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On the p^λ problem

by

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1. Introduction. The well known "H conjecture" (see [HaR]) states that n^2+1 is prime infinitely often. This is equivalent to the existence of infinitely many primes p satisfying $\{p^{1/2}\} < p^{-1/2}$. The current methods of analytic number theory are far from being sufficient to prove these conjectures.

However, Kubilius [Kub] and Ankeny [Ank] proved already about fifty years ago, assuming the truth of the Riemann Hypothesis for Hecke L-functions with Größencharacters over $\mathbb{Q}(i)$, that $p = n^2 + m^2$ is infinitely often prime with $m \ll \log p$. This implies that $\{p^{1/2}\} < p^{-1/2+\varepsilon}$ for infinitely many primes p. Of course, this is only a conditional result.

As demonstrated in [Ba1, 2], [Ha1, 2] and [BaH], it is also possible to obtain some *unconditional* non-trivial results on small fractional parts of $p^{1/2}$, or more generally, on small fractional parts of p^λ, λ being a fixed real number lying in the interval $(0,1)$. In particular, for $\lambda = 1/2$ Balog and Harman obtained $\{p^{1/2}\} < p^{-1/4+\varepsilon}$ for infinitely many primes p. This result has recently been beaten. Combining Kubilius's ideas with efficient sieve methods, Harman and Lewis [HaL] unconditionally showed that the exponent $1/4$ may be replaced by 0.262. However, their method works only for $\lambda = 1/2$, whereas the methods in [Ba1, 2], [Ha1, 2] and [BaH] are applicable to all λ in certain subintervals of $(0,1)$.

In the present paper, we focus our interest mainly on small exponents λ. Our starting point is the following result of Harman (Theorem 4 in [Ha2]).

THEOREM 1. *Suppose that $\varepsilon > 0$, $B > 0$ and $\lambda \in (0, 1/5]$ are given. Let $N \geq 3$. For every positive integer k define*

$$e_1(\lambda,k) := \frac{5k-(2k+4)\lambda}{12k+4}, \qquad e_2(\lambda,k) := \frac{5k}{12k-6} - \lambda$$

and

$$e(\lambda,k) := \min\{e_1(\lambda,k), e_2(\lambda,k)\}.$$

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Furthermore, define

$$E(\lambda) := \max_{k\in\mathbb{N}} e(\lambda, k).$$

Then for

(1) $$N^{-E(\lambda)+\varepsilon\lambda} \leq \delta \leq 1$$

we have

(2) $$\sum_{\substack{N<n\leq 2N\\ \{n^\lambda\}<\delta}} \Lambda(n) = \delta N\left(1 + O\left(\frac{1}{(\log N)^B}\right)\right)$$

as $N \to \infty$.

Here, as in the following, $\Lambda(n)$ denotes the von Mangoldt function.

As to be seen from the remark attached to Theorem 4 in [Ha2], this result essentially keeps its validity if one introduces an additional summation condition "$[n^\lambda] \in \mathbb{A}$" on the left side of (2), where $\mathbb{A}$ is any given subset of the set of positive integers (only the main term on the right side of (2) correspondingly changes). Harman's motivation to introduce this additional condition appears to be the special case when $\mathbb{A}$ is the set of primes.

Furthermore, in the same remark attached to Theorem 4 in [Ha2] it is noted that the condition $\lambda \leq 1/5$ may be replaced by $\lambda \leq 1/2$ without any change in the result.

To prove his result, Harman used density estimates for the set of non-trivial zeta zeros and an estimate for the $2k$th power moment of Dirichlet polynomials

$$\sum_{m\sim M} a_m m^{it}.$$

Hitherto, we have only considered small fractional parts of p^λ. A natural generalisation of this question is to consider small fractional parts of $\{p^\lambda - \theta\}$, where θ is a given real number. Unlike Theorem 4 in [Ha2], many results in [Ba1, 2] and [Ha1, 2] are formulated for $\{p^\lambda - \theta\}$ with a general real θ. To extend Theorem 4 in [Ha2] in order to cover this general case, one needs estimates for power moments of *shifted* Dirichlet polynomials

$$\sum_{m\sim M} a_m (m+\theta)^{it}.$$

In case θ is rational these shifted Dirichlet polynomials can be easily rewritten as ordinary ones: If $\theta = b/q$, where b, q are non-negative integers (without loss of generality, θ can supposed to be non-negative), then

$$\sum_{m\sim M} a_m (m+\theta)^{it} = q^{-it} \sum_{m\sim M} a_m (qm+b)^{it}.$$

Therefore, Harman's method works for all rational θ, not only for $\theta = 0$.

However, for irrational θ there seem to be no reasonable known estimates of the $2k$th moment of $\sum_{m\sim M} a_m(m+\theta)^{it}$ if $k > 2$. Harman obtained such estimates only for $k \le 2$. But in this case his power moment estimates for irrational θ are essentially the same as the known ones for rational θ. Thus, Theorem 1 keeps its validity also for irrational θ if we replace the function $E(\lambda)$ by

$$E^*(\lambda) := \max\{e(\lambda,1), e(\lambda,2)\}. \tag{3}$$

Summarising the above observations, Theorem 1 can be extended to the following

THEOREM 2. *Suppose that $\varepsilon > 0$, $B > 0$, $\lambda \in (0, 1/2]$ and a real θ are given. Let $N \ge 3$. Let $\mathbb{A}$ be an arbitrarily given subset of the set of positive integers. Define $E(\lambda)$ as in Theorem* 1 *and $E^*(\lambda)$ as in* (3)*. Suppose that condition* (1) *is satisfied if θ is rational and that*

$$N^{-E^*(\lambda)+\varepsilon\lambda} \le \delta \le 1 \tag{4}$$

is satisfied if θ is irrational. Then

$$\sum_{\substack{N<n\le 2N\\ \{n^\lambda-\theta\}<\delta\\ [n^\lambda]\in\mathbb{A}}} \Lambda(n) = \frac{\delta}{\lambda} \sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} n^{1/\lambda-1} + O\left(\frac{\delta N}{(\log N)^B}\right) \tag{5}$$

as $N \to \infty$.

It is easily verified that $E^*(\lambda) = 5/14 - 2\lambda/7$ for $\lambda \le 5/18$. Using zero density estimates and trivially estimating the shifted Dirichlet polynomials appearing in the method, or directly applying Huxley's prime number theorem, one obtains $5/12 - \lambda$ in place of $E^*(\lambda)$, which yields a better result than $E^*(\lambda)$ if $\lambda < 1/12$. This demonstrates that Harman's method is ineffective if θ is irrational and λ is close to 0.

The first aim of the present paper is to prove a substantially better result than the one obtained from Huxley's prime number theorem for irrational θ and λ close to 0. Our second aim is to improve Theorem 2 for rational θ. We shall prove the following

THEOREM 3. *Suppose that $\varepsilon > 0$, $B > 0$, $\lambda \in (0, 1/2]$ and a real θ are given. If θ is irrational, then suppose that $\lambda < 5/19$. Let $N \ge 3$. Let $\mathbb{A}$ be an arbitrarily given subset of the set of positive integers. For every positive integer k define*

$$f_1(\lambda,k) := \frac{5}{12} - \frac{k+6}{6(k+1)}\lambda, \qquad f_2(\lambda,k) := \frac{5}{11} - \frac{5k+1}{11}\lambda$$

and

$$f(\lambda,k) := \min\{f_1(\lambda,k), f_2(\lambda,k)\}.$$

Furthermore, define

$$F_\theta(\lambda) := \begin{cases} F(\lambda) & \textit{if } \theta \textit{ is rational}, \\ f(\lambda,1) & \textit{otherwise}, \end{cases}$$

where

$$F(\lambda) := \max_{k\in\mathbb{N}} f(\lambda,k).$$

Suppose that

$$N^{-F_\theta(\lambda)+\varepsilon\lambda} \le \delta \le 1. \tag{6}$$

Then we have the asymptotic estimate (5) *as* $N \to \infty$.

We note that

$$f(\lambda,1) = f_1(\lambda,1) = \frac{5}{12} - \frac{7\lambda}{12} \tag{7}$$

for all $\lambda > 0$.

In the next section we shall discuss Theorem 3 in detail and compare this result with Theorem 2. From the third section onwards we shall prove Theorem 3.

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2. Discussion of Theorem 3. In case θ is irrational it is supposed in Theorem 3 that $\lambda < 5/19$. We have $E^*(\lambda) = 5/14 - 2\lambda/7$ if $\lambda < 5/19$. Thus, we get $5/12 - 7\lambda/12 = f(\lambda,1) > E^*(\lambda)$ if $\lambda < 1/5$. Therefore, Theorem 3 yields a sharper result than Theorem 2 if θ is irrational and $\lambda < 1/5$. Moreover, since $f(\lambda,1) > 5/12 - \lambda$ for all $\lambda > 0$, Theorem 3 is always sharper than the result directly obtained from Huxley's prime number theorem (see the preceding section).

We now turn to the case when θ is rational. It is easily verified that $F(\lambda) = 5/12 - \lambda/6 + O(\lambda^2)$ as $\lambda \to 0$, whereas $E(\lambda) = 5/12 - \lambda/2 + O(\lambda^2)$ as $\lambda \to 0$. Thus, $F(\lambda) > E(\lambda)$ for every sufficiently small $\lambda > 0$. Therefore, Theorem 3 is sharper than Theorem 2 if θ is rational and λ is sufficiently small. We now make this observation more precise by analysing and comparing $E(\lambda)$ and $F(\lambda)$.

It is easily seen that there are sequences $(\eta_{1,k})$, $(\eta_{2,k})$ of real numbers with

$$1/2 = \eta_{2,1} > \eta_{1,2} > \eta_{2,2} > \eta_{1,3} > \eta_{2,3} > \ldots, \qquad \lim_{k\to\infty} \eta_{i,k} = 0 \quad (i = 1, 2),$$

such that

$$E(\lambda) = \begin{cases} e_1(\lambda, k-1) & \text{if } \eta_{2,k-1} \ge \lambda \ge \eta_{1,k}, \\ e_2(\lambda, k) & \text{if } \eta_{1,k} \ge \lambda \ge \eta_{2,k}, \end{cases}$$

where the functions $e_i(\lambda, k)$ $(i = 1, 2)$ are defined as in Theorem 1. So $E(\lambda)$ is a continuous piecewise linear function. To determine $\eta_{i,k}$ for $k \ge 2$, we simply have to solve the linear equations

$$e_1(x, k-1) = e_2(x, k), \qquad e_2(x, k) = e_1(x, k).$$

In this manner, we obtain

$$\eta_{1,k} = \left(\frac{5}{12} + \frac{1}{6(k-1)}\right)\frac{1}{k-1/2}, \qquad \eta_{2,k} = \frac{5}{12}\cdot\frac{1}{k-1/2}$$

if $k \ge 2$. Similarly, we obtain

$$F(\lambda) = \begin{cases} f_1(\lambda, k-1) & \text{if } \phi_{2,k-1} \ge \lambda \ge \phi_{1,k}, \\ f_2(\lambda, k) & \text{if } \phi_{1,k} \ge \lambda \ge \phi_{2,k}, \end{cases} \tag{8}$$

where $\phi_{2,1} = 1/2$ and

$$\phi_{1,k} = \frac{1}{2(6k - 1 - 11/k)}, \qquad \phi_{2,k} = \frac{1}{2(6k - 1 - 11/(k+1))}$$

if $k \ge 2$. Using these explicit expressions for $E(\lambda)$ and $F(\lambda)$, it is not difficult to calculate that $F(\lambda) > E(\lambda)$ whenever $\lambda < 5/66$ or $1/3 < \lambda < 1/2$. Consequently, for rational θ Theorem 3 yields a sharper result than Theorem 2 in these λ-ranges.

Some additional remarks. (a) We have $E^*(1/2) = E(1/2) = f(1/2, 1) = F(1/2) = 1/8$ and $E(\lambda) = F(\lambda) = 5/11 - \lambda$ if $5/66 = \eta_{2,6} \le \lambda \le 9/110 = \eta_{1,6}$.

(b) In the homogeneous case $\mathbb{A} = \mathbb{N}$, Harman and Balog were able to prove better results than Theorems 2 and 3 for $\lambda > 1/5$ (see Theorem 3 of [Ha2], Theorem 2 of [BaH] and the papers [Ba1, 2], [Ha1]).

(c) It is not difficult to prove that on the Riemann Hypothesis the function $F_\theta(\lambda)$ in Theorem 3 can be replaced by $(1-\lambda)/2$ for all λ in the interval $0 < \lambda < 1$ and all real θ.

3. Auxiliary results and outline of the method. We let the conditions of Theorem 3 be kept throughout the remaining part of the paper.

Without loss of generality, we continually suppose that $0 \le \theta < 1$. By ε and B we always mean the constants ε and B from Theorem 3.

We define $h = h_\theta(\lambda)$ to be the smallest positive integer such that

$$F_\theta(\lambda) = f(\lambda, h). \tag{9}$$

In particular, we always have $h = 1$ when θ is irrational. When θ is rational, we have $h \ge 2$ if and only if $\lambda < 1/11 = \phi_{1,2}$. In this case, we obtain

$$h\lambda < 2/11 \tag{10}$$

from (8) by a short calculation.

Using (9) and the definition of $F_\theta(\lambda)$ in Theorem 3, we obtain

$$\frac{5}{12} \le \lambda + f(\lambda, h) \le \frac{5}{11} \tag{11}$$

if $h \ge 2$ and

$$\frac{\lambda}{2} + f(\lambda, h) \le \frac{5}{11} \tag{12}$$

in any case. Moreover, we have

$$\lambda + f(\lambda, 1) = \lambda + f_1(\lambda, 1) = \frac{5(1+\lambda)}{12} \le \frac{5}{8} \tag{13}$$

for every $\lambda \le 1/2$ and

$$\lambda \le \frac{\lambda + f(\lambda, 1) - \varepsilon\lambda}{2} \tag{14}$$

if $\lambda < 5/19$ and $\varepsilon \le (5 - 19\lambda)/(12\lambda)$. We shall use the inequalities (10)–(14) in the course of this paper.

Next, we introduce some more notations. We write

$$D_y(u, s) := \sum_{\substack{N^\lambda < n \le u \\ n \in \mathbb{A}}} (n + y)^{s/\lambda - 1}$$

for any real $u > N^\lambda$, $y \ge 0$ and complex s. Moreover, we put

$$D_y(s) := D_y((2N)^\lambda, s).$$

As usual, by the symbol ϱ we denote the non-trivial zeta zeros, and we write γ for the imaginary part and β for the real part of ϱ. We define

$$S_\theta(u, \sigma) := \sum_{\substack{\varrho\,:\, 0 < \gamma \le T \\ \sigma \le \beta \le \sigma + 1/(\log N)}} |D_\theta(u, i\gamma)| \tag{15}$$

for any σ with $0 \le \sigma \le 1 - 1/(\log N)$, where the parameter T will be fixed at the beginning of the next section.

The first step of our method is to use the explicit formula of Landau in order to reduce the sum on the left side of (5) to sums of the form $S_\theta(u, \sigma)$.

PROPOSITION 1 (explicit formula). *For $x > 2$, $T_0 > 1$ we have*

$$\sum_{n<x} \Lambda(n) = x - \sum_{\varrho:|\gamma|\le T_0} \frac{x^\varrho}{\varrho} + O\biggl(\frac{x}{T_0}(\log xT_0)^2 + \log x\biggr).$$

We then estimate the sum $S_\theta(u,\sigma)$ in several σ-regions by different methods.

To control the range $0 \le \sigma \le 6/11$, we use the following mean value estimate for shifted Dirichlet polynomials which can be established in the same manner as the corresponding well known mean value estimate for ordinary (unshifted) Dirichlet polynomials (see [Ivi] for example).

PROPOSITION 2. *Suppose that $0 \le \theta < 1$, $K \ge 1$ and $T \ge 1$. Let (a_k) be an arbitrary sequence of complex numbers. Suppose that $|a_k| \le A$ for all $k \sim K$. Let (t_r) be an increasing sequence of positive real numbers such that $t_{r+1} - t_r \ge 1$ for every positive integer r. Let R be a positive integer. Suppose that $t_R \le T$. Then*

$$\sum_{r=1}^{R} \Bigl| \sum_{k\sim K} a_k (k+\theta)^{it_r} \Bigr|^2 \ll A^2 (T+K) K \log(2K).$$

Here, as in the following, the notation $k \sim K$ means $K < k \le 2K$.

To tackle $S_\theta(u,\sigma)$ in the range $1 - \Delta < \sigma \le 1$, Δ being defined as in (38), we use the second zero density estimate of the later Proposition 6 as well as Vinogradov's zero-free-region result.

PROPOSITION 3 (Vinogradov; see [Ivi]). *There is an absolute constant $C > 0$ such that*

$$\beta \le 1 - C(\log|\gamma|)^{-2/3}(\log\log|\gamma|)^{-1/3}$$

for every non-trivial zeta zero $\varrho = \beta + i\gamma$.

To calculate $S_\theta(u,\sigma)$ in the range $6/11 < \sigma \le 1 - \Delta$, we employ the following relation which is also the basis of the zero detection method for counting non-trivial zeta zeros (cf. [Ivi] for example).

PROPOSITION 4. *Suppose that $X, Y \ge 1$, $T > 1$, $\log N \ll \log T \ll \log Y \ll \log T \ll \log N$ and $\log X \ll \log N$. Define*

$$M_X(s) := \sum_{n\le X} \mu(n) n^{-s}, \qquad a(k) := \sum_{\substack{d\mid k\\ d\le X}} \mu(d).$$

Then $a(k) = 0$ if $1 < k \le X$,

$$\zeta(s) M_X(s) = \sum_{k=1}^{\infty} a(k) k^{-s}$$

if $\operatorname{Re} s > 1$ and

(16) $$1/2 < U_1(\varrho) \quad \text{or} \quad 1/2 < U_2(\varrho)$$

if N is sufficiently large and $\varrho=\beta+i\gamma$ is a non-trivial zeta zero satisfying $\beta\ge 1/2$ and $(\log N)^2<\gamma\le T$, where

$$U_1(\varrho):=\Big|\int\limits_{-(\log N)^2}^{(\log N)^2}\zeta(1/2+i(\gamma+t))M_X(1/2+i(\gamma+t))\times Y^{1/2-\beta+it}\Gamma(1/2-\beta+it)\,dt\Big|,\tag{17}$$

$$U_2(\varrho):=\Big|\sum_{X<k\le Y(\log N)^2}a(k)k^{-\varrho}e^{-k/Y}\Big|.\tag{18}$$

The sum on the right side of (18) *is supposed to equal* 0 *if* $X\ge Y(\log N)^2$. *We call every zero ϱ satisfying $1/2<U_j(\varrho)$ a zero of type j ($j=1,2$).*

We shall now proceed similarly to the original zero detection method, with the difference that here every non-trivial zeta zero ϱ is weighted by $|D_\theta(u,i\gamma)|$. Our method shall lead us to the problem of estimating mean values of products of shifted and ordinary (unshifted) Dirichlet polynomials. In the following, we state such mean value estimates.

THEOREM 4. *Suppose that $\alpha\ne 0$, $0\le\theta<1$, $T>0$, $K\ge 1$, $L\ge 1$. If $\theta\ne 0$, then additionally suppose that $L\le T^{1/2}$. Let (a_k) and (b_l) be arbitrary sequences of complex numbers. Suppose that $|a_k|\le A$ for all $k\sim K$ and $|b_l|\le B$ for all $l\sim L$. Then*

$$\int\limits_0^T\Big|\sum_{k\sim K}a_kk^{it}\Big|^2\Big|\sum_{l\sim L}b_l(l+\theta)^{i\alpha t}\Big|^2dt \ll A^2B^2(T+KL)KL\log^3(2KLT),\tag{19}$$

the implied $\ll$-constant depending only on α. If $\theta=0$, then $\log^3(2KLT)$ *on the right side of* (19) *may be replaced by* $\log^2(2KLT)$.

We shall rather need a discrete form of Theorem 4, namely

THEOREM 4*. *Let the conditions of Theorem* 4 *be kept. Moreover, let (t_r) be an increasing sequence of positive real numbers such that $t_{r+1}-t_r\ge 1$ for every positive integer r. Further, let R be a positive integer. Suppose that $t_R\le T$. Then*

$$\sum_{r=1}^R\Big|\sum_{k\sim K}a_kk^{it_r}\Big|^2\Big|\sum_{l\sim L}b_l(l+\theta)^{i\alpha t_r}\Big|^2 \ll A^2B^2(T+KL)KL\log^4(2KLT),\tag{20}$$

the implied $\ll$-constant depending only on α. If $\theta=0$, then $\log^4(2KLT)$ *on the right side of* (20) *may be replaced by* $\log^3(2KLT)$.

We postpone the proofs of Theorems 4, 4* to the last section, in which we shall also derive the following more general mean value estimate from Theorem 4*.

THEOREM 5. *Suppose that $\alpha \neq 0$, $0 \leq \theta < 1$, $T > 0$, $1 \leq K_1 < K_2$, $1 \leq L_1 < L_2$, $h \in \mathbb{N}$, $\varepsilon_0 > 0$ and $1/2 \leq \sigma \leq 1$. If θ is irrational, then additionally suppose that $L_2 \leq T^{1/2}$ and $h = 1$. Let (a_k) and (b_l) be arbitrary sequences of complex numbers. Suppose that $|a_k| \leq A$ and $|b_l| \leq B$ for all positive integers k and l. Let (t_r) be an increasing sequence of positive real numbers such that $t_{r+1} - t_r \geq 1$ for every positive integer r. Let R be a positive integer. Suppose that $t_R \leq T$. Then*

$$(21)\qquad \sum_{r=1}^{R} \Big| \sum_{K_1<k\leq K_2} a_k k^{-(\sigma+it_r)} \Big|^2 \Big| \sum_{L_1<l\leq L_2} b_l (l+\theta)^{i\alpha t_r - 1} \Big|^{2h} \ll A^2 B^{2h} (T K_1^{1-2\sigma} L_1^{-h} + K_2^{2(1-\sigma)}) (K_2 L_2 T)^{\varepsilon_0},$$

the implied $\ll$-constant depending only on α, θ, h and ε_0.

Theorem 5 is made for a direct application in the present paper.

In addition to these mean value estimates, we use the following well known fourth power moment estimate for the Riemann zeta function on the critical line.

PROPOSITION 5. *We have*

$$\int_0^T |\zeta(1/2 + it)|^4 \, dt \ll T \log^4 T.$$

Finally, we shall employ zero density estimates of Ingham and Huxley which themselves are consequences of the zero detection method.

PROPOSITION 6 (see [Ivi]). *For $T > 2$ we have*

$$\mathbf{N}(\sigma, T) \ll \begin{cases} T^{3(1-\sigma)/(2-\sigma)} (\log T)^5 & \text{if } 1/2 \leq \sigma \leq 3/4, \\ T^{3(1-\sigma)/(3\sigma-1)} (\log T)^{44} & \text{if } 3/4 \leq \sigma \leq 1, \end{cases}$$

where $\mathbf{N}(\sigma, T)$ denotes the number of zeta zeros $\varrho = \beta + i\gamma$ with $\beta \geq \sigma$ and $0 < \gamma \leq T$.

4. Reduction to sums over non-trivial zeta zeros. We define

$$T_0 := \frac{N^\lambda (\log N)^{B+2}}{\delta}$$

and

$$(22)\qquad T := N^{\lambda + f(\lambda,h) - \varepsilon\lambda} (\log N)^{B+2}.$$

We note that $T_0 \le T$ by (9) and condition (6) of Theorem 3. Furthermore, we state the following five bounds, which shall be used in the course of this paper. If $h \ge 2$, then we have

$$T \ll N^{5/11-\varepsilon\lambda/2} \tag{23}$$

as well as

$$N^{5/12-\varepsilon\lambda} \ll T \tag{24}$$

by (11). We always have

$$TN^{-\lambda/2} \ll N^{5/11-\varepsilon\lambda/2} \tag{25}$$

by (12). If $h = 1$, then we have

$$\begin{aligned} T &= N^{5(1+\lambda)/12-\varepsilon\lambda}(\log N)^{B+2} \\ &\ll N^{5/8-\varepsilon\lambda/2} \end{aligned} \tag{26}$$

by (13). If $h = 1$, $\lambda < 5/19$, $\varepsilon \le (5-19\lambda)/(12\lambda)$ and $N \ge 5$, then we have

$$(2N)^\lambda \le T^{1/2} \tag{27}$$

by (14).

By means of Proposition 1, we now decompose the sum on the left side of (5) into a main term and an error term involving non-trivial zeta zeros.

Lemma 1. *We have*

$$\begin{aligned} &\sum_{\substack{N<n\le 2N \\ \{n^\lambda-\theta\}<\delta \\ [n^\lambda]\in\mathbb{A}}} \Lambda(n) - \frac{\delta}{\lambda} D_0(1) \\ &\qquad \ll \delta(\log N)^{B+3} \sup_{0\le\sigma\le 1-1/(\log N)} N^\sigma \sup_{N^\lambda<u\le(2N)^\lambda} S_\theta(u,\sigma) \\ &\qquad\quad + \frac{\delta N}{(\log N)^B} + N^\lambda \log N, \end{aligned}$$

the implied $\ll$-constant depending only on λ and B.

Proof. Obviously, the sum in question can be written in the form

$$\begin{aligned} &\sum_{\substack{N<n\le 2N \\ \{n^\lambda-\theta\}<\delta \\ [n^\lambda]\in\mathbb{A}}} \Lambda(n) \\ &\qquad = \sum_{\substack{N^\lambda<n\le(2N)^\lambda \\ n\in\mathbb{A}}} \sum_{(n+\theta)^{1/\lambda}\le m<(n+\theta+\delta)^{1/\lambda}} \Lambda(m) + O(N^\lambda \log N). \end{aligned} \tag{28}$$

Combining this estimate and Proposition 1, and taking condition (6) into account, we get

$$\sum_{\substack{N<n\le 2N\\ \{n^\lambda-\theta\}<\delta\\ [n^\lambda]\in\mathbb{A}}} \Lambda(n) = \sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} ((n+\theta+\delta)^{1/\lambda}-(n+\theta)^{1/\lambda}) - \sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} \sum_{\varrho:|\gamma|\le T_0} \frac{(n+\theta+\delta)^{\varrho/\lambda}-(n+\theta)^{\varrho/\lambda}}{\varrho} + O\biggl(\frac{\delta N}{(\log N)^B}+N^\lambda\log N\biggr). \tag{29}$$

Using Taylor's formula, we approximate the first sum on the right side of (29) by

$$\sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} ((n+\theta+\delta)^{1/\lambda}-(n+\theta)^{1/\lambda}) = \frac{\delta}{\lambda}D_0(1)+O(\delta N^{1-\lambda}). \tag{30}$$

The fraction within the double sum on the right side of (29) can be written as an integral, namely

$$\frac{(n+\theta+\delta)^{\varrho/\lambda}-(n+\theta)^{\varrho/\lambda}}{\varrho} = \frac{1}{\lambda}\int_\theta^{\theta+\delta}(n+y)^{\varrho/\lambda-1}\,dy.$$

From that and the symmetry of the set of zeta zeros it follows that the double sum on the right side of (29) can be estimated by

$$\sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} \sum_{\varrho:|\gamma|\le T_0} \frac{(n+\theta+\delta)^{\varrho/\lambda}-(n+\theta)^{\varrho/\lambda}}{\varrho} \ll \delta \sup_{0\le y\le\delta} \sum_{\varrho:0<\gamma\le T_0} |D_{\theta+y}(\varrho)|. \tag{31}$$

Moreover, we have

$$\sum_{\varrho:0<\gamma\le T_0} |D_{\theta+y}(\varrho)| \ll (\log N) \sup_{0\le\sigma\le 1-1/(\log N)} \sum_{\substack{\varrho:0<\gamma\le T_0\\ \sigma\le\beta\le\sigma+1/(\log N)}} |D_{\theta+y}(\varrho)|. \tag{32}$$

The next step is to reduce the shifted Dirichlet polynomial $D_{\theta+y}(\varrho)$ to $D_\theta(i\gamma)$. By partial summation, we get

$$D_{\theta+y}(\varrho) = \sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} (n+\theta+y)^{\beta/\lambda}\biggl(1+\frac{y}{n+\theta}\biggr)^{i\gamma/\lambda-1}(n+\theta)^{i\gamma/\lambda-1} \tag{33}$$

$$= ((2N)^\lambda + \theta + y)^{\beta/\lambda}\left(1 + \frac{y}{(2N)^\lambda + \theta}\right)^{i\gamma/\lambda - 1} D_\theta(i\gamma)$$
$$- \int\limits_{N^\lambda}^{(2N)^\lambda} \frac{d}{du}\left((u + \theta + y)^{\beta/\lambda}\left(1 + \frac{y}{u + \theta}\right)^{i\gamma/\lambda - 1}\right) D_\theta(u, i\gamma)\, du.$$

For $N^\lambda \le u \le (2N)^\lambda$, $0 \le y \le \delta$, $0 < \gamma \le T_0$ and $0 \le \sigma \le \beta \le \sigma + 1/(\log N) \le 1$ we have

$$\text{(34)}\quad \frac{d}{du}\left((u + \theta + y)^{\beta/\lambda}\left(1 + \frac{y}{u + \theta}\right)^{i\gamma/\lambda - 1}\right)$$
$$= \frac{\beta}{\lambda}(u + \theta + y)^{\beta/\lambda - 1}\left(1 + \frac{y}{u + \theta}\right)^{i\gamma/\lambda - 1}$$
$$- \left(\frac{i\gamma}{\lambda} - 1\right)\frac{y}{(u + \theta)^2}(u + \theta + y)^{\beta/\lambda}\left(1 + \frac{y}{u + \theta}\right)^{i\gamma/\lambda - 2}$$
$$\ll N^{\beta - \lambda} + \delta T_0 N^{\beta - 2\lambda} \ll N^{\sigma - \lambda}(\log N)^{B+2}.$$

From (33) and (34), we obtain

$$\text{(35)}\quad |D_{\theta+y}(\varrho)| \ll N^\sigma (\log N)^{B+2}\Big(|D_\theta(i\gamma)| + N^{-\lambda}\int\limits_{N^\lambda}^{(2N)^\lambda} |D_\theta(u, i\gamma)|\, du\Big).$$

From (35), $T_0 \le T$ and the definition of $S_\theta(u, \sigma)$ in (15), we derive

$$\text{(36)}\quad \sum_{\substack{\varrho\,:\,0<\gamma\le T_0\\ \sigma\le\beta\le\sigma+1/(\log N)}} |D_{\theta+y}(\varrho)| \ll N^\sigma(\log N)^{B+2} \sup_{N^\lambda < u \le (2N)^\lambda} S_\theta(u, \sigma).$$

Combining (28)–(32) and (36), we obtain the desired estimate. ∎

By Lemma 1, in order to prove (5), we still have to show that

$$\text{(37)}\quad N^\sigma S_\theta(u, \sigma) \ll \frac{N}{(\log N)^{2B+3}}$$

for $0 \le \sigma \le 1 - 1/(\log N)$. This shall be the task of the next sections.

5. Estimation of $N^\sigma S_\theta(u, \sigma)$ for $0 \le \sigma \le 6/11$ and for $1 - \varDelta < \sigma \le 1$. In this section we establish (37) for $0 \le \sigma \le 6/11$ and for $1 - \varDelta < \sigma \le 1$, where

$$\text{(38)}\quad \varDelta := \begin{cases} 0.212 & \text{if } h \ge 2,\\ 1/36 & \text{if } h = 1.\end{cases}$$

LEMMA 2. *Without loss of generality assume that $\varepsilon \le f(\lambda,h)/\lambda$. Then for $0 \le \sigma \le 6/11$ we have*

$$N^\sigma S_\theta(u,\sigma) \ll N^{1-\varepsilon\lambda/3}.$$

Proof. By the Cauchy–Schwarz inequality, we have

$$S_\theta(u,\sigma) \ll \mathbf{N}(T)^{1/2}\Big(\sum_{\substack{\varrho\,:\,0<\gamma\le T\\ \sigma\le\beta\le\sigma+1/(\log N)}} |D_\theta(u,i\gamma)|^2\Big)^{1/2}, \tag{39}$$

where $\mathbf{N}(T)$ denotes the number of all non-trivial zeta zeros ϱ with $0<\gamma\le T$. By the well known properties of the set of zeta zeros, we can split the set of zeros ϱ satisfying the conditions $0<\gamma\le T$, $\sigma\le\beta\le\sigma+1/(\log N)$ into $O(\log T)$ subsets $\mathbf{S}$ satisfying the condition

$$\varrho_1,\varrho_2\in\mathbf{S},\ \varrho_1\ne\varrho_2 \ \Rightarrow\ |\mathrm{Im}\,\varrho_1-\mathrm{Im}\,\varrho_2|\ge 1.$$

Employing Proposition 2, we get

$$\sum_{\varrho\in\mathbf{S}} |D_\theta(u,i\gamma)|^2 \ll (T+N^\lambda)N^{-\lambda}(\log N) \tag{40}$$

for $N^\lambda < u \le (2N)^\lambda$.

Combining (25), (39), (40) and $\mathbf{N}(T)\ll T\log T$, and taking the condition $\varepsilon\le f(\lambda,h)/\lambda$ of Lemma 2 into account, we obtain the desired bound. ■

LEMMA 3. *For $1-\Delta<\sigma\le 1-1/(\log N)$ we have*

$$N^\sigma S_\theta(u,\sigma) \ll N\exp(-(\log N)^{1/4}).$$

Proof. By Proposition 3, there is no zeta zero ϱ with $0<\gamma\le T$ on the right side of the line $\mathrm{Re}\,s=\kappa(T)$, where

$$\kappa(T) := 1 - C(\log T)^{-2/3}(\log\log T)^{-1/3}.$$

Therefore, we can assume that $\sigma\le\kappa(T)$.

We first consider the case when $h\ge 2$. From the trivial estimate

$$D_\theta(u,i\gamma)\ll 1,$$

the second zero density estimate of Proposition 6 and (23), we obtain

$$N^\sigma S_\theta(u,\sigma) \ll N^{\sigma+(15(1-\sigma))/(11(3\sigma-1))}(\log N)^{44}. \tag{41}$$

We notice that

$$\frac{15}{11(3\sigma-1)} < 1-\frac{1}{3751} \tag{42}$$

if $\sigma>1-\Delta=0.788$. From (41), (42) and the above assumption $\sigma\le\kappa(T)$ follows

$$N^\sigma S_\theta(u,\sigma) \ll N^{1-(1-\kappa(T))/3751}(\log N)^{44}.$$

From this, we obtain

$$N^{\sigma}S_{\theta}(u,\sigma) \ll N\exp(-(\log N)^{1/4}) \tag{43}$$

by a short calculation. This completes the proof for the case when $h \geq 2$.

Now, let $h = 1$. Then, similarly to (41), we get

$$N^{\sigma}S_{\theta}(u,\sigma) \ll N^{\sigma+(15(1-\sigma))/(8(3\sigma-1))}(\log N)^{44} \tag{44}$$

by using (26). We notice that

$$\frac{15}{8(3\sigma-1)} < 1 - \frac{1}{46} \tag{45}$$

if $\sigma > 1-\Delta = 35/36$. In a similar manner to the case when $h \geq 2$, from (44) and (45), we anew obtain (43). This completes the proof. ∎

6. Zero detection method with weights. Next, we use a modified form of the zero detection method to handle the sum $S_{\theta}(u,\sigma)$ in the range $6/11 < \sigma \leq 1-\Delta$.

We note that by (22) the condition $\log N \ll \log T \ll \log N$ of Proposition 4 is satisfied if $\varepsilon < 1$.

LEMMA 4. *Suppose that* $1/2 \leq \sigma \leq 1 - 1/(\log N)$. *Then, under the conditions and using the definitions of Proposition* 4, *we have*

$$S_{\theta}(u,\sigma) \ll (V_1(u,\sigma)^{1/(2h)} + V_2(u,\sigma)^{1/2})N^{\varepsilon\lambda/400} + (\log N)^3$$

with

$$V_1(u,\sigma) := \mathbf{N}(\sigma,T)^{2h-3/2}T^{1/2}Y^{1-2\sigma} \times \int_{-(\log N)^2}^{(\log N)^2} \sum_{\varrho}{}^{(\sigma)} |M_X(1/2+i(\gamma+t))|^2 \cdot |D_{\theta}(u,i\gamma)|^{2h}\,dt \tag{46}$$

and

$$V_2(u,\sigma) := \mathbf{N}(\sigma,T) \times \sup_{X<v\leq Y(\log N)^2} \sum_{\varrho}{}^{(\sigma)} \Big|\sum_{X<k\leq v} a(k)e^{-k/Y}k^{-(\sigma+i\gamma)}\Big|^2 \cdot |D_{\theta}(u,i\gamma)|^2, \tag{47}$$

where the notation (σ) *attached to the summation symbol indicates the summation condition* "$(\log N)^2 < \gamma \leq T$ *and* $\sigma \leq \beta \leq \sigma + 1/(\log N)$". *The term* $V_2(u,\sigma)$ *is supposed to equal* 0 *if* $X \geq Y(\log N)^2$.

Proof. We first consider the contribution of zeta zeros with small imaginary part $\gamma \leq (\log N)^2$. By $\mathbf{N}((\log N)^2) \ll (\log N)^3$ and the trivial estimate $D_{\theta}(u,i\gamma) \ll 1$, we get

$$\sum_{\substack{\varrho\,:\,0<\gamma\leq(\log N)^2\\ \sigma\leq\beta\leq\sigma+1/(\log N)}} |D_{\theta}(u,i\gamma)| \ll (\log N)^3.$$

It remains to prove that

$$\sum_{\varrho}{}^{(\sigma)}|D_\theta(u,i\gamma)| \ll (V_1(u,\sigma)^{1/(2h)} + V_2(u,\sigma)^{1/2})N^{\varepsilon\lambda/400}. \tag{48}$$

By Proposition 4, for every sufficiently large N we have

$$\sum_{\varrho}{}^{(\sigma)}|D_\theta(u,i\gamma)| \le \Big(\sum_{\varrho}{}^{(\sigma,1)} + \sum_{\varrho}{}^{(\sigma,2)}\Big)|D_\theta(u,i\gamma)|, \tag{49}$$

where the notation (σ, j) attached to the summation symbol on the right side indicates that ϱ is a zero of type j (for the definition of "type j" see Proposition 4) satisfying the summation condition (σ), i.e. $(\log N)^2 < \gamma \le T$ and $\sigma \le \beta \le \sigma + 1/(\log N)$. We now separately consider the two sums on the right side of (49).

By Hölder's inequality and $1/2 < U_1(\varrho)$ for every zero ϱ of type 1, we get

$$\Big(\sum_{\varrho}{}^{(\sigma,1)}|D_\theta(u,i\gamma)|\Big)^h \ll \mathbf{N}(\sigma,T)^{h-1}\sum_{\varrho}{}^{(\sigma)}\ U_1(\varrho)\cdot|D_\theta(u,i\gamma)|^h. \tag{50}$$

Using the triangle and the Cauchy–Schwarz inequalities, and applying Stirling's formula to the Gamma factor contained in the integrand on the right side of (17), we derive

$$\begin{aligned}\Big(\sum_{\varrho}{}^{(\sigma)}U_1(\varrho)|D_\theta(u,i\gamma)|^h\Big)^2 \\ \ll (\log N)^2\mathbf{N}(\sigma,T)^{1/2}Y^{1-2\sigma}\Big(\sum_{\varrho}{}^{(\sigma)}\int\limits_{-(\log N)^2}^{(\log N)^2}|\zeta(1/2+i(\gamma+t))|^4\,dt\Big)^{1/2} \\ \times\Big(\int\limits_{-(\log N)^2}^{(\log N)^2}\sum_{\varrho}{}^{(\sigma)}|M_X(1/2+i(\gamma+t))|^2\cdot|D_\theta(u,i\gamma)|^{2h}\,dt\Big).\end{aligned} \tag{51}$$

Taking notice of $\log T \ll \log N$ and $\mathbf{N}(t+1)-\mathbf{N}(t) = O(\log(2t))$ for $t\ge 1$, and using Proposition 5, the term involving the ζ-function on the right side can be estimated by

$$\begin{aligned}\sum_{\varrho}{}^{(\sigma)}\int\limits_{-(\log N)^2}^{(\log N)^2}|\zeta(1/2+i(\gamma+t))|^4\,dt \\ \ll (\log N)^3\int\limits_{0}^{T+(\log N)^2}|\zeta(1/2+it)|^4\,dt \ll T(\log N)^7.\end{aligned} \tag{52}$$

Since $1/2 < U_2(\varrho)$ for every zero ϱ of type 2, we have

$$\sum_{\varrho}^{(\sigma,2)}|D_\theta(u,i\gamma)| \ll \sum_{\varrho}^{(\sigma)}U_2(\varrho)|D_\theta(u,i\gamma)|.$$

From that, applying partial summation to the term $U_2(\sigma)$ on the right side and taking $N^{1/(\log N)} = e$ into account, we obtain

$$\sum_{\varrho}^{(\sigma,2)}|D_\theta(u,i\gamma)| \ll \sup_{X<v\le Y(\log N)^2}\sum_{\varrho}^{(\sigma)}\Big|\sum_{X<k\le v}a(k)e^{-k/Y}k^{-(\sigma+i\gamma)}\Big|\cdot|D_\theta(u,i\gamma)|. \tag{53}$$

By the Cauchy–Schwarz inequality, we get

$$\Big(\sum_{\varrho}^{(\sigma)}\Big|\sum_{X<k\le v}a(k)e^{-k/Y}k^{-(\sigma+i\gamma)}\Big|\cdot|D_\theta(u,i\gamma)|\Big)^2 \ll \mathbf{N}(\sigma,T)\sum_{\varrho}^{(\sigma)}\Big|\sum_{X<k\le v}a(k)e^{-k/Y}k^{-(\sigma+i\gamma)}\Big|^2\cdot|D_\theta(u,i\gamma)|^2. \tag{54}$$

Combining (49)–(54), we obtain (48). This completes the proof. ∎

To bound $V_j(u,\sigma)$ $(j=1,2)$ by simple terms involving the parameters X and Y, we apply Theorem 5 after splitting the set of zeta zeros satisfying the condition (σ) into $O(\log T)$ subsets $\mathbf{S}$ such that $|\operatorname{Im}\varrho_1 - \operatorname{Im}\varrho_2| \ge 1$ for every pair $\varrho_1,\varrho_2 \in \mathbf{S}$ with $\varrho_1 \ne \varrho_2$. We take into consideration the fact that

$$|a(k)| \le \tau(k) \ll N^{\varepsilon_1}$$

for $X < k \le Y(\log N)^2$, where $\tau(k)$ denotes the number of divisors of k and ε_1 is any positive constant. Moreover, we point out that for irrational θ the additional conditions $u = L_2 \le T^{1/2}$ and $h = 1$ in Theorem 5 are really satisfied. Indeed, at the beginning of Section 3 we noticed that $h = 1$ if θ is irrational, and by (27) and the condition $\lambda < 5/19$ in Theorem 3, the inequality $u \le T^{1/2}$ is satisfied if $\varepsilon \le (5-19\lambda)/(12\lambda)$ and $N \ge 5$ (the latter two conditions may be supposed without loss of generality).

In this manner, we obtain the following result.

Lemma 5. *Suppose that* $1/2 \le \sigma \le 1-1/(\log N)$. *If* θ *is irrational*, *then suppose that* $\lambda < 5/19$ *and* $\varepsilon \le (5-19\lambda)/(12\lambda)$. *Then*, *on the conditions of Proposition* 4, *we have*

$$V_1(u,\sigma) \ll \mathbf{N}(\sigma,T)^{2h-3/2}T^{1/2}Y^{1-2\sigma}(TN^{-h\lambda}+X)N^{h\varepsilon\lambda/200},$$
$$V_2(u,\sigma) \ll \mathbf{N}(\sigma,T)(TN^{-\lambda}X^{1-2\sigma}+Y^{2(1-\sigma)})N^{\varepsilon\lambda/200},$$

the implied $\ll$*-constant only depending on* ε.

7. Estimation of $N^\sigma S_\theta(u,\sigma)$ for $6/11<\sigma\le 1-\Delta$. The final step of the proof of Theorem 3 is to show

LEMMA 6. *The estimate* (37) *holds true for* $6/11<\sigma\le 1-\Delta$.

Lemma 6 follows from the preceding Lemmas 4, 5 and the following

LEMMA 7. *Without loss of generality assume that* $\varepsilon\le 1/(10\lambda)$. *Then for any* σ *in the range* $6/11<\sigma\le 1-\Delta$ *there are parameters* X, Y *satisfying the conditions of Proposition* 4 *such that*

$$R_j(\sigma)\ll N^{2-\varepsilon\lambda/90}\qquad (j=1,\dots,4) \tag{55}$$

as $N\to\infty$, *where*

$$R_1(\sigma):=\mathbf{N}(\sigma,T)^{2-3/(2h)}T^{3/(2h)}N^{2\sigma-\lambda}Y^{(1-2\sigma)/h},$$
$$R_2(\sigma):=\mathbf{N}(\sigma,T)^{2-3/(2h)}T^{1/(2h)}N^{2\sigma}X^{1/h}Y^{(1-2\sigma)/h},$$
$$R_3(\sigma):=\mathbf{N}(\sigma,T)TN^{2\sigma-\lambda}X^{1-2\sigma},$$
$$R_4(\sigma):=\mathbf{N}(\sigma,T)N^{2\sigma}Y^{2(1-\sigma)},$$

the implied $\ll$*-constant in* (55) *depending only on* ε.

Proof. Firstly, we consider the case when $h=1$. We put

$$Y:=N^{1-\varepsilon\lambda/5}(1+\mathbf{N}(\sigma,T))^{-1/(2(1-\sigma))}$$

and

$$X:=1+Y^{2\sigma-1}N^{2(1-\sigma)(1-2\varepsilon\lambda/5)}T^{-1/2}(1+\mathbf{N}(\sigma,T))^{-1/2}.$$

We now derive some simple estimates for X and Y to verify the conditions of Proposition 4. Trivially, we have $1\le X$. By (26) and the well known bound

$$\mathbf{N}(\sigma,T)\ll T^{12(1-\sigma)/5}(\log T)^{44}$$

following from Proposition 6, we get

$$N^{(1-\lambda)/2}\ll Y\ll N.$$

This implies $\log N\ll\log Y\ll\log N$. Consequently, $\log T\ll\log Y\ll\log T$. The last condition to be verified is $\log X\ll\log N$. To prove this inequality, it suffices to show that $X\le Y$ for sufficiently large N, which follows from $1=o(Y)$ and

$$Y^{2\sigma-1}N^{2(1-\sigma)(1-2\varepsilon\lambda/5)}T^{-1/2}(1+\mathbf{N}(\sigma,T))^{-1/2}=o(Y) \tag{56}$$

as $N\to\infty$. The bound (56) can be easily obtained from the definition of Y, the bound $\mathbf{N}(\sigma,T)\ll T(\log T)$ and $\sigma\le 1-\Delta=35/36$. Therefore, all conditions of Proposition 4 to X, Y are satisfied.

Next, we calculate the order of magnitude of the terms $R_j(\sigma)$. From the definitions of X, Y and $\sigma\le 35/36$, we obtain

$$R_4(\sigma) \ll N^{2-\varepsilon\lambda/90}, \tag{57}$$

$$R_2(\sigma) = N^{2-\varepsilon\lambda/45} + R_1(\sigma)T^{-1}N^{\lambda}, \tag{58}$$

$$R_1(\sigma) \ll (1+\mathbf{N}(\sigma,T))^{\sigma/(2(1-\sigma))}T^{3/2}N^{1-\lambda+\varepsilon\lambda/5}. \tag{59}$$

From Proposition 6, we derive

$$\sup_{1/2\le\sigma\le 35/36}(1+\mathbf{N}(\sigma,T))^{\sigma/(1-\sigma)} \ll T^{\varepsilon\lambda/5}\Big(\sup_{1/2\le\sigma\le 3/4}T^{3\sigma/(2-\sigma)} + \sup_{3/4\le\sigma\le 35/36}T^{3\sigma/(3\sigma-1)}\Big). \tag{60}$$

The function $g_1(\sigma) := \sigma/(2-\sigma)$ is increasing on the interval $[1/2, 3/4]$, and the function $g_2(\sigma) := \sigma/(3\sigma-1)$ is decreasing on the interval $[3/4, 35/36]$. Therefore, from (60) follows

$$\sup_{1/2\le\sigma\le 35/36}(1+\mathbf{N}(\sigma,T))^{\sigma/(1-\sigma)} \ll T^{9/5+\varepsilon\lambda/5}. \tag{61}$$

Combining the first line of (26), (59) and (61), we get

$$R_1(\sigma) \ll N^{2-\varepsilon\lambda/2}. \tag{62}$$

From $N^{\lambda} \le T$, (58) and (62), we obtain

$$R_2(\sigma) \ll N^{2-\varepsilon\lambda/45}. \tag{63}$$

The last step is to verify the bound

$$R_3(\sigma) \ll N^{2-\varepsilon\lambda/5}. \tag{64}$$

From $X \le Y$ (which we have seen above) and the definitions of X and Y, we conclude

$$\left(\frac{Y}{X}\right)^{2\sigma-1} \le \frac{Y}{X} \le \left(\frac{T}{1+\mathbf{N}(\sigma,T)}\right)^{1/2} N^{\varepsilon\lambda/5},$$

from which follows

$$\begin{aligned} R_3(\sigma) &= \mathbf{N}(\sigma,T)TN^{2\sigma-\lambda}X^{1-2\sigma} \\ &\le \mathbf{N}(\sigma,T)^{1/2}T^{3/2}N^{2\sigma-\lambda}Y^{1-2\sigma}N^{\varepsilon\lambda/5} = R_1(\sigma)N^{\varepsilon\lambda/5}. \end{aligned}$$

Combining this inequality and (62), we get (64).

By (57), (62), (63) and (64), the bound (55) is satisfied for $j = 1, \dots, 4$. This completes the proof for the case when $h = 1$.

Secondly, we consider the case when $h \ge 2$. We observe that $N^{h\lambda} \le T$ by (10), (24) and the assumption $\varepsilon \le 1/(10\lambda)$ of Lemma 7. Here we put

$$X := TN^{-h\lambda}, \quad Y := N^{1-\varepsilon\lambda/4}(1+\mathbf{N}(\sigma,T))^{-1/(2(1-\sigma))}$$

unlike the case when $h = 1$. Using Proposition 6, (23) and $\varepsilon \le 1/(10\lambda)$, it is easily verified that X and Y satisfy the conditions of Proposition 4. Further, it is an immediate consequence of the definition of Y and the condition $6/11 < \sigma \le 1 - \Delta = 0.788$ that (55) holds true for $j = 4$. Here, as in the following, we use the condition $6/11 < \sigma \le 0.788$ in order to obtain the correct ε-terms.

From the definitions of X and Y follows

$$(65)\qquad R_1(\sigma) + R_2(\sigma) \ll \mathbf{N}(\sigma, T)^{2-3/(2h)+(2\sigma-1)/(2h(1-\sigma))} T^{3/(2h)} N^{2\sigma(1-1/h)+1/h-\lambda+\varepsilon\lambda/(7h)}.$$

By Proposition 6, we have

$$(66)\qquad \mathbf{N}(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon\lambda/10},$$

where

$$A(\sigma) = \begin{cases} 3/(2-\sigma) & \text{if } 1/2 \le \sigma \le 3/4, \\ 3/(3\sigma - 1) & \text{if } 3/4 < \sigma \le 1. \end{cases}$$

Combining (22), (65) and (66), and taking (11) and $h \ge 2$ into consideration, we get

$$(67)\qquad R_1(\sigma) + R_2(\sigma) \ll N^{r_1(\sigma)+c_1-\varepsilon\lambda/10},$$

where

$$r_1(\sigma) := (\lambda + f(\lambda, h))\left(2 - \frac{2}{h} + \left(\frac{5}{2h} - 2\right)\sigma\right)A(\sigma) + 2\left(1 - \frac{1}{h}\right)\sigma,$$

$$c_1 := (\lambda + f(\lambda, h))\,\frac{3}{2h} + \frac{1}{h} - \lambda.$$

Our next aim is to show that $r_1(\sigma)$ is increasing on the interval $6/11 < \sigma < 3/4$ and decreasing on the interval $3/4 < \sigma \le 0.788$. For $6/11 < \sigma < 3/4$ we have

$$r_1'(\sigma) = -(\lambda + f(\lambda, h))\left(2 - \frac{3}{h}\right)\frac{3}{(2-\sigma)^2} + 2\left(1 - \frac{1}{h}\right).$$

From that, (11) and $h \ge 2$, we obtain

$$r_1'(\sigma) \ge \frac{14}{55} + \frac{34}{55h} > 0$$

for $6/11 < \sigma < 3/4$. Hence, $r_1(\sigma)$ is increasing on this interval. For $3/4 < \sigma \le 0.788$ we have

$$r_1'(\sigma) = -(\lambda + f(\lambda, h))\left(12 - \frac{21}{2h}\right)\frac{1}{(3\sigma-1)^2} + 2\left(1 - \frac{1}{h}\right).$$

From that and (11), we obtain

$$r_1'(\sigma) < -0.6 + \frac{0.4}{h} < 0$$

for $3/4 < \sigma \le 0.788$. Hence, $r_1(\sigma)$ is decreasing on this interval.

We note that the function $r_1(\sigma)$ is continuous on the interval $(6/11, 0.788]$ since $A(\sigma)$ is continuous on this interval. From that and the above observations, we conclude that the exponent $r_1(\sigma) + c_1 - \varepsilon\lambda/10$ on the right side of (67) takes its maximum at the point $\sigma_0 = 3/4$. Furthermore, from

$$\lambda + f(\lambda, h) \leq \lambda + f_1(\lambda, h) = \frac{5}{12} + \frac{5h\lambda}{6(h+1)},$$

we obtain

$$r_1(3/4) + c_1 \leq 2$$

by a short calculation. From that and (67), we derive (55) for $j = 1, 2$.

Finally, we evaluate the term $R_3(\sigma)$. From (22), (66) and the definition of X, we obtain

$$R_3(\sigma) \ll N^{r_2(\sigma) - (h+1)\lambda - \varepsilon\lambda/2}, \tag{68}$$

where

$$r_2(\sigma) := (\lambda + f(\lambda, h))(2 + A(\sigma))(1 - \sigma) + 2(1 + h\lambda)\sigma.$$

For $6/11 < \sigma < 3/4$ we have

$$r_2'(\sigma) = -(\lambda + f(\lambda, h))\left(2 + \frac{3}{(2 - \sigma)^2}\right) + 2(1 + h\lambda).$$

From that and (11), we obtain

$$r_2'(\sigma) > 0$$

for $6/11 < \sigma < 3/4$. Hence, $r_2(\sigma)$ is increasing on this interval. At the end of this section, we shall separately prove that $r_2(\sigma)$ is decreasing on the interval $3/4 < \sigma \leq 0.788$.

Like $r_1(\sigma)$, the function $r_2(\sigma)$ is continuous on the interval $(6/11, 0.788]$. Consequently, the exponent $r_2(\sigma) - (h+1)\lambda - \varepsilon\lambda/2$ on the right side of (68) takes its maximum at the point $\sigma_0 = 3/4$. Furthermore, from

$$\lambda + f(\lambda, h) \leq \lambda + f_2(\lambda, h) = \frac{5}{11} + \frac{(10 - 5h)\lambda}{11},$$

we obtain

$$r_2(3/4) - (h+1)\lambda \leq 2$$

by a short calculation. From that and (68), we derive (55) for $j = 3$. This completes the proof of Lemma 7. ∎

By proving Lemma 7 we have also completed the proof of Theorem 3.

It remains to show that $r_2'(\sigma) < 0$ for $3/4 < \sigma \leq 0.788$ if $h \geq 2$. On this interval, we have

$$r_2'(\sigma) = -(\lambda + f(\lambda, h))\left(2 + \frac{6}{(3\sigma - 1)^2}\right) + 2(1 + h\lambda).$$

Thus, $r_2'(\sigma) < 0$ for $3/4 < \sigma \le 0.788$ is equivalent to

$$2\left(2 + \frac{6}{1.364^2}\right)^{-1} =: \xi < \frac{\lambda + f(\lambda, h)}{1 + h\lambda},$$

where $\xi \approx 0.3828$.

By definition, for $k \in \mathbb{N}$ we have

$$\frac{\lambda + f(\lambda, k)}{1 + k\lambda} = \min\left\{\frac{\lambda + f_1(\lambda, k)}{1 + k\lambda}, \frac{\lambda + f_2(\lambda, k)}{1 + k\lambda}\right\}$$

with

$$\frac{\lambda + f_1(\lambda, k)}{1 + k\lambda} = \frac{5}{12}\left(1 - \left(1 - \frac{1}{1 + k\lambda}\right)\left(1 - \frac{2}{1 + k}\right)\right),$$

$$\frac{\lambda + f_2(\lambda, k)}{1 + k\lambda} = \frac{5}{11} \cdot \frac{1 + (2 - k)\lambda}{1 + k\lambda}.$$

For fixed $k \ge 2$ the functions $g_i(\lambda) := (\lambda + f_i(\lambda, k))/(1 + k\lambda)$ $(i = 1, 2)$ are obviously decreasing for $\lambda > 0$. Furthermore, by (8) and $h \ge 2$, we have

$$\lambda \le \phi_{1,h} = \frac{1}{2(6h - 1 - 11/h)}, \qquad f(\phi_{1,h}, h) = f_2(\phi_{1,h}, h).$$

Hence, it suffices to prove that

$$\xi < z_h := \frac{5}{11} \cdot \frac{1 + (2 - h)/(2(6h - 1 - 11/h))}{1 + h/(2(6h - 1 - 11/h))}. \tag{69}$$

Inequality (69) holds true for $h = 2, 3, 4$. Furthermore, it is easily seen that the sequence (z_h) is decreasing for $h \ge 4$, and we have

$$\lim_{h\to\infty} z_h = \frac{5}{13} > \xi.$$

This completes the proof. ∎

8. Proofs of Theorems 4, 4* and 5. Theorem 4* can be derived from Theorem 4 in a standard way using the inequality

$$|f(x)| \le \int_{x-1/2}^{x+1/2} (|f(t)| + |f'(t)|)\, dt,$$

which is valid for every continuously differentable function $f : [x - 1/2, x + 1/2] \to \mathbb{C}$.

To derive Theorem 5 from Theorem 4*, we proceed as follows: In case θ is rational we write the shifted Dirichlet polynomial on the left side of (21) as an ordinary one via the relation

$$\sum_{L_1 < l \le L_2} b_l (l + \theta)^{i\alpha t_r - 1} = q^{-i\alpha t_r + 1} \sum_{L_1 < l \le L_2} b_l (ql + m)^{i\alpha t_r - 1},$$

where $\theta = m/q$, m and q being non-negative integers. We then write the $2h$th power of the absolute value of the Dirichlet polynomial on the right side in the form

$$\Big| \sum_{L_1<l\le L_2} b_l (ql+m)^{i\alpha t_r - 1} \Big|^{2h} = \Big| \sum_{(qL_1+m)^h<n\le(qL_2+m)^h} c_n n^{i\alpha t_r} \Big|^2,$$

where

$$c_n := n^{-1} \sum_{\substack{L_1<l_1,\dots,l_h\le L_2 \\ n=(ql_1+m)\dots(ql_h+m)}} b_{l_1}\dots b_{l_h}.$$

We note that

$$|c_n| \le B^h n^{-1} d_h(n) \ll B^h n^{\varepsilon_2 - 1},$$

where $d_h(n)$ denotes the divisor function of order h and ε_2 is any positive constant. Now, we divide each of the Dirichlet polynomials

$$\sum_{K_1<k\le K_2} a_k k^{-(\sigma+it_r)} \quad \text{and} \quad \sum_{(qL_1+m)^h<n\le(qL_2+m)^h} c_n n^{i\alpha t_r}$$

into $O(\log K_2)$ and $O(\log L_2)$ partial sums over ranges of the form $K < k \le 2K$ and $L < l \le 2L$ respectively, use the Cauchy–Schwarz inequality, multiply out the two resulting sums of squares of absolute values of Dirichlet polynomials, and sum up over r. In this manner, we obtain a sum of terms having the same shape as the one on the left side of (20), where now $\theta = 0$. Applying Theorem 4* with $\theta = 0$ to these terms, we obtain the desired bound.

When θ in Theorem 5 is irrational, it is supposed that $h = 1$. Now, we just split up the ordinary and the shifted Dirichlet polynomial on the left side of (21) in the same manner as above, use the Cauchy–Schwarz inequality, multiply out, sum up over r and apply Theorem 4*. In this way, we again obtain the desired bound. This completes the proof of Theorem 5.

We now turn to proving Theorem 4. If $\theta = 0$, Theorem 4 is nothing but a slight modification of Theorem 1 in [BaH]. However, in the case when $\theta \ne 0$ Theorem 4 actually appears to be a new result, which we shall prove in the following.

Without loss of generality, we assume that $A = B = 1$ and $\alpha > 0$. We denote the integral in question on the left side of (19) by I.

Multiplying out the integrand contained in I, integrating the resulting fourfold sum term by term and using the standard inequalities

$$\int_0^T x^{it}\, dt \ll \min\{T, |\log x|^{-1}\}$$

for $x > 0$ and

$$|\log \omega| \gg |\omega - 1|$$

for $2^{-(\alpha+1)} \le \omega \le 2^{\alpha+1}$, we obtain

$$I \le \sum_{k_1,k_2\sim K}\sum_{l_1,l_2\sim L} \min\left\{T, \left|\frac{k_1(l_2+\theta)^\alpha}{k_2(l_1+\theta)^\alpha}-1\right|^{-1}\right\} \tag{70}$$
$$\le 2^\alpha \sum_{k_1,k_2\sim K}\sum_{l_1,l_2\sim L} \min\left\{T, \left|\frac{k_1}{k_2}-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right|^{-1}\right\}$$
$$\le 2^\alpha \sum_{d\le 2K}\sum_{\substack{k_1,k_2\sim K/d\\(k_1,k_2)=1}}\sum_{l_1,l_2\sim L} \min\left\{T, \left|\frac{k_1}{k_2}-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right|^{-1}\right\}.$$

Let $M := [2+\alpha+(\log T)/(\log 2)]$. In the following, we suppose that $H \ge 1/2$ and $0 < Z \le 2^M/T \le 2^{2+\alpha}$. By $G(H,Z)$ we denote the number of solutions to

$$\left|\frac{k_1}{k_2}-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right| \le Z$$

with $k_1, k_2 \sim H$, $(k_1,k_2)=1$ and $l_1, l_2 \sim L$. We then have

$$\sum_{\substack{k_1,k_2\sim H\\(k_1,k_2)=1}}\sum_{l_1,l_2\sim L} \min\left\{T, \left|\frac{k_1}{k_2}-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right|^{-1}\right\} \ll T\sum_{m=0}^{M} G(H,2^m/T)2^{-m}. \tag{71}$$

Let $S(H)$ be the set of all fractions k_1/k_2 with $k_1, k_2 \sim H$, $(k_1,k_2)=1$. This set is well-spaced with spacing $1/(4H^2)$. Hence,

$$G(H,Z) = \sum_{l_1,l_2\sim L}\left|\left\{u\in S(H) : \left|\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha - u\right| \le Z\right\}\right| \ll L^2(ZH^2+1). \tag{72}$$

By a short calculation, from (72), we derive

$$T\sum_{m=0}^{M} G(H,2^m/T)2^{-m} \ll (H^2L^2+TL^2)\log(2T). \tag{73}$$

We now estimate the left side of (73) in an alternative way. Using the Cauchy–Schwarz inequality and taking the above-mentioned spacing properties of the set $S(H)$ into account, we obtain

$$G(H,Z) = \sum_{u\in S(H)}\left|\left\{l_1,l_2\sim L : \left|u-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right| \le Z\right\}\right| \tag{74}$$
$$\ll H\left(\sum_{u\in S(H)}\left|\left\{l_1,l_2\sim L : \left|u-\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha\right| \le Z\right\}\right|^2\right)^{1/2}$$

$$\ll H\left((ZH^2+1)\left|\left\{l_1,l_2,l_1',l_2' \sim L : \left|\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha - \left(\frac{l_1'+\theta}{l_2'+\theta}\right)^\alpha\right| \le 2Z\right\}\right|\right)^{1/2}.$$

Using Taylor's formula, $1/2 \le (l_1+\theta)/(l_2+\theta) \le 2$ and $1/2 \le (l_1'+\theta)/(l_2'+\theta) \le 2$, we deduce that there is a positive constant c depending only on α such that

$$(75)\qquad \left|\left\{l_1,l_2,l_1',l_2' \sim L : \left|\left(\frac{l_1+\theta}{l_2+\theta}\right)^\alpha - \left(\frac{l_1'+\theta}{l_2'+\theta}\right)^\alpha\right| \le 2Z\right\}\right| \le \left|\left\{l_1,l_2,l_1',l_2' \sim L : \left|\frac{l_1+\theta}{l_2+\theta} - \frac{l_1'+\theta}{l_2'+\theta}\right| \le cZ\right\}\right|.$$

Taking the first inequality on page 145 of [Ha2] into account, we get

$$(76)\qquad \left|\left\{l_1,l_2,l_1',l_2' \sim L : \left|\frac{l_1+\theta}{l_2+\theta} - \frac{l_1'+\theta}{l_2'+\theta}\right| \le cZ\right\}\right|$$
$$\le |\{l_1,l_2,l_1',l_2' \sim L : |(l_1+\theta)(l_2'+\theta) - (l_1'+\theta)(l_2+\theta)| \le cZ(2L+\theta)^2\}|$$
$$\ll (ZL^2+1)L^2\log^2(2L).$$

The implied $\ll$-constant does not depend on θ. Combining (74), (75) and (76), we get

$$(77)\qquad G(H,Z) \ll (ZH^2L^2 + Z^{1/2}(H+L)HL + HL)\log(2L).$$

By a short calculation, from (77), we derive

$$(78)\qquad T\sum_{m=0}^{M} G(H,2^m/T)2^{-m} \ll (H^2L^2 + T^{1/2}(H+L)HL + THL)\log(2L)\log(2T).$$

Combining (73) and (78), and taking the condition $L \le T^{1/2}$ in Theorem 4 into account, we get

$$(79)\qquad T\sum_{m=0}^{M} G(H,2^m/T)2^{-m} \ll (H^2L^2 + THL + T\min\{H^2,L^2\})\log(2L)\log(2T).$$

From (70), (71) and (79), we obtain

$$(80)\qquad I \ll \left(K^2L^2 + TKL\log(2K) + T\sum_{d\le 2K}\min\left\{\frac{K^2}{d^2},L^2\right\}\right) \times \log(2L)\log(2T).$$

If $K \le L$, then we have

$$\sum_{d \le 2K} \min\left\{\frac{K^2}{d^2}, L^2\right\} \le \sum_{d \le 2K} \frac{K^2}{d^2} \ll K^2 \le KL.$$

Otherwise, we have

$$\sum_{d \le 2K} \min\left\{\frac{K^2}{d^2}, L^2\right\} \le \sum_{d \le K/L} L^2 + \sum_{K/L < d \le 2K} \frac{K^2}{d^2} \ll KL.$$

Therefore, from (80) follows

$$I \ll (K^2L^2 + TKL)\log(2K)\log(2L)\log(2T).$$

This implies the result of Theorem 4. ∎

We note that if the condition $L \le T^{1/2}$ in Theorem 4 could be removed for all $\theta \ne 0$, then the condition $\lambda \le 5/19$ in Theorem 3 could be removed for all irrational θ.

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