak{r}_i} \frac{dt}{t}.\end{aligned}$$

Since $\mathfrak{r}_i \subseteq \mathfrak{M}_{\Delta_i}$ we consider (for $(a,q) \neq (1,1)$)

$$\int_{\mathfrak{M}_\Delta(q,a)\cap(0,1]} \frac{dt}{t} = \int_{\frac{a}{q}-\frac{1}{q}N^{\Delta-d}}^{\frac{a}{q}+\frac{1}{q}N^{\Delta-d}} \frac{dt}{t} = \log\frac{1+\frac{1}{a}N^{\Delta-d}}{1-\frac{1}{a}N^{\Delta-d}} \ll \frac{1}{a}\,N^{\Delta-d}. \tag{23}$$

It follows that

$$\int_{(N^{1-d},1]\cap\mathfrak{M}_\Delta} \frac{dt}{t} \ll \sum_{1\le a\le q\le N^\Delta} \frac{1}{a}\,N^{\Delta-d} \ll N^{\Delta-d}\sum_{1<q\le N^\Delta} \log q \ll N^{2\Delta-d}\log N.$$

Altogether we obtain

$$\begin{aligned}\int_{(N^{d-1},1]\cap\mathfrak{m}_\Delta} |S_N(t)|\,\frac{dt}{t} &\ll N^{s-d} + \sum_{i=1}^{n} N^{s-d-(\omega-2)\Delta_{i-1}+2(\Delta_i-\Delta_{i-1})}\log N\\ &\ll N^{s-d} + N^{s-d-(\omega-2)\Delta+2\varepsilon} \ll N^{s-d},\end{aligned}$$

if we choose $\Delta_i - \Delta_{i-1} < \varepsilon$ sufficiently small. This proves (22) for every $\Delta \in (0,1]$.

Lemma 4.4 (major arcs). *If* $\Omega(F) > 2$ *and* $0 < \Delta < 1/4$ *then*

$$\int_{\mathfrak{M}_\Delta} |S_N(t)|\,dt \ll N^{s-d}, \tag{24}$$

$$\int_{(N^{1-d},1]\cap\mathfrak{M}_\Delta} |S_N(t)|\,\frac{dt}{t} \ll N^{s-d}. \tag{25}$$

Proof. If F is a polynomial with integer coefficients and t is close to a rational number with small denominator, then $S_N(t)$ can be evaluated asymptotically. It is well known (cf. [6, p. 26, Lemma 5.A]) that for every $t \in \mathfrak{M}_\Delta(q,a)$, we have

$$S_N(t) = S\Big(\frac{a}{q}\Big)G_0\Big(t-\frac{a}{q}\Big) + O(qN^{s-1+\Delta}), \tag{26}$$

where

$$S\left(\frac{a}{q}\right) = q^{-s} \sum_{n\in q(0,1]^s\cap\mathbb{Z}^s} e\left(\frac{a}{q}F(n)\right), \qquad G_0(t) = \int_{NB} e(tF(u))\,du.$$

Since a/q with $(a,q)=1$ lies in $\mathfrak{M}_1(q,a)$ with $N=q$, the definition of $\Omega(F)$ implies

$$S\left(\frac{a}{q}\right) \ll q^{-\omega} \tag{27}$$

for every $\omega<\Omega(F)$. Additionally, by Lemma 4.1, $G_0(t)\ll N^s\min(1,|tN^d|^{-\omega})$ for $\omega<\Omega(F)$. Since $\Omega(F)>2$ we can choose $\omega>2$. Using these estimates it is easy to prove (24) and (25). We demonstrate (25). Since

$$\left|t-\frac{a}{q}\right| \le \frac{1}{q}N^{\Delta-d} \quad \text{for } t\in\mathfrak{M}_\Delta(q,a),$$

it follows that $t\ge a/(2q)$. Hence

$$\begin{aligned}\int_{\mathfrak{M}_\Delta(q,a)\cap(0,1]} &|S_N(t)|\,\frac{dt}{t}\\ &\ll \left|S\left(\frac{a}{q}\right)\right|\frac{q}{a}\int_{|u|\le\frac{1}{q}N^{\Delta-d}}|G_0(u)|\,du + qN^{s-1+\Delta}\int_{\mathfrak{M}_\Delta(q,a)\cap(0,1]}\frac{dt}{t}.\end{aligned}$$

The substitution $u=N^{-d}v$ yields

$$\begin{aligned}\int_{|u|\le\frac{1}{q}N^{\Delta-d}}|G_0(u)|\,du &= N^{-d}\int_{|v|\le\frac{1}{q}N^{\Delta}}|G_0(N^{-d}v)|\,dv\\ &\ll N^{s-d}\int_{|v|\le\frac{1}{q}N^{\Delta}}\min(1,|v|^{-\omega})\,dv \ll N^{s-d}.\end{aligned}$$

Together with (23) and (27) we obtain

$$\begin{aligned}\int_{(N^{1-d},1]\cap\mathfrak{M}_\Delta}|S_N(t)|\,\frac{dt}{t} &\ll N^{s-d}\sum_{1\le a\le q\le N^\Delta}(a^{-1}q^{1-\omega}+a^{-1}qN^{2\Delta-1})\\ &\ll N^{s-d}(1+N^{4\Delta-1})\log N \ll N^{s-d}.\end{aligned}$$

5. Proof of Theorem 2. Let $F_0(X)=\sum_{i=1}^s\lambda_iX_i^d$ with integer coefficients $\lambda_i>0$. It is known that $\Omega(F_0)\ge s2^{1-d}$ (see [6, p. 24] and the remarks following (11)). Hence Theorem 3 implies $P_{F_0}(R)\ll R^{s/d-1}$ if $s>d2^{d-1}$. For large d this can be substantially improved by Vinogradov's mean value theorem. We prove that (A)–(D) of the Proposition are satisfied if $s>\varrho_0(d)$, where $\varrho_0(d)$ is an explicitly computable function which satisfies $\varrho_0(d)\sim 2d^3\log d$ for $d\to\infty$.

First we prove that (C) and (D) are satisfied if $s > d^2$, $d > 2$. To do this we establish (7) and (8) with $\omega = s/d$. By [2, Theorem 2.2] (the second derivative test), it follows that

$$\sum_{M<n\le M'} e(t(n+u)^d) \ll (|t|M^{d-2})^{-1/2} + M(|t|M^{d-2})^{1/2}$$

uniformly for $u \in [-1,1]$ and $1 \le M < M' \le 2M$. Splitting $[0,N]$ into dyadic intervals of the form $(2^{j-1}U, 2^jU]$ with $U = |t|^{-1/d}$ we obtain

$$\begin{aligned}\sum_{0\le n\le N} e(t(n+u)^d) &\ll 1 + U + \sum_j (|t|^{-1/2}(2^jU)^{1-d/2} + |t|^{1/2}(2^jU)^{d/2})\\ &\ll 1 + U + |t|^{-1/2}U^{1-d/2} + |t|^{1/2}N^{d/2}\\ &\ll |t|^{-1/d} + |t|^{1/2}N^{d/2}.\end{aligned}$$

It follows that

$$\sum_{n\in NB'} e(tF_0(n+u)) \ll (|t|^{-1/d} + |t|^{1/2}N^{d/2})^s \ll |t|^{-s/d}$$

if $|t| \le N^{1-d}$. This proves (7) with $\omega = s/d$. To prove (D) observe that for $t > 0$,

$$\int_0^N e(tx^d)\,dx = t^{-1/d}d^{-1}\int_0^{tN^d} \xi^{1/d-1}e(\xi)\,d\xi \ll t^{-1/d}$$

(the last integral is bounded by an absolute constant). This proves (8) with $\omega = s/d$.

Next we prove (A) and (B). Let

$$f(t) = \sum_{1\le n\le N} e(tn^d),$$

then $S_N(t) = \prod_{i=1}^s (1 + 2f(\lambda_i t))$. By Hölder's inequality it is sufficient to prove

$$\int_{(0,1]} |f(t)|^s\,dt \ll N^{s-d} \quad \text{and} \quad \int_{(\lambda_i N^{1-d},1]} |f(t)|^s\,\frac{dt}{t} \ll N^{s-d}. \tag{28}$$

To estimate the special function $f(t)$ one can work with larger major arcs. Let $N = \lceil (R+1)^{1/d}\rceil + 1/2$ and set

$$\mathfrak{M}(q,a) = \left\{ t \in \mathbb{R}/\mathbb{Z} \;\middle|\; \left|t - \frac{a}{q}\right| \le \frac{P}{qR} \right\}, \qquad P = \frac{N}{2d}.$$

Write $\mathfrak{M}$ for the union of the $\mathfrak{M}(q,a)$ with $1 \le a \le q \le P$ and $(a,q) = 1$, and set $\mathfrak{m} = (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}$.

Lemma 5.1 (major arcs). *If $s > 2d$ and $c > 0$ then*

$$\int_{\mathfrak{M}} |f(t)|^s\,dt \ll N^{s-d} \quad \text{and} \quad \int_{(cN^{1-d},1]\cap\mathfrak{M}} |f(t)|^s\,\frac{dt}{t} \ll N^{s-d}.$$

Proof. By [9, Theorem 4.1], for $t \in \mathfrak{M}(q,a)$ and any $\varepsilon > 0$,

$$f(t) = \frac{1}{q} S\left(\frac{a}{q}\right) v\left(t - \frac{a}{q}\right) + O(q^{1/2+\varepsilon}),$$

where, by [9, Theorem 4.2 and Lemma 2.8],

$$\frac{1}{q} S\left(\frac{a}{q}\right) \ll q^{-1/d} \quad \text{and} \quad v(t) \ll \min(N, |t|^{-1/d}).$$

This yields

$$\int\limits_{(cN^{1-d},1]\cap\mathfrak{M}} |f(t)|^s \frac{dt}{t} \ll \sum_{1\le a\le q\le P} \left(q^{-s/d} \int\limits_{|u|\le P/(qR)} |v(u)|^s\, du + q^{s/2+\varepsilon} \frac{P}{qR}\right) \frac{q}{a}.$$

Since

$$\int\limits_{|u|\le P/(qR)} |v(u)|^s\, du \ll N^{s-d} + \int\limits_{(N^{-d},P/(qR)]} u^{-s/d}\, du \ll N^{s-d},$$

we obtain, for $s > 2d$,

$$\int\limits_{(cN^{1-d},1]\cap\mathfrak{M}} |f(t)|^s \frac{dt}{t} \ll N^{s-d} \sum_{q\le N} q^{1-s/d} \log q + N^{1-d} \sum_{q\le N} q^{s/2+2\varepsilon} \ll N^{s-d}.$$

This proves the second assertion of the lemma. The first one follows in the same way.

Finally, we estimate the contribution of the minor arcs to (28). Since

$$\int\limits_{(\lambda_i N^{d-1},1]\cap\mathfrak{m}} |f(t)|^s \frac{dt}{t} \ll N^{1-d} \int\limits_{\mathfrak{m}} |f(t)|^s\, dt$$

(28) is a consequence of Lemma 5.1 and the following lemma.

LEMMA 5.2 (minor arcs). *There is an explicitly computable function $\varrho_0(d)$, which satisfies $\varrho_0(d) \sim 2d^3 \log d$ for $d \to \infty$, such that for $s \ge \varrho_0(d)$,*

$$\int\limits_{\mathfrak{m}} |f(t)|^s\, dt \ll N^{s-2d+1}.$$

Proof. We use Wooley's refinement of Vinogradov's mean value theorem. The original form of the mean value theorem yields Lemma 5.2 with $\varrho_0(d) \sim 4d^3 \log d$. By [9, Theorem 5.6], there is an explicitly computable function $\sigma(d)$ such that for $t \in \mathfrak{m}$,

$$f(t) \ll N^{1-\sigma(d)} \log N.$$

We have $\sigma(d) \sim (2d^2 \log d)^{-1}$ for $d \to \infty$. Furthermore, by [9, Theorem 5.5 and (5.37)], for every integer $l \ge 1$,

$$\int\limits_{(0,1]} |f(t)|^{2dl}\, dt \ll N^{2dl-d+\eta_l(d)},$$

where

$$\eta_l(d) = \frac{1}{2}d(d-1)\left(1 - \frac{5}{4d}\right)^{l-1}.$$

These estimates imply, for every $l \geq 1$,

$$\int_{\mathfrak{m}} |f(t)|^s \, dt \ll (\sup_{t \in \mathfrak{m}} |f(t)|^{s-2dl}) \int_{(0,1]} |f(t)|^{2dl} \, dt$$
$$\ll N^{(s-2dl)(1-\sigma(d))+2dl-d+\eta_l(d)} (\log N)^{s-2dl}.$$

There is an l such that the right hand side is $\ll N^{s-2d+1}$ if

$$s > \min_l \left\{ \frac{\eta_l(d)}{\sigma(d)} + 2dl \right\} + \frac{d-1}{\sigma(d)} = \varrho_0(d),$$

say. By [9, Theorem 5.7], the minimum is $\ll d^2 \log d$, thus $\varrho_0(d) \sim 2d^3 \log d$ for $d \to \infty$.

We remark that for small d Theorem 2 can be further sharpened. For instance, Hua's lemma ([9, Lemma 2.5]) can be used to prove $P_{F_0}(R) \ll R^{s/d-1}$ for $s > 2^{d+1} - 2$.

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