Lattice points in bodies with algebraic boundary

by

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1. Introduction. Let $F$ be a polynomial of even degree $d$ in $s$ variables with integer coefficients. Assume that the leading homogeneous part $F^{(d)}$ in the decomposition $F = F^{(d)} + G$ with $\deg(G) < d$ is positive definite. Then $D_F(R) = \{ x \in \mathbb{R}^d \mid F(x) \leq R \}$ is compact. Denote by $A_F(R)$ the number of lattice points of the standard lattice $\mathbb{Z}^s$ which are contained in $D_F(R)$. Then $A_F(R)$ is approximately equal to $\text{vol}(D_F(R))$. It is easy to see that the discrepancy $P_F(R) = A_F(R) - \text{vol}(D_F(R))$ satisfies

$$P_F(R) = O(R^{s/d-1}).$$

One only has to observe that $A_F(R + \varepsilon) = A_F(R)$ for $R \in \mathbb{N}$ and $0 < \varepsilon < 1$, but $\text{vol}(D_F(R + \varepsilon)) - \text{vol}(D_F(R)) \gg R^{s/d-1}$. Our aim is to give a sharp upper bound for $P_F(R)$. To formulate the main result we introduce the invariant $h(F)$ of $F$, defined as the smallest integer $h$ such that $F^{(d)}$ has a representation

$$F^{(d)} = \sum_{i=1}^{h} A_i B_i$$

with homogeneous polynomials $A_i, B_i \in \mathbb{Q}[X_1, \ldots, X_s]$ of positive degree.

Theorem 1. Assume that $h(F) > \varrho(d)$ where $\varrho(2) = 4$, $\varrho(4) = 288$ and $\varrho(d) = d(d-1)2^{d-1}(\log 2)^{-d}d!$ for $d > 4$. Then for $R \geq 1$,

$$P_F(R) = O(R^{s/d-1}).$$

In the case $d = 2$ it is easy to see that $h(F) = s$. Thus Theorem 1 contains as a special case the well known theorem of Walfisz [10] and Landau [4] who proved (2) for rational quadratic forms of dimension $s > 4$. If $F^{(d)}$ is non-singular, i.e. the only solution of $\frac{\partial}{\partial x_i}(F^{(d)}(x)) = 0, 1 \leq i \leq s$, in $\mathbb{C}^s$ is $x = 0$, then $h(F) \geq s/2$ (cf. [7, p. 282]). In this case the theorem gives the exact order of $P_F(R)$ if $s > 2\varrho(d)$. The proof of Theorem 1 uses a variant of the Hardy–Littlewood method. For general $F$ this method was first used by

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Schmidt in his famous work on diophantine equations [6], [8]. For special $F$ the estimate (2) can be true for much smaller $s$. As an example we prove

**Theorem 2.** Let $F_0(X) = \sum_{i=1}^{s} \lambda_i X_i^d$ with $d \geq 2$ even and integer coefficients $\lambda_i > 0$. Then $P_{F_0}(R) = O(R^{s/d-1})$, provided that $s \geq \min(d2^{d-1}, g_0(d))$. Here $g_0$ denotes an explicitly computable function which satisfies $g_0(d) \sim 2d^3\log d$ for $d \to \infty$.

As noted by Randol [5] Theorem 2 cannot be true if $s < d^2 - d + 1$. See Krätzel [3] for a detailed study of $P_{F_0}(R)$ for small $s$. With some obvious modifications our proof shows that Theorem 2 remains true for real coefficients $\lambda_i > 0$.

Recently, Bentkus and Götze [1] studied $P_F(R)$ for polynomials $F$ with real coefficients and leading homogeneous part

$$F(d)(X) = \sum_{i=1}^{s_0} \lambda_i X_i^d + P(X) \quad (\lambda_i > 0).$$

Here $P$ denotes a homogeneous polynomial of degree $d$ such that the degree of $P$ viewed as a polynomial in $(X_1, \ldots, X_{s_0})$ is strictly smaller than $d$. They proved (2) under the assumptions that $s_0 = s$ and $s > \alpha(d)$ or $s_0 < s$ and $s_0 > 2^d\alpha(d)$, where $\alpha(2) = 8$, $\alpha(4) = 1512$ and $\alpha(d) = d2^{d-1}e^{3d\log d}$ for $d > 4$. The condition (3) on the leading homogeneous part of $F$ is rather restrictive. Bentkus and Götze already remarked that one should expect that (2) is true for general $F$ if $h(F)$ is sufficiently large. The main advantage of their method is that it applies to polynomials with real coefficients, whereas we have to assume that $F$ has integer coefficients.

**2. The Hardy–Littlewood method.** Let $B = (-1, 1]^s$. Assume that $R \in \mathbb{N}$ and $D_F(R) \subseteq R^{1/d}B$ for $R \geq c(F)$ sufficiently large. Otherwise consider $cF$ instead of $F$, where $c \in \mathbb{N}$ is sufficiently large, and use $A_F(R) = A_{cF}(cR)$. To count the number of lattice points in $D_F(R)$ we introduce the auxiliary function $\chi = I_{(-R-1/2,R+1/2)} \ast \delta$ which is the convolution of the indicator function with a symmetric probability density $\delta \in C^\infty(\mathbb{R})$ satisfying $\text{supp}(\delta) \subseteq [-1/2,1/2]$. Then $\chi(u) = 1$ if $|u| \leq R$, $\chi(u) = 0$ if $|u| \geq R + 1$ and $0 \leq \chi(u) \leq 1$ if $R < |u| < R + 1$. By Fourier inversion one obtains

$$\chi(u) = \int_{\mathbb{R}} \hat{\chi}(t)e(-tu) dt = \int_{\mathbb{R}} \hat{\chi}(t)e(tu) dt,$$

where

$$\hat{\chi}(t) = \int_{\mathbb{R}} \chi(u)e(tu) du = \tilde{I}_{(-R-1/2,R+1/2)}(t)\hat{\delta}(t).$$
Here \( e(x) = e^{2\pi ix} \) as usual. Furthermore,
\[
\hat{I}_{(-R-1/2,R+1/2)}(t) = \frac{1}{\pi t} \sin(2\pi t(R + 1/2)).
\]
Applying \( j \)-fold partial integration one obtains \( \hat{\delta}(t) \ll_j (|t| + 1)^{-j} \) for \( j \geq 0 \).
Hence
\[
\hat{\chi}(t) \ll \frac{1}{|t|} (1 + |t|)^{-j} \quad (j \geq 0).
\]
Set \( N = \lceil (R + 1)^{1/d} \rceil + 1/2 \). Then \( F(k) \leq R \) implies \( k \in NB \) and (4) yields
\[
\chi(F(n)) = \int S_N(t)\hat{\chi}(t) \, dt
\]
with
\[
S_N(t) = \sum_{n \in NB \cap \mathbb{Z}^s} e(tF(n)).
\]
This should be compared with the following integral which counts the number of lattice points on the boundary of \( D_F(R) \):
\[
\int_0^1 S_N(t)e(-tR) \, dt.
\]
It is not surprising that the properties of \( S_N(t) \) known from the Hardy–Littlewood method can be used to analyse \( A_F(R) \). The main difference comes from the behaviour of \( \hat{\chi}(t) \) for small \( t \). Note that \( S_N(t) \) is one-periodic if \( F \) has integer coefficients. The following proposition deals with these small values of \( t \).

**Proposition.** Assume that for \( N \geq 1 \):

(A) \[ \int_{(0,1]} |S_N(t)| \, dt \ll N^{s-d}. \]

(B) \[ \int_{(N^{1-d},1]} \frac{|S_N(t)|}{t} \, dt \ll N^{s-d}. \]

(C) There exists an \( \omega > d \) such that for \( |t| \leq N^{1-d} \)

\[ \sum_{n \in NB \cap \mathbb{Z}^s} e(tF(n + u)) \ll N^{s-\omega d}|t|^{-\omega} \]

uniformly in \( u \in B \) and all boxes \( B' \subseteq B \) with sides parallel to the coordinate axes.

(D) There exists an \( \omega > d \) such that for \( |t| \geq N^{-d} \),

\[ \int_{NB'} e(tF(x)) \, dx \ll N^{s-\omega d}|t|^{-\omega} \]

uniformly in all boxes \( B' \subseteq B \) with sides parallel to the coordinate axes.

Then \( P_F(R) \ll R^{s/d-1} \).
The proof of this Proposition is given in Section 3. Here we describe the “axiomatic” form of the Hardy–Littlewood method given by Schmidt [6]. If $F$ is a polynomial with integer coefficients, $S_N(t)$ can be evaluated asymptotically in a neighbourhood of a rational number with small denominator. The union of these neighbourhoods is called the major arcs. To be precise let $0 < \Delta \leq 1$ and set, for $1 \leq a \leq q \leq N^\Delta$ with $(a, q) = 1$, 

$$M_\Delta(q, a) = \left\{ t \in \mathbb{R}/\mathbb{Z} \mid \left| t - \frac{a}{q} \right| < \frac{1}{q} N^\Delta - d \right\}.$$ 

Then the major arcs and minor arcs are defined by

$$M_\Delta = \bigcup_{1 \leq a \leq q \leq N^\Delta \atop (a, q) = 1} M_\Delta(q, a) \quad \text{and} \quad m_\Delta = (\mathbb{R}/\mathbb{Z}) \setminus M_\Delta.$$ 

Note that $M_\Delta$ is the union of disjoint intervals if $N$ is sufficiently large.

If $F$ is homogeneous, i.e. $F = F^{(d)}$, we define $\Omega(F)$ as the supremum of all $\omega > 0$ such that for all $\Delta \in (0, 1]$ and $t \in m_\Delta,$

$$\sum_{n \in NB' \cap \mathbb{Z}^s} e(t F(n + u)) \ll_{F, \omega} N^{s - \omega \Delta}$$

uniformly for all $u \in B$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes. If $F$ is an arbitrary polynomial with leading form $F^{(d)}$ we define $\Omega(F)$ as the supremum of all $\omega > 0$ such that for all $\Delta \in (0, 1]$ and $t \in m_\Delta,$

$$\sum_{n \in NB' \cap \mathbb{Z}^s} e(t F^{(d)}(n) + P(n)) \ll_{F, \omega} N^{s - \omega \Delta}$$

uniformly for all polynomials $P \in \mathbb{R}[X_1, \ldots, X_s]$ with $\deg(P) < d$ and all boxes $B' \subseteq B$ with sides parallel to the coordinate axes.

$\Omega(F)$ is similar to the invariant $\omega(F)$ introduced by Schmidt [6]. The latter is defined as the supremum of all $\omega > 0$ such that for all $\Delta \in (0, 1]$ and $t \in m_\Delta,$ (9) is true with $u = 0$ uniformly for all boxes $B' \subseteq B$. We prove that the assumption $\Omega(F) > d$ implies $\text{(A)}$–$\text{(D)}$ of the above Proposition.

**Theorem 3.** If $\Omega(F) > d$ then $P_F(R) \ll R^{s/d - 1}$.

Theorem 1 follows immediately from Theorem 3 and the following inequality:

$$\Omega(F) \geq \frac{h(F)}{\tau(d)}.$$ 

Here $\tau(2) = 2$, $\tau(4) = 72$ and $\tau(d) < (d - 1)2^{d-1}(\log 2)^{-d}d!$ in general. With $\Omega(F)$ replaced by $\omega(F)$ this is Theorem 6.A in [6, p. 86]. We have to verify that Schmidt’s inequality remains true with our modified invariant $\Omega(F)$. To see this note that Schmidt’s proof starts with a $d$-fold application
of Weyl’s inequality. This transforms the exponential sum in the definition of $\Omega(F)$ into an exponential sum of the form $\sum e(G_d(n_1, \ldots, n_d))$, where $G(X) = tF^{(d)}(X) + P(X)$ and $G_d$ is the unique symmetric multilinear form which satisfies $G^{(d)}(X) = \frac{(-1)^d}{d!}G_d(X_1, \ldots, X_d)$. If $P$ is a polynomial of degree strictly less than $d$, then $P_d = 0$. It follows that $G_d = tF^{(d)}_d$. Hence the new exponential sum does not depend on $P$. From this moment on, one proceeds as in [6]. Note that $\Omega(F)$ and the above lower bound on $\Omega(F)$ depend only on the leading form of $F$.

3. Proof of the Proposition. Assume that conditions (A)–(D) of the Proposition are satisfied. The representation (6), together with (5), (A) and (B), yields

$$A_F(R) = \int_{|t| \leq N^{1-d}} S_N(t) \tilde{\chi}(t) \, dt$$

$$+ O \left( \int_{(N^{1-d},1]} |S_N(t)| \frac{dt}{t} + \sum_{j=1}^{\infty} \int_{(j,j+1]} |S_N(t)| \, dt \right)$$

$$= \int_{|t| \leq N^{1-d}} S_N(t) \tilde{\chi}(t) \, dt + O(N^{s-d}).$$

If $|t| \leq N^{1-d}$ we use an asymptotic expansion of $S_N(t)$. There are several ways to obtain it. We use the following expansion of a sufficiently smooth complex-valued function $g : \mathbb{R}^s \to \mathbb{C}$ due to Bentkus and Götze [1]. Let $J \in \mathbb{N}$, and $x, u_1, \ldots, u_J \in \mathbb{R}^s$. Then

$$g(x) = g(x + u_1) + \sum_{j=1}^{J-1} g_j + r_J,$$

where for $1 \leq j < J$,

$$g_j = \sum_{|\alpha| = j} c(\alpha)g^{(j)}(x + u_{m+1})[u_1^{\alpha_1} \ldots u_m^{\alpha_m}]$$

and

$$r_J = \sum_{|\alpha| = J} c'(\alpha) \int_0^1 (1 - \tau)^{\alpha_{m-1}}g^{(J)}(x + \tau u_m)[u_1^{\alpha_1} \ldots u_m^{\alpha_m}] \, d\tau.$$

The summation extends over all $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ with $1 \leq m \leq j$ and $|\alpha| = \sum_{i=1}^m \alpha_i = j$. Furthermore, $g^{(j)}(x)[u_1^{\alpha_1} \ldots u_m^{\alpha_m}]$ denotes the $j$-fold directional derivative

$$g^{(j)}(x)[u_1^{\alpha_1} \ldots u_m^{\alpha_m}] = \frac{\partial^j}{\partial \lambda_1^{\alpha_1} \ldots \partial \lambda_m^{\alpha_m}} g(x + \lambda_1 u_1 + \ldots + \lambda_m u_m) \bigg|_{\lambda_1 = \ldots = \lambda_m = 0}$$
and
\[ c(\alpha) = \frac{(-1)^m}{\alpha_1! \cdots \alpha_m!}, \quad c'(\alpha) = \frac{(-1)^m}{\alpha_1! \cdots \alpha_{m-1}!(\alpha_m - 1)!}. \]

This expansion can be obtained by iteratively applying Taylor expansions, first to \( \lambda \mapsto g(x + \lambda u_1) \) and then for every summand \( g^{(\alpha_1)}(x)[u_1^{\alpha_1}] \) in the resulting expansion to \( \lambda \mapsto g^{(\alpha_1)}(x + \lambda u_2)[u_1^{\alpha_1}] \). After \( J \) such steps one obtains (13).

We use (13) with \( g(x) = e(tF(x)) \). Summing over \( x \in NB \cap \mathbb{Z}^s \) and integrating over \((u_1, \ldots, u_J) \in T^J \) with \( T = (-1/2, 1/2]^s \), yields

\[ S_N(t) = G_0(t) + \sum_{j=1}^{J-1} G_j(t) + R_J(t), \tag{14} \]

where

\[
G_0(t) = \int \sum_{x \in NB \cap \mathbb{Z}^s} g(x + u_1) \, du_1 = \int_{NB} g(x) \, dx,
\]

\[
G_j(t) = \sum_{|\alpha| = j} c(\alpha) \int_{T^m} \left( \int_{NB} g^{(j)}(x)[u_1^{\alpha_1} \cdots u_m^{\alpha_m}] \, dx \right) d(u_1, \ldots, u_m),
\]

\[
R_J(t) = \sum_{|\alpha| = J} c'(\alpha) \int_{0}^{1} (1 - \tau)^{\alpha_m - 1} \times \int_{T^m} \sum_{x \in NB \cap \mathbb{Z}^s} g^{(j)}(x + \tau u_m)[u_1^{\alpha_1} \cdots u_m^{\alpha_m}] \, d(u_1, \ldots, u_m) \, d\tau.
\]

With the choice \( J = d \) we prove that

\[ \int_{|t| \leq N^{1-d}} R_d(t) \hat{\chi}(t) \, dt \ll N^{s-d} \tag{15} \]

and for \( 0 \leq j < d, \)

\[ \int_{|t| > N^{1-d}} G_j(t) \hat{\chi}(t) \, dt \ll N^{s-d}. \tag{16} \]

From this it follows that

\[
\int_{|t| \leq N^{1-d}} S_N(t) \hat{\chi}(t) \, dt = \sum_{j=0}^{d-1} \int_{|t| \leq N^{1-d}} G_j(t) \hat{\chi}(t) \, dt + O(N^{s-d})
\]

\[ = \sum_{j=0}^{d-1} H_j + O(N^{s-d}), \]

where

\[ H_j = \int_{\mathbb{R}} G_j(t) \hat{\chi}(t) \, dt. \]
Together with (12) and the definition of $N$ we obtain

$$A_F(R) = \sum_{j=0}^{d-1} H_j + O(R^{s/d-1}).$$

$H_0$ yields the main term since

$$H_0 = \int G_0(t) \hat{\chi}(t) \, dt = \int \int e(tF(x)) \hat{\chi}(t) \, dt \, dx = \int \chi(F(x)) \, dx$$

$$= \int_{F(x) \leq R} dx + O\left( \int_{R<F(x) \leq R+1} dx \right) = \text{vol}(D_F(R)) + O(R^{s/d-1}).$$

In the remaining part of this section we prove (15), (16) and $H_j = 0$ for $j \geq 1$. This will complete the proof of the Proposition. We begin with the following lemma which can be proved by induction.

**Lemma 3.1.** Let $g(x) = e(tF(x))$ and $x, u_1, \ldots, u_j \in \mathbb{R}^s$. Then

$$g^{(j)}(x)[u_1, \ldots, u_j] = g(x) \sum_{l=1}^{j} (2\pi it)^l P_{j,l}(x),$$

where $P_{j,l}, 1 \leq l \leq j$, are polynomials with deg($P_{j,l}$) $\leq ld - j$ whose coefficients are linear in $u_1, \ldots, u_j$. They can be determined recursively by

$$P_{j+1,1}(x) = \sum_{i=1}^{s} \frac{\partial}{\partial x_i} (P_{j,1}(x)) u_{j+1}^{(i)};$$

$$P_{j+1,l}(x) = \sum_{i=1}^{s} \frac{\partial}{\partial x_i} (P_{j,l}(x)) u_{j+1}^{(i)} + P_{j,l-1}(x) \sum_{i=1}^{s} \frac{\partial F}{\partial x_i}(x) u_{j+1}^{(i)} \quad (2 \leq l \leq j),$$

$$P_{j+1,j+1}(x) = P_{j,j}(x) \sum_{i=1}^{s} \frac{\partial F}{\partial x_i}(x) u_{j+1}^{(i)},$$

and

$$P_{1,1}(x) = \sum_{i=1}^{s} \frac{\partial F}{\partial x_i}(x) u_1^{(i)}.$$

Here $u_j^{(i)}$ denotes the $i$th component of $u_j$.

To prove (15) we consider the cases $|t| \leq N^{-d}$ and $N^{-d} < |t| \leq N^{1-d}$ separately. If $|t| \leq N^{-d}$ we estimate $g^{(d)}$ trivially. Since $P_{j,l}(x) \ll N^{ld-j}$ uniformly in $u_1, \ldots, u_j \in T$ and $x \in 2NB$, (17) and $|t|N^d \leq 1$ imply $g^{(j)}(x)[u_1, \ldots, u_j] \ll |t|N^{d-j}$. Hence $R_d(t) \ll |t|^N$. Together with $\hat{\chi}(t) \ll |t|^{-1}$ this yields

$$\left( \int_{|t| \leq N^{-d}} R_d(t)\hat{\chi}(t) \, dt \right) \ll \int_{|t| \leq N^{-d}} N^s \, dt \ll N^{s-d}. $$
In the case $N^{-d} < |t| \leq N^{1-d}$ we use assumption (C). Since the estimate in (C) is uniform in all boxes $B' \subseteq B$ with sides parallel to the coordinate axes we can apply partial summation. This yields, for an arbitrary polynomial $P$,

$$\sum_{n \in NB} e(tF(n + u))P(n + u) \ll N^{\deg(P) + s - \omega d}|t|^{-\omega}$$

uniformly in $u \in T$. Together with (17) we obtain

$$\sum_{n \in NB \cap Z^s} g^{(d)}(n + \tau u_m)[u_1^{a_1} \ldots u_m^{a_m}]$$

$$= \sum_{l=1}^{d}(2\pi it)^l \sum_{n \in NB \cap Z^s} P_{d,l}(n + \tau u_m)e(tF(n + \tau u_m))$$

$$\ll N^{-d+s-\omega d}|t|^{-\omega} \sum_{l=1}^{d}(|t|N^d)^l \ll N^{d^2-d+s-\omega d}|t|^{d-\omega}.$$ 

Since $\omega > d$ it follows that

$$\int_{(N^{-d},N^{1-d})} R_d(t) \hat{\chi}(t) \, dt \ll N^{d^2-d+s-\omega d} \int_{(N^{-d},N^{1-d})} t^{d-\omega-1} \, dt \ll N^{s-d}.$$ 

This together with (18) implies (15).

To prove (16) we use (D). Since the estimate in (D) is uniform in all boxes $B' \subseteq B$ we can apply partial integration. This gives, for an arbitrary polynomial $P$ and $|t| \geq N^{-d}$,

$$\int_{NB} P(x)e(tF(x)) \, dx \ll N^{\deg(P) + s - \omega d}|t|^{-\omega}.$$ 

Hence Lemma 3.1 implies, for $|t| \geq N^{-d}$ (uniformly in $u_1, \ldots, u_m \in T$),

$$\int_{NB} g^{(j)}(x)[u_1^{a_1} \ldots u_m^{a_m}] \, dx = \sum_{l=1}^{j} (2\pi it)^l P_{j,l}(x)e(tF(x)) \, dx$$

$$\ll N^{s-j-\omega d}|t|^{-\omega} \sum_{l=1}^{j} (|t|N^d)^l \ll N^{s+j(d-1)-\omega d}|t|^{j-\omega}.$$ 

For $0 \leq j < d$ this together with (5) yields

$$\int_{|t| > N^{1-d}} G_j(t) \hat{\chi}(t) \, dt \ll N^{s+j(d-1)-\omega d} \left( \int_{(N^{1-d},1]} t^{j-\omega-1} \, dt + \int_{(1,\infty)} t^{-2} \, dt \right) \ll N^{s-\omega}.$$ 

Since $\omega > d$ this implies (16).

Finally, we prove

**Lemma 3.2.** $H_j = 0$ for $j \geq 1$. 
Proof. By Lemma 3.1 and the definition of \( H_j \) we obtain, for \( j \geq 1 \),

\[
H_j = \int_{\mathbb{R}} G_j(t) \hat{\chi}(t) \, dt \\
= \sum_{|\alpha|=j} c(\alpha) \int_{\mathbb{R}} \int_{T^m} \int_{NB} g^{(j)}(x)[u_1^{\alpha_1} \ldots u_m^{\alpha_m}] \hat{\chi}(t) \, dx \, du_1 \ldots du_m \, dt \\
= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \int_{NB} \sum_{l=1}^j P_{j,l}(x) e(tF(x)) \chi^{(l)}(t) \, dt \, dx \, du_1 \ldots du_m \\
= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \int_{1} P_{j,l}(x) \chi^{(l)}(F(x)) \, dx \, du_1 \ldots du_m \\
= \sum_{|\alpha|=j} c(\alpha) \int_{T^m} \int_{1} P_{j,l}(x) \chi^{(l)}(F(x)) \, dx \, du_1 \ldots du_m.
\]

Here we used \( \int_{\mathbb{R}^s} (2\pi i t)^l \hat{\chi}(t) \, dt \) and the fact that \( \chi^{(l)}(F(x)) = 0 \) if \( x \notin NB \). In the case \( j = 1 \) Lemma 3.1 yields

\[
H_1 = -\int_{T} \int_{\mathbb{R}^s} P_{1,1}(x) \chi^{(1)}(F(x)) \, dx \, du_1 \\
= -\int_{\mathbb{R}^s} \sum_{i=1}^s \frac{\partial F}{\partial x_i}(x) \chi^{(1)}(F(x)) \, dx \int_{T} u_1^{(i)} \, du_1 = 0.
\]

Remember that \( T = (-1/2, 1/2)^s \). For \( j \geq 1 \) we prove that

\[
(19) \quad \sum_{l=1}^{j+1} \int_{\mathbb{R}^s} P_{j+1,l}(x) \chi^{(l)}(F(x)) \, dx = 0.
\]

This implies \( H_j = 0 \) for \( j \geq 2 \). To prove (19) set

\[
H_{j,l} = \int_{\mathbb{R}^s} \sum_{i=1}^s \frac{\partial}{\partial x_i} (P_{j,l}(x)) u_{j+1}^{(i)} \chi^{(l)}(F(x)) \, dx.
\]

Using partial integration one obtains, for \( 2 \leq l \leq j + 1 \),

\[
\int_{\mathbb{R}^s} P_{j,l-1}(x) \sum_{i=1}^s \frac{\partial F}{\partial x_i}(x) u_{j+1}^{(i)} \chi^{(l)}(F(x)) \, dx \\
= \sum_{i=1}^s u_{j+1}^{(i)} \int_{\mathbb{R}^s} P_{j,l-1}(x) \frac{\partial}{\partial x_i} \chi^{(l-1)}(F(x)) \, dx \\
= -\sum_{i=1}^s u_{j+1}^{(i)} \int_{\mathbb{R}^s} \frac{\partial}{\partial x_i} (P_{j,l-1}(x)) \chi^{(l-1)}(F(x)) \, dx = -H_{j,l-1}.
\]
This together with the representation of $P_{j+1,l}$ in Lemma 3.1 implies

$$
\int_{\mathbb{R}^s} P_{j+1,l}(x) \chi^{(1)}(F(x)) \, dx = H_{j,l},
\int_{\mathbb{R}^s} P_{j+1,j+1}(x) \chi^{(j+1)}(F(x)) \, dx = -H_{j,j}.
$$

Adding these $j+1$ equations yields (19). This completes the proof of Lemma 3.2 and the proof of the Proposition.

4. Proof of Theorem 3. We have to prove that $\Omega(F) > d$ implies (A)–(D) of the Proposition. We start with (D). It is only here that we use, for inhomogeneous $F$, the more sophisticated definition (10) instead of (9).

**Lemma 4.1.** If $0 < \omega < \Omega(F)$ then

$$
\int_{NB'} e(tF(u)) \, du \ll N^s \min(1, (|t|N^d)^{-\omega})
$$

uniformly for all boxes $B' \subseteq B$ with sides parallel to the coordinate axes.

**Proof.** The estimate is trivial for $|t| \leq N^{-d}$. If $|t| > N^{-d}$ the substitution $u = Q^{-1}x$ with $QN \geq 1$ yields

$$
\int_{QB'} e(tF(u)) \, du = Q^{-s} \int_{QN_{B'}} e(tF(Q^{-1}x)) \, dx
= Q^{-s} \left( \sum_{n \in QNB' \cap \mathbb{Z}^s} e(tF(Q^{-1}n)) + O(|t|N^d(QN)^{s-1}) \right).
$$

To prove (20) cover $QNB'$ by cubes of the form $n + T$, $T = [-1/2, 1/2]^s$. There are at most $O((QN)^{s-1})$ cubes which intersect the boundary of $QNB'$. Furthermore, for $x \in n + T$ with $n \in QNB'$, one finds

$$
e(tF(Q^{-1}x)) = e(tF(Q^{-1}n)) + O(|t|N^{-d-1})
$$

since $\frac{\partial}{\partial x_i}(tF(Q^{-1}x)) \ll |t|Q^{-1} \frac{\partial F}{\partial x_i}(Q^{-1}x) \ll |t|Q^{-1}N^{d-1}$. This proves (20). The exponential sum in (20) has the form

$$
\sum_{n \in QNB' \cap \mathbb{Z}^s} e(tQ^{-d}F'(d)(n) + P(n))
$$

with a polynomial $P \in \mathbb{R}[X_1, \ldots, X_s]$ of degree strictly smaller than $d$. For $0 < \Delta \leq 1$ choose $Q$ such that $|t|Q^{-d} = (QN)^{\Delta-d}$. Then $QN \geq 1$ and $|t|Q^{-d}$ lies on the boundary of $\mathfrak{M}\Delta(1,1)$. By the definition (10) of $\Omega(F)$ the exponential sum is $\ll (QN)^{s-\omega \Delta}$. If $F$ is homogeneous the same follows
from the alternative definition (9). Now (20) implies
\[ \int_{NB'} e(tF(u)) \, du \ll Q^{-s}(QN)^{s-\omega} + \frac{|t|Q^{-1}N^{s+d-1}}{N^{s-\omega}d|t|^{-\omega} + N^s(|t|N^d)^{1-1/\Delta}}. \]
Both terms on the right hand side are equal if we set \( \Delta = (1+\omega)^{-1} \in (0,1] \).

**Lemma 4.2.** \( \Omega(F) > d \) implies that condition (C) of the Proposition is satisfied.

**Proof.** Condition (C) is trivially satisfied if \( |t| \leq N^{-d} \). If \( N^{-d} < |t| \leq N^{1-d} \) choose \( \Delta(t) \) such that \( |t| = N^{\Delta(t)-d} \), i.e. \( \Delta(t) = d + \log |t|/\log N \). The condition \( N^{-d} < |t| \leq N^{1-d} \) ensures \( \Delta(t) \in (0,1] \). With this choice \( t \) lies on the boundary of \( \mathfrak{M}_{\Delta(t)}(1,1) \). Hence \( t \in \mathfrak{m}_{\Delta(t)} \) and the definition (10) or (9) implies, for every \( \Omega(F) > \omega > d \),
\[ \sum_{n \in NB' \cap \mathbb{Z}^s} e(tF(n + u)) \ll N^{s-\omega\Delta(t)} \ll N^{s-\omega d|t|^{-\omega}} \]
uniformly for all \( u \in B \) and all boxes \( B' \subseteq B \) with sides parallel to the coordinate axes. This proves (C).

To verify conditions (A) and (B) of the Proposition, we split the domain of integration into a part covered by minor arcs and a second part covered by major arcs.

**Lemma 4.3 (minor arcs).** If \( \Omega(F) > d \) and \( 0 < \Delta < 1 \) then
\begin{align*}
(21) & \quad \int_{\mathfrak{m}_{\Delta}} |S_N(t)| \, dt \ll N^{s-d}, \\
(22) & \quad \int_{(N^{1-d},1] \cap \mathfrak{m}_{\Delta}} |S_N(t)| \, \frac{dt}{t} \ll N^{s-d}.
\end{align*}

**Proof.** We prove (22). The proof of (21) is analogous; see [6, p. 24, Lemma 4.B], for an even sharper estimate. Choose \( \omega \) such that \( \Omega(F) > \omega > d \). If \( \Delta = 1 \) the definition of \( \Omega(F) \) implies \( S_N(t) \ll N^{s-\omega} \) for all \( t \in \mathfrak{m}_1 \). Hence
\[ \int_{(N^{1-d},1] \cap \mathfrak{m}_1} |S_N(t)| \, \frac{dt}{t} \ll N^{s-\omega} \int_{(N^{1-d},1]} \frac{dt}{t} \ll N^{s-\omega} \log N \ll N^{s-d}. \]

If \( 0 < \Delta < 1 \) we split \( (\Delta,1] \) into subintervals \( (\Delta_{i-1}, \Delta_i] \), where \( \Delta = \Delta_0 < \Delta_1 < \ldots < \Delta_n = 1 \). Then
\[ \mathfrak{m}_{\Delta} = ((\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}_1) \cup \bigcup_{i=1}^n \mathfrak{M}_{\Delta_i} \setminus \mathfrak{M}_{\Delta_{i-1}} = \mathfrak{m}_1 \cup \bigcup_{i=1}^n \mathfrak{r}_i, \]
where \( r_i = \mathcal{M}_\Delta \setminus \mathcal{M}_{\Delta - 1} \subseteq \mathcal{M}_\Delta \). Since \( \mathcal{M}_\Delta \) has Lebesgue measure
\[
\lambda(\mathcal{M}_\Delta) \ll \sum_{1 \leq a \leq q \leq N^\Delta} q^{-1} N^{\Delta - d} \ll N^{2\Delta - d},
\]
it follows that \( \lambda(r_i) \ll N^{2\Delta_i - d} \). Furthermore, the definition of \( \Omega(F) \) yields for \( t \in r_i \subseteq \mathcal{M}_{\Delta - 1} \) the estimate \( S_N(t) \ll N^{s - \omega \Delta_i - 1} \). Hence we obtain
\[
\int_{(N^{d-1},1) \cap \mathcal{M}_\Delta} |S_N(t)| \frac{dt}{t} \ll \int_{(N^{d-1},1) \cap \mathcal{M}_1} |S_N(t)| \frac{dt}{t} + \sum_{i=1}^n \int_{(N^{d-1},1) \cap r_i} |S_N(t)| \frac{dt}{t}
\ll N^{s-d} + \sum_{i=1}^n N^{s - \omega \Delta_i - 1} \int_{(N^{d-1},1) \cap r_i} \frac{dt}{t}.
\]
Since \( r_i \subseteq \mathcal{M}_\Delta \), we consider (for \( (a,q) \neq (1,1) \))
\[
(23) \quad \int_{\mathcal{M}_\Delta(q,a) \cap (0,1]} \frac{dt}{t} = \int_{\mathcal{M}_\Delta(q,a) \cap (0,1]} \frac{dt}{t} = \log \frac{1 + \frac{1}{a} N^{\Delta - d}}{1 - \frac{1}{a} N^{\Delta - d}} \ll \frac{1}{a} N^{\Delta - d}.
\]
It follows that
\[
\int_{(N^{1-d},1) \cap \mathcal{M}_\Delta} \frac{dt}{t} \ll \sum_{1 \leq a \leq q \leq N^\Delta} \frac{1}{a} N^{\Delta - d} \ll N^{\Delta - d} \sum_{1 < q \leq N^\Delta} \log q \ll N^{2\Delta - d} \log N.
\]
Altogether we obtain
\[
\int_{(N^{d-1},1) \cap \mathcal{M}_\Delta} |S_N(t)| \frac{dt}{t} \ll N^{s-d} + \sum_{i=1}^n N^{s-d - (\omega - 2) \Delta_i - 2(\Delta_i - \Delta_{i-1})} \log N
\ll N^{s-d} + N^{s-d - (\omega - 2) \Delta + 2\varepsilon} \ll N^{s-d},
\]
if we choose \( \Delta_i - \Delta_{i-1} < \varepsilon \) sufficiently small. This proves (22) for every \( \Delta \in (0,1] \).

**Lemma 4.4 (major arcs).** If \( \Omega(F) > 2 \) and \( 0 < \Delta < 1/4 \) then
\[
(24) \quad \int_{\mathcal{M}_\Delta} |S_N(t)| \, dt \ll N^{s-d},
\]
\[
(25) \quad \int_{(N^{1-d},1) \cap \mathcal{M}_\Delta} |S_N(t)| \frac{dt}{t} \ll N^{s-d}.
\]

**Proof.** If \( F \) is a polynomial with integer coefficients and \( t \) is close to a rational number with small denominator, then \( S_N(t) \) can be evaluated asymptotically. It is well known (cf. [6, p. 26, Lemma 5.A]) that for every \( t \in \mathcal{M}_\Delta(q,a) \), we have
\[
(26) \quad S_N(t) = S\left(\frac{a}{q}\right)G_0\left(t - \frac{a}{q}\right) + O(qN^{s-1+\Delta}),
\]
where
\[ S\left( \frac{a}{q} \right) = q^{-s} \sum_{n \in (0,1)^* \cap \mathbb{Z}^s} e\left( \frac{a}{q} F(n) \right), \quad G_0(t) = \int_{NB} e(t F(u)) \, du. \]

Since \( a/q \) with \( (a,q) = 1 \) lies in \( \mathfrak{M}_1(q,a) \) with \( N = q \), the definition of \( \Omega(F) \) implies
\[ (27) \quad S\left( \frac{a}{q} \right) \ll q^{-\omega} \]
for every \( \omega < \Omega(F) \). Additionally, by Lemma 4.1, \( G_0(t) \ll N^s \min(1, |t N^d|^{-\omega}) \) for \( \omega < \Omega(F) \). Since \( \Omega(F) > 2 \) we can choose \( \omega > 2 \). Using these estimates it is easy to prove (24) and (25). We demonstrate (25). Since
\[ \left| t - \frac{a}{q} \right| \leq \frac{1}{q} N^{\Delta-d} \quad \text{for } t \in \mathfrak{M}_\Delta(q,a), \]
it follows that \( t \geq a/(2q) \). Hence
\[ \int_{\mathfrak{M}_\Delta(q,a) \cap (0,1]} |S_N(t)| \frac{dt}{t} \ll \left| S\left( \frac{a}{q} \right) \right| \frac{q}{a} \int_{|u| \leq \frac{1}{q} N^{\Delta-d}} |G_0(u)| \, du \ll q N^{s-1+\Delta} \int_{\mathfrak{M}_\Delta(q,a) \cap (0,1]} \frac{dt}{t}. \]
The substitution \( u = N^{-d} v \) yields
\[ \int_{|u| \leq \frac{1}{q} N^{\Delta-d}} |G_0(u)| \, du = N^{-d} \int_{|v| \leq \frac{1}{q} N^\Delta} |G_0(N^{-d} v)| \, dv \ll N^{s-d} \int_{|v| \leq \frac{1}{q} N^\Delta} \min(1, |v|^{-\omega}) \, dv \ll N^{s-d}. \]
Together with (23) and (27) we obtain
\[ \int_{(N^{1-d},1] \cap \mathfrak{M}_\Delta} |S_N(t)| \frac{dt}{t} \ll N^{s-d} \sum_{1 \leq a \leq N^\Delta} \left( a^{-1} q^{-1-\omega} + a^{-1} q N^{2\Delta-1} \right) \ll N^{s-d}(1 + N^{4\Delta-1}) \log N \ll N^{s-d}. \]

5. Proof of Theorem 2. Let \( F_0(X) = \sum_{i=1}^s \lambda_i X_i^d \) with integer coefficients \( \lambda_i > 0 \). It is known that \( \Omega(F_0) \geq s 2^{1-d} \) (see [6, p. 24] and the remarks following (11)). Hence Theorem 3 implies \( P_{F_0}(R) \ll R^{s/d-1} \) if \( s > d 2^{d-1} \). For large \( d \) this can be substantially improved by Vinogradov’s mean value theorem. We prove that (A)–(D) of the Proposition are satisfied if \( s > \varrho_0(d) \), where \( \varrho_0(d) \) is an explicitly computable function which satisfies \( \varrho_0(d) \sim 2d^3 \log d \) for \( d \to \infty \).
First we prove that (C) and (D) are satisfied if \( s > d \). To do this we establish (7) and (8) with \( \omega = s/d \). By [2, Theorem 2.2] (the second derivative test), it follows that

\[
\sum_{M < n \leq M'} e(t(n + u)^d) \ll (|t|M^{d-2})^{-1/2} + M(|t|M^{d-2})^{1/2}
\]

uniformly for \( u \in [-1, 1] \) and \( 1 \leq M < M' \leq 2M \). Splitting \([0, N]\) into dyadic intervals of the form \((2^j - 1, 2^j] \) with \( U = |t|^{-1/d} \) we obtain

\[
\sum_{0 \leq n \leq N} e(t(n + u)^d) \ll 1 + U + \sum_j (|t|^{-1/2}(2^j U)^{1-d/2} + |t|^{1/2}(2^j U)^{d/2})
\]

\[
\ll |t|^{-1/d} + |t|^{1/2}N^{d/2}.
\]

It follows that

\[
\sum_{n \in NB'} e(tF_0(n + u)) \ll (|t|^{-1/d} + |t|^{1/2}N^{d/2})^s \ll |t|^{-s/d}
\]

if \(|t| \leq N^{1-d}\). This proves (7) with \( \omega = s/d \). To prove (D) observe that for \( t > 0 \),

\[
\int_0^N e(tx^d) \, dx = t^{-1/d} \int_0^{tN^d} \xi^{1/d-1}e(\xi) \, d\xi \ll t^{-1/d}
\]

(the last integral is bounded by an absolute constant). This proves (8) with \( \omega = s/d \).

Next we prove (A) and (B). Let

\[
f(t) = \sum_{1 \leq n \leq N} e(tn^d),
\]

then \( S_N(t) = \prod_{i=1}^s (1 + 2f(\lambda_it)) \). By Hölder’s inequality it is sufficient to prove

\[
\int_{(0,1]} |f(t)|^s \, dt \ll N^{s-d} \quad \text{and} \quad \int_{(\lambda_iN^{1-d},1]} |f(t)|^s \frac{dt}{t} \ll N^{s-d}.
\]

To estimate the special function \( f(t) \) one can work with larger major arcs. Let \( N = [(R + 1)^{1/d}] + 1/2 \) and set

\[
\mathcal{M}(q,a) = \left\{ t \in \mathbb{R}/\mathbb{Z} \ \bigg| \ |t - a/q| \leq P/qR \right\}, \quad P = \frac{N}{2d}.
\]

Write \( \mathcal{M} \) for the union of the \( \mathcal{M}(q,a) \) with \( 1 \leq a \leq q \leq P \) and \( (a,q) = 1 \), and set \( m = (\mathbb{R}/\mathbb{Z}) \setminus \mathcal{M} \).

**Lemma 5.1 (major arcs).** If \( s > 2d \) and \( c > 0 \) then

\[
\int_{\mathcal{M}} |f(t)|^s \, dt \ll N^{s-d} \quad \text{and} \quad \int_{(cN^{1-d},1] \cap \mathbb{R}} |f(t)|^s \frac{dt}{t} \ll N^{s-d}.
\]
Proof. By [9, Theorem 4.1], for $t \in \mathfrak{M}(q, a)$ and any $\varepsilon > 0$, 

$$f(t) = \frac{1}{q} S \left( \frac{a}{q} \right) v \left( t - \frac{a}{q} \right) + O(q^{1/2+\varepsilon}),$$

where, by [9, Theorem 4.2 and Lemma 2.8],

$$\frac{1}{q} S \left( \frac{a}{q} \right) \ll q^{-1/d} \quad \text{and} \quad v(t) \ll \min(N, |t|^{-1/d}).$$

This yields

$$\int_{(cN^{1-d}, 1) \cap \mathfrak{M}} |f(t)|^s \frac{dt}{t} \ll \sum_{1 \leq a \leq q \leq P} \left( \int_{|u| \leq P/(qR)} |v(u)|^s \, du + q^{s/2 + \varepsilon} \frac{P}{qR} \right) \frac{q}{a}. $$

Since

$$\int_{|u| \leq P/(qR)} |v(u)|^s \, du \ll N^{s-d} + \int_{(N^{-d}, P/(qR))} u^{-s/d} \, du \ll N^{s-d},$$

we obtain, for $s > 2d$,

$$\int_{(cN^{1-d}, 1) \cap \mathfrak{M}} |f(t)|^s \frac{dt}{t} \ll N^{s-d} \sum_{q \leq N} q^{1-s/d} \log q + N^{1-d} \sum_{q \leq N} q^{s/2 + 2\varepsilon} \ll N^{s-d}.$$ 

This proves the second assertion of the lemma. The first one follows in the same way.

Finally, we estimate the contribution of the minor arcs to (28). Since

$$\int_{(\lambda, N^{d-1}, 1) \cap \mathfrak{m}} |f(t)|^s \frac{dt}{t} \ll N^{1-d} \int_{\mathfrak{m}} |f(t)|^s \, dt$$

(28) is a consequence of Lemma 5.1 and the following lemma.

**Lemma 5.2 (minor arcs).** There is an explicitly computable function $\varrho_0(d)$, which satisfies $\varrho_0(d) \sim 2d^3 \log d$ for $d \to \infty$, such that for $s \geq \varrho_0(d)$,

$$\int_{\mathfrak{m}} |f(t)|^s \, dt \ll N^{s-2d+1}.$$ 

**Proof.** We use Wooley’s refinement of Vinogradov’s mean value theorem. The original form of the mean value theorem yields Lemma 5.2 with $\varrho_0(d) \sim 4d^3 \log d$. By [9, Theorem 5.6], there is an explicitly computable function $\sigma(d)$ such that for $t \in \mathfrak{m}$,

$$f(t) \ll N^{1-\sigma(d)} \log N.$$ 

We have $\sigma(d) \sim (2d^2 \log d)^{-1}$ for $d \to \infty$. Furthermore, by [9, Theorem 5.5 and (5.37)], for every integer $l \geq 1$,

$$\int_{(0, 1]} |f(t)|^{2dl} \, dt \ll N^{2dl-d+\eta(d)},$$
where
\[ \eta_l(d) = \frac{1}{2} d(d - 1) \left(1 - \frac{5}{4d}\right)^{l-1}. \]
These estimates imply, for every \( l \geq 1 \),
\[
\int \int |f(t)|^s dt \ll \left( \sup_{t \in \mathbb{R}} |f(t)|^{s-2dl} \right) \int \int |f(t)|^{2dl} dt \\
\ll N^{(s-2dl)(1-\sigma(d)) + 2dl - d + \eta_l(d)(\log N)^{s-2dl}}.
\]
There is an \( l \) such that the right hand side is \( \ll N^{s-2d+1} \) if
\[ s > \min_l \left\{ \frac{\eta_l(d)}{\sigma(d)} + 2dl \right\} + \frac{d - 1}{\sigma(d)} = \varrho_0(d), \]
say. By [9, Theorem 5.7], the minimum is \( \ll d^2 \log d \), thus \( \varrho_0(d) \sim 2d^3 \log d \) for \( d \to \infty \).

We remark that for small \( d \) Theorem 2 can be further sharpened. For instance, Hua’s lemma ([9, Lemma 2.5]) can be used to prove \( P_{F_0}(R) \ll R^{s/d-1} \) for \( s > 2d+1 - 2 \).

References


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