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On Fourier coefficients of modular forms of different weights

by

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1. Introduction. Let f and g be modular forms on a congruence subgroup Γ of $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ of weights k_1 and k_2, respectively. We shall suppose that $k_1, k_2 > 1$ and that either both k_1 and k_2 are integral or both are half-integral, with the usual assumption that $\Gamma \subset \Gamma_0(4)$ in the latter case. For basic facts on half-integral weight modular forms we refer the reader to [9]. We denote by $a(n)$ resp. $b(n)$ $(n \geq 0)$ the Fourier coefficients of f resp. g.

A rather intrinsic question then is to ask for the least index n such that $a(n) \neq b(n)$ provided that $f \neq g$. More generally, if $a(n)$ and $b(n)$ for all n are contained in the ring of integers $\mathcal{O}_K$ of a number field K and $\wp$ is a prime ideal of $\mathcal{O}_K$, then if $f \not\equiv g \pmod{\wp}$ (meaning that there exists at least one n with $a(n) \not\equiv b(n) \pmod{\wp}$), one may ask for the least n with $a(n) \not\equiv b(n) \pmod{\wp}$.

If $k_1 = k_2$, then as is well known the valence formula for modular forms implies that there exists $n \leq (k_1/12)[\Gamma(1) : \Gamma]$ such that $a(n) \neq b(n)$ if $f \neq g$. Under the additional hypothesis of integrality of $a(n)$ and $b(n)$ as above, by a fundamental result of Sturm [10] the same result is true modulo $\wp$. Note that in the above discussion the half-integral weight case can be deduced from the integral weight case by taking squares.

In the following we will suppose that $k_1 \neq k_2$. In this case, if k_1 and k_2 are integral and in addition one assumes that f and g are normalized cuspidal Hecke eigenforms on $\Gamma_0(N)$, very good bounds on the least n with $a(n) \neq b(n)$ are known [1, 6]. Note that the paper [6] also contains some statements in the case of unequal integral weights for arbitrary f and g whose proof unfortunately is not correct, as was first pointed out by the author of [6] and by J. Sengupta.

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The main purpose of this paper is to give some generalizations of Sturm's result when the weights are different, with bounds however depending on $\wp$, and also to give some consequences in characteristic zero.

We shall start by showing that if p is the rational prime with $\wp\,|\,p$ and $a(n), b(n) \in \mathcal{O}_K$ for all n, then if p is odd there exists $n \leq (\max\{k_1, k_2\} \cdot (p^2-1)/12)[\Gamma(1) : \Gamma]$ such that $a(n) \not\equiv b(n) \pmod{\wp}$ provided that $f \not\equiv g \pmod{\wp}$ (Thm. 1, Sect. 2). The proof easily follows from the existence of a certain Eisenstein series of weight 1 on $\Gamma_1(p)$ with certain congruence properties modulo p. The usefulness of this Eisenstein series in the study of congruences of modular forms was first pointed out by Shimura (cf. e.g. [5, Chap. XV, Sect. 1]). A similar result is also (trivially) true if $p = 2$.

From Theorem 1, using results of Serre [7] and Katz [4] on modular forms modulo p, one can also obtain some results in characteristic zero for $k_1 \neq k_2$. For example, suppose that $\Gamma = \Gamma_0(N)$, that k_1 and k_2 are even integral and that $a(n) \in K$ for all n. Then using a theorem of Heath-Brown [3] on the least prime in an arithmetic progression, we shall show that there exists $n \ll_N \max\{k_1, k_2\} \log^{11} |k_1 - k_2|$ such that $a(n) \neq b(n)$ where the implied constant in $\ll_N$ depends on N. In fact, we shall prove a slightly more general result, allowing f and g to have Dirichlet characters modulo N (Thm. 2, Sect. 3). Theorem 2 has an obvious application to quadratic forms (Corollary, Sect. 3).

One can obtain bounds that are sometimes slightly better than those of Theorem 1 if e.g. $\Gamma = \Gamma_0(N)$ and if in addition one assumes that k_1, k_2 are integral and f and g are eigenforms of the usual Hecke operator $T(p)$, with eigenvalues non-zero modulo $\wp$ (Thm. 3, Sect. 4). The proof uses the first Rankin–Cohen bracket on modular forms.

Ideally, one would hope that similar assertions to those of Theorem 1 would hold with bounds independent of $\wp$. However, it seems to be unclear how to prove this. On the other hand, in the case of $\Gamma = \Gamma_0(N)$ and integral weights, and if f and g are normalized cuspidal Hecke eigenforms, under certain simple conditions on N we shall show that in fact there are infinitely many prime ideals $\wp$ of $\mathcal{O}_{K_{f,g}}$ for which such a result is true (Thm. 4, Sect. 5). Here $K_{f,g}$ is the number field generated over $\mathbb{Q}$ by all the Fourier coefficients $a(n), b(n)\,(n \geq 1)$. The proof is an easy modification of a beautiful and simple argument due to M. Ram Murty [6] in characteristic zero.

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Notation. The letter Γ always denotes a congruence subgroup of $\Gamma(1)$. For $N \in \mathbb{N}$ we let as usual $\Gamma_0(N)$, resp. $\Gamma_1(N)$, be the congruence subgroups of $\Gamma(1)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \equiv 0 \pmod{N}$, resp. $c \equiv 0 \pmod{N}$ and $a \equiv d \equiv 1 \pmod{N}$.

For $z \in \mathcal{H}$, the complex upper half-plane, we put $q = e^{2\pi i z}$. If f is a modular form on $\Gamma \subset \Gamma(1)$ and $M \in \mathbb{N}$ is minimal such that $\left(\begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, then f has a Fourier expansion $f = \sum_{n \geq 0} a(n) q_M^n$ where $q_M = e^{2\pi i z/M}$ $(z \in \mathcal{H})$.

The letter K always denotes a number field and $\wp$ is a prime ideal of the ring of integers $\mathcal{O}_K$ of K. If $f = \sum_{n\geq 0} a(n) q_M^n$ and $g = \sum_{n \geq 0} b(n) q_M^n$ are power series in q_M with $\wp$-integral coefficients and $a(n) \equiv b(n) \pmod{\wp}$ for all n, then we shall write $f \equiv g \pmod{\wp}$.

2. A generalization of Sturm's result to the case of different weights. We shall prove

THEOREM 1. *Let f and g be modular forms on Γ of weights k_1 and k_2 respectively, where $k_1, k_2 > 1$, $k_1 \neq k_2$ and either both k_1, k_2 are integral or both are half-integral. Suppose that f resp. g have Fourier coefficients $a(n)$ resp. $b(n)$ in $\mathcal{O}_K$. Let p be the rational prime with $\wp \mid p$. Then if $f \not\equiv g \pmod{\wp}$, there exists*

$$n \leq \frac{\max\{k_1, k_2\}}{12} \cdot \begin{cases} [\Gamma(1) : \Gamma \cap \Gamma_1(p)] & \text{if } p > 2, \\ [\Gamma(1) : \Gamma \cap \Gamma_1(4)] & \text{if } p = 2 \end{cases}$$

such that $a(n) \not\equiv b(n) \pmod{\wp}$. In particular there exists

$$n \leq \begin{cases} \dfrac{\max\{k_1, k_2\}}{12} (p^2 - 1) \cdot [\Gamma(1) : \Gamma] & \text{if } p > 2, \\ \max\{k_1, k_2\} \cdot [\Gamma(1) : \Gamma] & \text{if } p = 2 \end{cases}$$

with $a(n) \not\equiv b(n) \pmod{\wp}$.

Proof. First suppose that p is odd. Let ζ be a primitive $(p-1)$th root of unity. Then p splits completely in $\mathbb{Q}(\zeta)$. Choose a prime ideal $\wp_1$ of $\mathbb{Q}(\zeta)$ lying above p. Since the numbers $1, \zeta, \ldots, \zeta^{p-2}$ are different modulo $\wp_1$ and reduction modulo $\wp_1$ induces an isomorphism $(\mathcal{O}_{\mathbb{Q}(\zeta)}/\wp_1)^* \cong (\mathbb{Z}/p\mathbb{Z})^*$, we can define a Dirichlet character χ modulo p by requiring that

$$\chi(m) m \equiv 1 \pmod{p}$$

for all m prime to p.

Let

$$G_{1,\chi} := B_{1,\chi} - 2 \sum_{n \geq 1} \Big(\sum_{d \mid n} \chi(d) \Big) q^n$$

be the Eisenstein series of weight 1 and character χ on $\Gamma_0(p)$, where

$$B_{1,\chi} = \frac{1}{p} \sum_{m=1}^{p-1} \chi(m) m$$

is the usual modified 1st Bernoulli number. In particular $G_{1,\chi}$ is on $\Gamma_1(p)$. Note that

$$\sum_{m=1}^{p-1} \chi(m)m \not\equiv 0 \pmod{\wp_1}.$$

Put

$$E_{1,\chi} := \frac{1}{B_{1,\chi}} G_{1,\chi}.$$

Then $E_{1,\chi}$ is a modular form of weight 1 on $\Gamma_1(p)$ with $\wp_1$-integral Fourier coefficients and

$$E_{1,\chi} \equiv 1 \pmod{p}. \tag{1}$$

This construction is of course well known.

Let $\mathcal{P}$ be a prime ideal of the composite field $K\mathbb{Q}(\zeta)$ lying above $\wp$, hence above p. Then $\mathcal{Q} := \mathcal{P} \cap \mathcal{O}_{\mathbb{Q}(\zeta)} \supset \mathcal{P} \cap \mathbb{Z} = p\mathbb{Z}$, i.e. $\mathcal{Q}$ is an ideal of $\mathbb{Q}(\zeta)$ dividing p. By (1) we therefore find that $E_{1,\chi} \equiv 1 \pmod{\mathcal{Q}}$, hence $E_{1,\chi} \equiv 1 \pmod{\mathcal{P}}$.

Suppose without loss of generality that $k_1 > k_2$. From our assumption $f \not\equiv g \pmod{\wp}$ it then follows that $f \not\equiv g E_{1,\chi}^{k_1-k_2} \pmod{\mathcal{P}}$. Hence by Sturm's result (cf. Introduction) there exists $n \le (k_1/12)[\Gamma(1) : \Gamma \cap \Gamma_1(p)]$ such that $a(n) \not\equiv b(n) \pmod{\mathcal{P}}$. Since

$$[\Gamma(1) : \Gamma \cap \Gamma_1(p)] \le [\Gamma(1) : \Gamma_1(p)][\Gamma(1) : \Gamma] = (p^2-1)[\Gamma(1) : \Gamma],$$

this proves our assertion for p odd.

Now assume that $p = 2$. Let $\theta = 1 + \sum_{n\ge1} q^{n^2}$ be the basic theta function of weight 1/2 on $\Gamma_0(4)$. Then θ^2 has weight 1 and is on $\Gamma_1(4)$. Multiplying g with θ^2 and proceeding as above, taking into account that $[\Gamma(1) : \Gamma_1(4)] = 12$, this proves our assertion if $p = 2$.

3. A result in characteristic zero. As indicated in the Introduction, from Theorem 1 one can obtain some conditional results also in characteristic zero. For this, one has to use the fact that if f resp. g are modular forms of integral weights k_1 resp. k_2 on the principal congruence subgroup $\Gamma(N)$ of level N with Fourier coefficients in $\mathcal{O}_K$ and $f \equiv g \pmod{\wp}$, $f \not\equiv 0 \pmod{\wp}$ where $\wp$ is a prime ideal of $\mathcal{O}_K$ with $(\wp, 2N) = 1$, then $k_1 \equiv k_2 \pmod{(p-1)}$ where $\wp \mid p$ (see [4, 5, 7]). As an example we will prove

THEOREM 2. *Let f resp. g be modular forms of integral weights k_1 resp. k_2 on $\Gamma_0(N)$ with Dirichlet characters χ_1 resp. χ_2 modulo N, and with Fourier coefficients $a(n)$ resp. $b(n)$ $(n \ge 0)$. Suppose that $k_1, k_2 \ge 2$ and $k_1 \ne k_2$. Suppose furthermore that f has Fourier coefficients in a number field K and $f \ne 0$. Let M be the product of the different prime divisors of N.*

Then there exists

$$n \ll \begin{cases} \max\{k_1, k_2\} \cdot \log^{11} |k_1 - k_2| \cdot M^{11} \dfrac{\phi(N)}{\#(\ker \chi_1 \cap \ker \chi_2)} [\Gamma(1) : \Gamma_0(N)] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |k_1 - k_2| > 1, \\ \max\{k_1, k_2\} \cdot \log^2(2N) \cdot \dfrac{\phi(N)}{\#(\ker \chi_1 \cap \ker \chi_2)} [\Gamma(1) : \Gamma_0(N)] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |k_1 - k_2| = 1, \end{cases}$$

such that $a(n) \neq b(n)$, *where the constant implied in* $\ll$ *is absolute.*

Proof. Assume that there exist f and g satisfying the given conditions (for appropriate $k_1, k_2, N, \chi_1, \chi_2$) and that

$$a(n) = b(n) \quad (\forall n \leq C) \tag{2}$$

where C is a positive (absolute) multiple of

$$\max\{k_1, k_2\} \cdot \log^{11} |k_1 - k_2| \cdot M^{11} \frac{\phi(N)}{\#(\ker \chi_1 \cap \ker \chi_2)} [\Gamma(1) : \Gamma_0(N)]$$

or of

$$\max\{k_1, k_2\} \cdot \log^2(2N) \cdot \frac{\phi(N)}{\#(\ker \chi_1 \cap \ker \chi_2)} [\Gamma(1) : \Gamma_0(N)]$$

according as $|k_1 - k_2|$ is > 1 or 1, and that this multiple can be chosen arbitrarily large.

Observe that f and g can be viewed as modular forms of weights k_1 resp. k_2 on the subgroup $\Gamma_0(N, \chi_1, \chi_2)$ of $\Gamma_0(N)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $a \in \ker \chi_1 \cap \ker \chi_2$. The index of this subgroup in $\Gamma_0(N)$ is $\phi(N)/\#(\ker \chi_1 \cap \ker \chi_2)$.

Since $k_2 \geq 2$, the space of cusp forms of weight k_2 on $\Gamma_0(N, \chi_1, \chi_2)$ has a basis consisting of functions with rational Fourier expansions [8, Thm. 3.5.2]. By the theory of Eisenstein series, the space of modular forms of weight k_2 on $\Gamma_0(N, \chi_1, \chi_2)$ has therefore a basis of functions $\{g_1, \ldots, g_r\}$ with Fourier coefficients in $\mathbb{Q}(\zeta_N)$ where ζ_N is a primitive Nth root of unity. If $b_m(n)$ $(n \geq 0)$ is the nth Fourier coefficients of g_m $(1 \leq m \leq r)$, then by the valence formula (cf. Introduction), since

$$C \geq \frac{k_2}{12} \frac{\phi(N)}{\#(\ker \chi_1 \cap \ker \chi_2)} [\Gamma(1) : \Gamma_0(N)] \tag{$*$}$$

by our assumption, the $(M+1, r)$-matrix $(b_m(n))_{0 \leq n \leq M,\, 1 \leq n \leq r}$ has rank r. Writing g in terms of our basis and taking into account (2) and the fact that $a(n) \in K$ for all n, we conclude that $b(n) \in K\mathbb{Q}(\zeta_N)$ for all n.

Again by the valence formula, since $f \neq 0$ and by $(*)$ with k_2 replaced by k_1, at least one of the coefficients $a(n) = b(n)$ for $n \leq C$ must be non-zero. Dividing out by that and then taking the trace (sum of Galois conjugates) of f and g from $K\mathbb{Q}(\zeta_N)$ down to $\mathbb{Q}$, we see that without loss of generality we can assume that $a(n), b(n) \in \mathbb{Q}$ for all n (note that the traces are not zero).

Since the weights are at least 2, f and g have bounded denominators, hence multiplying with appropriate non-zero integers we can assume that f and g have integral coefficients.

Let p be a prime. If $f \equiv 0 \pmod{p}$, then by (2) and Sturm's result also $g \equiv 0 \pmod{p}$. Dividing out by p and continuing in this way, since $f \neq 0$ we can assume that $f \not\equiv 0 \pmod{p}$.

We will now dispose of the prime p appropriately and then apply Theorem 1.

Put

$$a := |k_1 - k_2|.$$

Let us first suppose that $a > 1$. We then claim that there exists a prime $\ell \leq c_1 \log a$ with $(\ell, a) = 1$, where $c_1 > 1$ is an absolute constant (compare the reasoning in [6, proof of Thm. 4]). Indeed, for $x \geq 2$ in the usual notation let

$$\vartheta(x) = \sum_{\ell \leq x} \log \ell$$

with the sum extending over all primes $\ell \leq x$. Then by the classical result of Chebyshev there exists $c_2 > 0$ such that

$$\vartheta(x) > c_2 x \qquad (\forall x \geq 2).$$

Hence we can find $c_1 \geq 2/\log a$ such that

$$\vartheta(c_1 x) > x \qquad (\forall x \geq 2).$$

Therefore

$$\prod_{\ell \leq c_1 \log a} \ell = \exp(\vartheta(c_1 \log a)) > a,$$

which implies our claim.

By [3], there exists an absolute constant $c_3 > 0$ and a prime p satisfying

$$p \equiv 1 \pmod{\ell M}, \qquad p < c_3 \, (\ell M)^{11/2}.$$

Since $\ell \mid (p-1)$ and $(\ell, a) = 1$, it follows that $p - 1$ does not divide a. Also $(p, N) = 1$ and $p > 2$. Finally

$$p^2 - 1 < c_3^2 \, (\ell M)^{11} \leq c_3^2 c_1^{11} \cdot M^{11} \log^{11} a.$$

Next if $a = 1$, then from what we saw above there exists a prime p with $p \leq c_1 \log(2N)$ and $(p, 2N) = 1$. Hence

$$p^2 - 1 < c_1^2 \log^2(2N).$$

We now apply Theorem 1 with $K = \mathbb{Q}$ and $\wp = p$. Then (2) implies that $f \equiv g \pmod{p}$. However, $f \not\equiv 0 \pmod{p}$, hence by [4, 7] since $(p, N) = 1$

we infer that $k_1 \equiv k_2 \pmod{(p-1)}$, which is a contradiction. This proves Theorem 2.

REMARKS. (i) Note that

$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + 1/p) \ll N \log\log N$$

where the latter bound follows by elementary reasoning.

(ii) Note that in special situations the arguments in the proof of Theorem 2 can be much shortened and sharper results may be derived. For example, assuming that $k_1 \equiv k_2 + 2 \pmod 4$, $(N,5) = 1$ and $\chi_1 = \chi_2 = 1$, we deduce taking $p = 5$ that $a(n) \neq b(n)$ for some n satisfying the sharper bound $n \le 2\max\{k_1, k_2\} \cdot [\Gamma_1 : \Gamma_0(N)]$.

(iii) It seems very desirable to replace M^{11} in the bound of Theorem 2 for $|k_1 - k_2| > 1$ by a smaller constant depending on N. Note that the Generalized Riemann Hypothesis (GRH) implies the existence of a prime p with

$$p \equiv 1 \pmod{\ell M}, \qquad p \ll (\ell M)^2 \log^2(\ell M)$$

[3, Sect. 1]. Hence in Theorem 2 for $|k_1 - k_2| > 1$ under GRH one can replace $\log^{11}|k_1 - k_2| \cdot M^{11}$ by $\log^4|k_1 - k_2| \cdot \log^4(\log(|k_1 - k_2|M))M^4$.

If Q is a positive-definite integral quadratic form in an even number of variables $2k$ and $n \in \mathbb{N}_0$, we denote by $r_Q(n)$ the number of representations of n by Q. Recall that the theta series

$$\theta_Q(z) = \sum_{n \ge 0} r_Q(n) q^n \qquad (z \in \mathcal{H})$$

is a modular form of weight k on $\Gamma_0(N)$ with a real quadratic character modulo N where N is the level of Q. In particular, $\ker\chi \supset (\mathbb{Z}/N\mathbb{Z})^{*2}$. Elementary considerations show that

$$[(\mathbb{Z}/N\mathbb{Z})^* : (\mathbb{Z}/N\mathbb{Z})^{*2}] = 2^{t+\max\{e-3,0\}}$$

where t is the number of odd prime divisors of N and 2^e is the exact 2-power dividing N. Hence we obtain from Theorem 2 (keeping in mind Remarks (i) and (iii)), for example, the following

COROLLARY. *Let Q_1 and Q_2 be two positive-definite integral quadratic forms of level N in an even number of variables $2k_1$ and $2k_2$ respectively, with $k_1 > k_2 + 1$ and $k_2 \ge 2$. Then with the above notation there exists $n \in \mathbb{N}$ with*

$$n \ll \max\{k_1, k_2\} \cdot \log^{11}|k_1 - k_2| \cdot 2^{t+\max\{e-3,0\}} M^{11} N \log\log N$$

such that $r_{Q_1}(n) \neq r_{Q_2}(n)$, where the constant implied in $\ll$ is absolute.

Under GRH the above bound can be improved to

$$n \ll \max\{k_1, k_2\} \cdot \log^4 |k_1 - k_2| \times \log^4(\log(|k_1 - k_2|M)) \cdot 2^{t+\max\{e-3,0\}} M^4 N \log\log N.$$

REMARKS. (i) It is clear that by imposing more special conditions on Q_1 and Q_2, the above bounds can be much improved; compare Remark (ii) after Theorem 2.

(ii) We do not know if a result of the above type can be proved using only the arithmetic theory of quadratic forms and no modular forms theory.

4. The case of eigenforms of $T(p)$. Here we want to give a slight improvement of the assertions of Theorem 1 e.g. in the case $\Gamma = \Gamma_0(N)$ if f and g are eigenforms of integral weights of the usual Hecke operator $T(p)$.

THEOREM 3. *Let f resp. g be modular forms on $\Gamma_0(N)$ of integral weights k_1 resp. k_2 with $k_1, k_2 \geq 2$, $k_1 \neq k_2$, and suppose that f and g have Fourier coefficients $a(n)$ resp. $b(n)$ $(n \geq 0)$ in $\mathcal{O}_K$. Let p be the rational prime with $\wp \mid p$ and suppose that $p \geq 5$. Furthermore suppose that $f|T(p) \equiv \alpha_p f \pmod{\wp}$, $g|T(p) \equiv \beta_p g \pmod{\wp}$ with $\alpha_p, \beta_p \in (\mathcal{O}_K/\wp)^*$ where the slash denotes the usual action of $T(p)$ (in weights k_1 and k_2 respectively). Then if $f \not\equiv g \pmod{\wp}$, there exists*

$$n \leq \frac{k_1 + k_2 + 2 + p(p-1)}{12} [\Gamma(1) : \Gamma_0(N)]$$

with $a(n) \not\equiv b(n) \pmod{\wp}$.

REMARK. Note that in Theorem 3 we do not require that f or g are eigenforms of all Hecke operators nor that $(\wp, N) = 1$.

Proof of Theorem 3. First note that because of our assumption $k_1, k_2 \geq 2$, the operator $T(p)$ acts on the Fourier coefficients of modular forms reduced modulo $\wp$ as the operator usually denoted by $U(p)$, i.e. replaces the nth Fourier coefficient modulo $\wp$ by the pnth coefficient modulo $\wp$.

We let

$$E_{p-1} = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{n \geq 1} \sigma_{p-2}(n) q^n,$$

the normalized Eisenstein series of weight $p-1$ on $\Gamma(1)$, where B_{p-1} is the $(p-1)$th Bernoulli number and $\sigma_{p-2}(n) = \sum_{d|n} d^{p-2}$. Then E_{p-1} has p-integral Fourier coefficients and $E_{p-1} \equiv 1 \pmod{p}$.

Determine non-negative integers a and b such that

$$k_1 + a(p-1) \equiv k_2 + b(p-1) \not\equiv 0 \pmod{p}.$$

Clearly we can do this in such a way that $a + b \leq p$.

Put

$$\kappa_1 := k_1 + a(p-1), \quad \kappa_2 := k_2 + b(p-1).$$

Then

$$F := fE_{p-1}^a, \quad G := gE_{p-1}^b$$

are modular forms of weights κ_1 resp. κ_2 on $\Gamma_0(N)$ with $\wp$-integral Fourier coefficients and $F \equiv f \pmod{\wp}$, $G \equiv g \pmod{\wp}$.

Write

$$\theta = q\,\frac{d}{dq} = \frac{1}{2\pi i}\,\frac{d}{dz}.$$

Denote by

$$H := \kappa_2 \theta F \cdot G - \kappa_1 F \cdot \theta G$$

the first Rankin–Cohen bracket of F and G. Then as is well known and easy to see, H is a modular form (in fact a cusp form) of weight $\kappa_1 + \kappa_2 + 2$ on $\Gamma_0(N)$. Observe that H has $\wp$-integral Fourier coefficients.

Now assume that

(3) $$a(n) \equiv b(n) \pmod{\wp} \qquad (\forall n \le C)$$

where

$$C := \frac{k_1 + k_2 + 2 + p(p-1)}{12}\,[\Gamma(1) : \Gamma_0(N)].$$

Since $\kappa_1 \equiv \kappa_2 \pmod{\wp}$, we see that H has order of vanishing modulo $\wp$ greater than C. Since by construction

$$C \ge \frac{\kappa_1 + \kappa_2 + 2}{12}\,[\Gamma_1 : \Gamma_0(N)],$$

it follows from Sturm's result that

(4) $$H \equiv 0 \pmod{\wp}.$$

If $g \equiv 0 \pmod{\wp}$, then also $f \equiv 0 \pmod{\wp}$, by (3) and again by Sturm's result since

$$C \ge \frac{k_1}{12}\,[\Gamma_1 : \Gamma_0(N)],$$

hence $f \equiv g \equiv 0 \pmod{\wp}$, and we have a contradiction.

Therefore $g \not\equiv 0 \pmod{\wp}$, and we find from (4) that

$$\theta\left(\frac{f}{g}\right) \equiv 0 \pmod{\wp}.$$

By (3) again and the fact that

$$C \ge \frac{k_2}{12}\,[\Gamma_1 : \Gamma_0(N)],$$

it follows that

(5) $$f \equiv g \cdot A|V(p) \pmod{\wp}$$

where A is a power series in q with $\wp$-integral coefficients and $V(p)$ is the operator on $(\mathcal{O}_K/\wp)[[q]]$ given by

$$\sum_{n\geq 0} c(n)q^n|V(p) = \sum_{n\geq 0} c(n)q^{pn}.$$

Applying $U(p)$ on both sides of (5) we conclude that

$$f|U(p) \equiv g|U(p)\cdot A \pmod{\wp},$$

hence our hypothesis implies that

$$A \equiv \frac{\alpha_p}{\beta_p}\,\frac{f}{g} \pmod{\wp}. \tag{6}$$

Therefore, combining (5) and (6) we obtain

$$\frac{f}{g} \equiv \frac{\alpha_p}{\beta_p}\cdot\left(\frac{f}{g}\right)\bigg|V(p) \pmod{\wp}.$$

Hence f/g is constant modulo $\wp$ and so $f \equiv g \pmod{\wp}$, because there must exist $n \leq C$ with $a(n) \equiv b(n) \not\equiv 0 \pmod{\wp}$ since otherwise $g \equiv 0 \pmod{\wp}$. This is a contradiction and proves Theorem 3.

5. An example in the case of Hecke eigenforms. The purpose of this section is to prove

THEOREM 4. *Let f resp. g be normalized Hecke eigenforms on $\Gamma_0(N)$ of even integral weights k_1 resp. k_2 with $k_1, k_2 \geq 2$, $k_1 \neq k_2$, and with Fourier coefficients $a(n)$ resp. $b(n)$ $(n \geq 1)$. Suppose that $(N, 30) = 1$. Let $K_{f,g}$ be the number field generated over $\mathbb{Q}$ by the $a(n)$ and $b(n)$ for all $n \geq 1$. Then there are infinitely many prime ideals $\wp$ of $K_{f,g}$ such that $f \not\equiv g \pmod{\wp}$ implies the existence of*

$$n \leq \max\left\{900, \frac{\max\{k_1, k_2\}}{12}\,[\Gamma_1 : \Gamma_0(N)]\right\} \tag{$**$}$$

with $a(n) \not\equiv b(n) \pmod{\wp}$.

Proof. As stated in the Introduction, the proof is a modification of an argument due to M. Ram Murty in characteristic zero.

If ℓ is a prime with $(N, \ell) = 1$, then we have

$$a(\ell^2) = a(\ell)^2 - \ell^{k_1-1}, \qquad b(\ell^2) = b(\ell)^2 - \ell^{k_2-1}. \tag{7}$$

By [2] there exist infinitely many prime numbers p such that 2, 3 or 5 is a primitive root modulo p. Take a prime ideal $\wp$ of $\mathcal{O}_{K_{f,g}}$ lying above such a p and assume that $a(n) \equiv b(n) \pmod{\wp}$ for all n satisfying $(**)$. Then for $\ell \in \{2, 3, 5\}$ it follows from (7) and the multiplicativity of the coefficients $a(n)$ and $b(n)$ that

$$\ell^{k_1-1} \equiv \ell^{k_2-1} \pmod{\wp},$$

hence

$$\ell^{k_1-k_2} \equiv 1 \pmod{p}.$$

Since ℓ is a primitive root modulo p, we conclude that $k_1 \equiv k_2 \pmod{p-1}$.

For $p \geq 5$, let E_{p-1} be the normalized Eisenstein series of weight $p-1$ on Γ_1 as in Section 4. Assuming $k_1 > k_2$, we then infer from (7) that

$$f \equiv g E_{p-1}^{(k_1-k_2)/(p-1)} \pmod{\wp}$$

and from Sturm's result obtain the contradiction $f \equiv g \pmod{\wp}$.

References

[1] D. Goldfeld and J. Hoffstein, *On the number of Fourier coefficients that determine a modular form*, in: A Tribute to Emil Grosswald: Number Theory and Related Analysis, M. Knopp and M. Sheingorn (eds.), Contemp. Math. 143, Amer. Math. Soc., 1993, 385–393.

[2] D. R. Heath-Brown, *Artin's conjecture for primitive roots*, Quart. J. Math. Oxford Ser. (2) 37 (1986), 27–38.

[3] —, *Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression*, Proc. London Math. Soc. (3) 64 (1992), 265–338.

[4] N. Katz, *p-adic properties of modular schemes and modular forms*, in: Modular Functions of One Variable III, W. Kuyk and J.-P. Serre (eds.), Lecture Notes in Math. 350, Springer, 1973, 69–190.

[5] S. Lang, *Introduction to Modular Forms*, Grundlehren Math. Wiss. 222, Springer, 1976.

[6] M. Ram Murty, *Congruences between modular forms*, in: Analytic Number Theory, Y. Motohashi (ed.), London Math. Soc. Lecture Note Ser. 247, Cambridge Univ. Press, 1997.

[7] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, in: Modular Functions of One Variable III, W. Kuyk and J.-P. Serre (eds.), Lecture Notes in Math. 350, Springer, 1973, 191–268.

[8] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Forms*, Publ. Math. Soc. Japan 11, Iwanami Shoten and Princeton Univ. Press, 1971.

[9] —, *On modular forms of half-integral weight*, Ann. of Math. (2) 97 (1973), 440–481.

[10] J. Sturm, *On the congruence of modular forms*, in: Number Theory (New York, 1984–1985), D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn and M. B. Nathanson (eds.), Lecture Notes in Math. 1240, Springer, 1987.

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