ACTA ARITHMETICA
115.4 (2004)

On higher-power moments of $E(t)$

by

Wenguang Zhai (Jinan)

1. Main result. Let $\zeta(s)$ denote the Riemann zeta-function. For $t > 2$, define

$$E(t) := \int_0^t |\zeta(1/2+iu)|^2\,du - t\log(t/2\pi) - (2\gamma-1)t. \tag{1.1}$$

It is an important problem to study the upper bound of $E(t)$. The latest result is

$$E(t) = O(t^{72/227}\log^{629/227} t), \tag{1.2}$$

due to Huxley [3]. We have the conjecture

$$E(t) = O(t^{1/4+\varepsilon}), \tag{1.3}$$

which is supported by the mean square formula

$$\int_2^T E^2(t)\,dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}}\,T^{3/2} + O(T\log^5 T) \tag{1.4}$$

proved by Meurman [8].

Tsang [9] studied the third- and fourth-power moments of $E(t)$. He proved that the asymptotic formulas

$$\int_2^T E^3(t)\,dt = \frac{6}{7}\,(2\pi)^{-3/4}c_1T^{7/4} + O(T^{7/4-\delta_1+\varepsilon}), \tag{1.5}$$

$$\int_2^T E^4(t)\,dt = \frac{3}{8\pi}\,c_2T^2 + O(T^{2-\delta_2+\varepsilon}) \tag{1.6}$$

hold with $\delta_1 > 0$ and $\delta_2 > 0$, where

$$c_1 = \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}},$$

2000 *Mathematics Subject Classification*: 11N37, 11M06.

Key words and phrases: higher-power moment, Atkinson's formula.

This work is supported by National Natural Science Foundation of China (10301018).

$$c_2 = \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}+\sqrt{n_4}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1 n_2 n_3 n_4)^{3/4}}.$$

Tsang [9] proved that (1.5) holds for $\delta_1 = 1/36$, but did not specify the permissible value of δ_2 in (1.6). Ivić [4] proved that (1.5) holds with $\delta_1 = 1/14$ and (1.6) holds with $\delta_2 = 1/23$. Recently following Ivić's approach, the author [10] proved that (1.5) holds with $\delta_1 = 1/12$ and (1.6) holds with $\delta_2 = 2/41$.

Tsang [9] began with Atkinson's formula [1] and used the averaging technique over a short interval. Ivić's argument was different from Tsang's. He used a theorem of Jutila [6] (see also Theorem 15.6 of Ivić [5]) to transform the problem into the higher-power moments of $\Delta^*(x)$, the error term of $\frac{1}{2}\sum_{n\le 4x}(-1)^n d(n)$, where $d(n)$ is the Dirichlet divisor function. The higher-power moments of $\Delta^*(x)$ are easier to handle than those of $E(t)$, since $\Delta^*(x)$ has the Voronoï formula.

Heath-Brown [2] proved that for any $3 \le k \le 9$ ($k \in \mathbb{N}$), the limit

$$\lim_{T\to\infty} T^{-1-k/4} \int_2^T E^k(t)\,dt$$

exists. The author [11] got an asymptotic formula for $\int_2^T E^k(t)\,dt$ for any $5 \le k \le 9$, where Jutila's theorem [6] and power moment results for $E(t)$ and $\Delta(x)$, the error term of the Dirichlet divisor problem, were used.

However, the exponent $1/12$ in the third-power moment of $E(t)$ is the limit of Jutila's theorem. In order to reduce this exponent, we have to go back to Atkinson's formula and not use Jutila's theorem. In this paper, we shall use a different approach, which is a generalization of that in [11], to study the higher-power moments of $E(t)$. In this approach, we use Atkinson's formula for $E(t)$ only. Since for $k \ge 4$ the results obtained by this approach are the same as the previous results (see Zhai [11] for details), we only consider the case $k = 3$.

THEOREM. *We have*

$$\int_2^T E^3(t)\,dt = \frac{6}{7}(2\pi)^{-3/4} c_1 T^{7/4} + O(T^{7/4-83/393+\varepsilon}). \tag{1.7}$$

REMARK. It is well known that many properties of $E(t)$ are similar to those of $\Delta(x)$. We also have a similar conjecture

$$\Delta(x) \ll x^{1/4+\varepsilon}, \tag{1.8}$$

which seems easier than the conjecture (1.3) by a result of Jutila [7], who proved that if (1.8) is true, then $E(t) = O(t^{3/10+\varepsilon})$.

Theorem 1 of [11] shows that if (1.8) is true, then for any $k \geq 3$ we have

$$\int_2^T \Delta^k(t)\,dt = C_k T^{1+k/4} + O(T^{\eta_k}), \tag{1.9}$$

where C_k and $\eta_k < 1 + k/4$ are explicit constants. This means that (1.8) is equivalent to the following conjecture: (1.9) is true for any $k \geq 3$.

Theorem 5 of [11] shows that if both (1.3) and (1.8) are true, then for any $k \geq 3$ we can get the asymptotic formula

$$\int_2^T E^k(t)\,dt = C'_k T^{1+k/4} + O(T^{\eta'_k}), \tag{1.10}$$

where C'_k and $\eta'_k < 1+k/4$ are explicit constants. Combining the approaches of this paper and [11], we know that the conjecture (1.8) can be removed in the above conclusion. Thus we deduce that the conjecture (1.3) is equivalent to the following conjecture: (1.10) is true for any $k \geq 3$.

Acknowledgements. The author deeply thanks the referee for his valuable suggestions and comments.

Notations. Throughout this paper, $\{x\}$ denotes the fractional part of x, $\|x\|$ denotes the distance from x to the integer nearest to x, $n \sim N$ means $N < n \leq 2N$, ε always denotes a small positive constant which may be different at different places.

2. Some preliminary lemmas

Lemma 2.1. *We have*

$$E(t) = \Sigma_1(t) + \Sigma_2(t) + O(\log^2 t)$$

with

$$\Sigma_1(t) := \frac{1}{\sqrt{2}} \sum_{n \leq N} h(t,n) \cos(f(t,n)), \tag{2.1}$$

$$\Sigma_2(t) := -2 \sum_{n \leq N'} d(n) n^{-1/2} \left(\log \frac{t}{2\pi n} \right)^{-1} \cos\left(t \log \frac{t}{2\pi n} - t + \frac{\pi}{4} \right), \tag{2.2}$$

$$h(t,n) := (-1)^n d(n) n^{-1/2} \left(\frac{t}{2\pi n} + \frac{1}{4} \right)^{-1/4} (g(t,n))^{-1}, \tag{2.3}$$

$$g(t,n) := \operatorname{arsinh}\left(\left(\frac{\pi n}{2t} \right)^{1/2} \right), \tag{2.4}$$

$$f(t,n) := 2tg(t,n) + (2\pi nt + \pi^2 n^2)^{1/2} - \pi/4, \tag{2.5}$$

$$At \leq N \leq A't, \ \ N' := t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2}, \tag{2.6}$$

where $0 < A < A'$ *are any fixed constants.*

Proof. This is the famous Atkinson formula; see Ivić [5, Theorem 15.1]. ∎

LEMMA 2.2. *Suppose $Y > 1$. Define*

$$c_1^* := \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}} \frac{(-1)^{n_1+n_2+n_3} d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}},$$

$$c_1^*(Y) := \sum_{\substack{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}\\ n_1,n_2,n_3\le Y}} \frac{(-1)^{n_1+n_2+n_3} d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}},$$

$$c_1(Y) := \sum_{\substack{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}\\ n_1,n_2,n_3\le Y}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}}.$$

Then

$$c_1 = c_1^*, \quad c_1(Y) = c_1^*(Y), \quad |c_1 - c_1(Y)| \ll Y^{-1+\varepsilon}.$$

Proof. The estimate $|c_1 - c_1(Y)| \ll Y^{-1+\varepsilon}$ appears on page 70 of Tsang [9]. The equalities $c_1 = c_1^*$ and $c_1(Y) = c_1^*(Y)$ follow from the fact that if $\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}$, then $n_1+n_2+n_3$ must be an even number. ∎

LEMMA 2.3. *Suppose $Y > 1$. Then*

$$H_1(Y) := \sum_{\substack{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}\\ n_1,n_2,n_3\le Y}} \frac{d(n_1)d(n_2)d(n_3)n_3^{3/4}}{(n_1n_2)^{3/4}} \ll Y^{1/2+\varepsilon}.$$

Proof. By a classical result of Besicovitch, if $\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}$, then $n_j = m_j^2 h$, $m_1+m_2=m_3$, $\mu(h)\neq 0$. Thus we get

$$\begin{aligned} H_1(Y) &\ll \sum_{(m_1+m_2)^2h\le Y} \frac{d(m_1^2h)d(m_2^2h)d((m_1+m_2)^2h)(m_1+m_2)^{3/2}}{h^{3/4}(m_1m_2)^{3/2}} \\ &\ll \sum_{h<Y} h^{-3/4+\varepsilon} \sum_{m_2\le m_1\ll (Y/h)^{1/2}} m_1^{\varepsilon} m_2^{-3/2+\varepsilon} \ll Y^{1/2+\varepsilon} \end{aligned}$$

if we notice $d(n) \ll n^{\varepsilon}$. ∎

LEMMA 2.4. *Let $N, M, K \ge 10$, $D = \max(N, M, K)$, $0 < |\Delta| \ll D^{1/2}$. Let*

$$\mathcal{A}(N,M,K;\Delta) := \sum_{\substack{n\sim N,\, m\sim M,\, k\sim K\\ |\sqrt{n}+\sqrt{m}-\sqrt{k}|\le \Delta}} 1.$$

Then

$$D^{-\varepsilon}\mathcal{A}(N,M,K;\Delta) \ll \Delta D^{-1/2} NMK + D^{-1/2}(NMK)^{1/2}.$$

Proof. This is Lemma 2.5 of [10]. ∎

LEMMA 2.5. *If $\sqrt{n}+\sqrt{m}-\sqrt{k}\neq 0$, then*

$$|\sqrt{n}+\sqrt{m}-\sqrt{k}|\gg \frac{1}{\sqrt{nmk}},$$

where the implied constant is absolute.

Proof. If n is not a square, then

$$\|\sqrt{n}\|\gg 1/\sqrt{n}. \tag{2.7}$$

We omit the proof of (2.7) since it is elementary and easy. Let $\alpha=\sqrt{n}+\sqrt{m}-\sqrt{k}$. We suppose $|\alpha|<1/10$, otherwise the lemma is trivial. Squaring $\alpha+\sqrt{k}=\sqrt{n}+\sqrt{m}$ we get

$$\alpha^2+2\sqrt{k}\,\alpha=n+m+\sqrt{4nm}-k. \tag{2.8}$$

If nm is a square, then the right-hand side of (2.8) is a non-zero integer and then $|\alpha^2+2\sqrt{k}\,\alpha|\geq 1$, which implies $|\alpha|\gg 1/\sqrt{k}$. If nm is not a square, then from (2.8) we have $|\alpha^2+2\sqrt{k}\,\alpha|\gg\|\sqrt{4nm}\|$, which combined with (2.7) implies $|\alpha|\gg 1/\sqrt{nmk}$. ∎

LEMMA 2.6. *Suppose $(i_1,i_2)\in\{0,1\}^2$ and $Y\geq 10$ is a real number. For $(n_1,n_2,n_3)\in\mathbb{N}^3$, define*

$$\alpha_3:=\sqrt{n_1}+(-1)^{i_1}\sqrt{n_2}+(-1)^{i_2}\sqrt{n_3},$$
$$H(Y;i_1,i_2):=\sum_{\substack{n_j\leq Y,\,1\leq j\leq 3\\ \alpha_3\neq 0}}\frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|}.$$

Then

$$H(Y;i_1,i_2)\ll Y^{1/4+\varepsilon}.$$

Proof. By a splitting argument and $d(n)\ll n^\varepsilon$ we get, for some $1\ll N_j\ll Y$ $(1\leq j\leq 3)$,

$$\begin{aligned}Y^{-\varepsilon}H(Y;i_1,i_2)&\ll\sum_{\substack{n_j\sim N_j,\,1\leq j\leq 3\\ \alpha_3\neq 0}}\frac{1}{(n_1n_2n_3)^{3/4}|\alpha_3|}\\&\ll(N_1N_2N_3)^{-3/4}\sum_{\substack{n_j\sim N_j,\,1\leq j\leq 3\\ \alpha_3\neq 0}}\frac{1}{|\alpha_3|}.\end{aligned}$$

If $(i_1,i_2)=(0,0)$, then trivially

$$Y^{-\varepsilon}H(Y;0,0)\ll\frac{(N_1N_2N_3)^{1/4}}{\max(N_1,N_2,N_3)^{1/2}}\ll\min(N_1,N_2,N_3)^{1/4}\ll Y^{1/4}.$$

Now suppose $(i_1,i_2)\neq(0,0)$. By Lemma 2.5 we have $|\alpha_3|\gg 1/(N_1N_2N_3)^{1/2}$. By a splitting argument again we infer for some $1/(N_1N_2N_3)^{1/2}\ll\Delta\ll$

$\max(N_1, N_2, N_3)^{1/2}$ that

$$Y^{-\varepsilon} H(Y; i_1, i_2) \ll \frac{(N_1N_2N_3)^{-3/4}}{\Delta} \sum_{\substack{n_j \sim N_j,\, 1\le j\le 3 \\ \Delta < |\alpha_3| \le 2\Delta}} 1.$$

By Lemmas 2.4 and 2.5 we get

$$\begin{aligned} Y^{-\varepsilon} H(Y; i_1, i_2) &\ll \frac{(N_1N_2N_3)^{-3/4}}{\Delta} \frac{\Delta N_1N_2N_3 + (N_1N_2N_3)^{1/2}}{\max(N_1, N_2, N_3)^{1/2}} \\ &\ll \frac{(N_1N_2N_3)^{1/4}}{\max(N_1, N_2, N_3)^{1/2}} + \frac{(N_1N_2N_3)^{-1/4}}{\Delta \max(N_1, N_2, N_3)^{1/2}} \\ &\ll \frac{(N_1N_2N_3)^{1/4}}{\max(N_1, N_2, N_3)^{1/2}} \ll \min(N_1, N_2, N_3)^{1/4} \ll Y^{1/4}. \ \blacksquare \end{aligned}$$

LEMMA 2.7. *Suppose $f_j(t)$ $(1 \le j \le k)$ and $g(t)$ are continuous, monotonic real-valued functions on $[a, b]$ and let $g(t)$ have a continuous, monotonic derivative on $[a, b]$. If $|f_j(t)| \le A_j$ $(1 \le j \le k)$, $|g'(t)| \gg \Delta$ for any $t \in [a, b]$, then*

$$\int_a^b f_1(t) \cdots f_k(t) e(g(t))\, dt \ll A_1 \cdots A_k \Delta^{-1}.$$

Proof. This is Lemma 15.3 of Ivić [5]. ∎

LEMMA 2.8. *Suppose $(i_1, i_2) \in \{0, 1\}^2$, $T \ge 100$ is a large real number, $1 \le Z_j < Y_j \le T^{1/2}$ $(1 \le j \le 3)$ are three real numbers such that there are at least two Z_j satisfying $Z_j \ge T^{1/3-\varepsilon}$, $Y = \max(Y_1, Y_2, Y_3)$. Define*

$$F(t; n_1, n_2, n_3; i_1, i_2) := f(t, n_1) + (-1)^{i_1} f(t, n_2) + (-1)^{i_2} f(t, n_3),$$

$$S_{i_1,i_2}(t) := \sum_{\substack{Z_j < n_j \le Y_j,\, 1\le j\le 3 \\ \alpha_3 \ne 0}} h(t, n_1) h(t, n_2) h(t, n_3) \cos(F(t; n_1, n_2, n_3; i_1, i_2)).$$

Then

$$\int_T^{2T} S_{i_1,i_2}(t)\, dt \ll T^{1+\varepsilon} Y + T^{17/12+\varepsilon}. \tag{2.9}$$

Proof. It is easy to check that for any $n \le T/\pi$, the function $h(t, n)$ is a product of monotonic functions and

$$h(t, n) = \frac{2^{3/4}}{\pi^{1/4}} \frac{(-1)^n d(n)}{n^{3/4}} t^{1/4} \left(1 + O\left(\frac{n}{t}\right)\right). \tag{2.10}$$

For any $n \le T^{1/2}$ it is easy to check that

$$f(t, n) = 2^{3/2} (\pi n t)^{1/2} - \frac{\pi}{4} + \frac{\pi^{3/2}}{3\sqrt{2}} \frac{n^{3/2}}{t^{1/2}} + f_1(t, n), \tag{2.11}$$

where

$$(2.12)\qquad f_1(t,n)=O\bigg(\frac{n^{5/2}}{t^{3/2}}\bigg),\quad f_1'(t,n)=O\bigg(\frac{n^{5/2}}{t^{5/2}}\bigg),\quad f_1''(t,n)=O\bigg(\frac{n^{5/2}}{t^{7/2}}\bigg).$$

So we have

$$(2.13)\qquad F'(t;n_1,n_2,n_3;i_1,i_2) = \frac{(2\pi)^{1/2}\alpha_3}{t^{1/2}} - \frac{\pi^{3/2}}{3\cdot 2^{3/2}}\,\frac{\beta_3}{t^{3/2}} + O\bigg(\frac{\max(n_1,n_2,n_3)^{5/2}}{t^{5/2}}\bigg),$$

where $\beta_3 := n_1^{3/2} + (-1)^{i_1} n_2^{3/2} + (-1)^{i_2} n_3^{3/2}$.

If $(i_1,i_2)=(0,0)$, then from (2.10) and Lemma 2.7 we get

$$(2.14)\qquad \int_T^{2T} S_{0,0}(t)\,dt \ll T^{5/4} \sum_{Z_j<n_j\le Y_j} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}(\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3})} \ll T^{5/4}Y^{1/4}\log^3 Y \ll T^{11/8+\varepsilon}.$$

Now suppose $(i_1,i_2)\neq(0,0)$. Without loss of generality, suppose $(i_1,i_2) = (0,1)$. By a splitting argument there exist $Z_j \le M_j < M_j' \le 2M_j \le Y_j$ $(1\le j\le 3)$ such that

$$(2.15)\qquad \log^{-3} T \int_T^{2T} S_{0,1}(t)\,dt \ll |I|,$$

where

$$I := \sum_{\substack{M_j<n_j\le M_j',\,1\le j\le 3\\ \alpha_3\neq 0}} \int_T^{2T} h(t,n_1)h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;0,1))\,dt.$$

Write $I = I_1 + I_2$, with

$$I_1 := \sum_{\substack{M_j<n_j\le M_j',\,1\le j\le 3\\ |\alpha_3|\ge 1/10}} \int_T^{2T} h(t,n_1)h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;0,1))\,dt,$$

$$I_2 := \sum_{\substack{M_j<n_j\le M_j',\,1\le j\le 3\\ 0<|\alpha_3|<1/10}} \int_T^{2T} h(t,n_1)h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;0,1))\,dt.$$

If $|\alpha_3|\ge 1/10$, then it is easily seen that $F'(t;n_1,n_2,n_3;0,1) \gg |\alpha_3|T^{-1/2}$ via (2.13). By (2.10) and Lemmas 2.7 and 2.6 we get

$$(2.16)\qquad I_1 \ll T^{5/4} \sum_{\substack{M_j<n_j\le M_j',\,1\le j\le 3\\ \alpha_3\neq 0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|} \ll T^{5/4+\varepsilon}Y^{1/4} \ll T^{11/8+\varepsilon}.$$

Now we estimate I_2. Suppose n_1, n_2, n_3 are three integers which satisfy $M_j < n_j \le M_j'$ $(1 \le j \le 3)$, $|\sqrt{n_1} + \sqrt{n_2} - \sqrt{n_3}\,| < 1/10$. We first estimate the integral

$$\int(n_1, n_2, n_3) = \int_T^{2T} h(t, n_1)h(t, n_2)h(t, n_3)\cos(F(t; n_1, n_2, n_3; 0, 1))\,dt.$$

Suppose $H \ge 100$ is a parameter to be determined later and divide the interval $[T, 2T]$ into two disjoint parts J_1 and J_2, where

$$J_1 = \{t \in [T, 2T] : |F'(t; n_1, n_2, n_3; 0, 1)| \le |\alpha_3|/HT^{1/2}\},$$
$$J_2 = \{t \in [T, 2T] : |F'(t; n_1, n_2, n_3; 0, 1)| > |\alpha_3|/HT^{1/2}\}.$$

Correspondingly, let

$$\int_{J_1} = \int_{J_1} h(t, n_1)h(t, n_2)h(t, n_3)\cos(F(t; n_1, n_2, n_3; 0, 1))\,dt,$$
$$\int_{J_2} = \int_{J_2} h(t, n_1)h(t, n_2)h(t, n_3)\cos(F(t; n_1, n_2, n_3; 0, 1))\,dt.$$

If J_1 is empty, then $J_2 = [T, 2T]$. By (2.10) and Lemma 2.7 we get

$$\int_{J_1} = 0, \tag{2.17}$$

$$\int_{J_2} \ll \frac{HT^{5/4}d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|}. \tag{2.18}$$

We suppose now that J_1 is not empty. Let

$$G(t) = t^{1/2}F'(t; n_1, n_2, n_3; 0, 1), \qquad T_1 = \inf J_1, \qquad T_2 = \sup J_1.$$

From $n_3^{1/2} = n_1^{1/2} + n_2^{1/2} - \alpha_3$ we get

$$\begin{aligned}\beta_3 &= n_1^{3/2} + n_2^{3/2} - n_3^{3/2}\\ &= -3(n_1n_2)^{1/2}(n_1^{1/2} + n_2^{1/2}) + 3(n_1^{1/2} + n_2^{1/2})^2\alpha_3 - 3(n_1^{1/2} + n_2^{1/2})\alpha_3^2 + \alpha_3^3,\end{aligned}$$

which implies

$$|\beta_3| \asymp (n_1n_2n_3)^{1/2} \tag{2.19}$$

if we notice $|\alpha_3| < 1/10$.

From (2.12), (2.13) and (2.19), we get

$$G'(t) \asymp \beta_3/T^2, \qquad \alpha_3/\beta_3 \asymp 1/T.$$

Thus from the relation $G(T_2) - G(T_1) = O(|\alpha_3|H^{-1})$ and the mean value theorem we get $|J_1| = T_2 - T_1 \ll T/H$, which combined with (2.10) implies

$$\int_{J_1} \ll \frac{T^{7/4}d(n_1)d(n_2)d(n_3)}{H(n_1n_2n_3)^{3/4}}. \tag{2.20}$$

Since $J_2 = [T, T_1) \cup (T_2, 2T]$, by (2.10) and Lemma 2.7 we get (2.18) again. From (2.18) and (2.20) we have

$$I_2 \ll \Sigma_3 + \Sigma_4, \tag{2.21}$$

where

$$\Sigma_3 = \frac{T^{7/4}}{H} \sum_{\substack{M_j < n_j \le M_j',\, 1\le j\le 3\\ \alpha_3/\beta_3 \asymp 1/T}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}},$$

$$\Sigma_4 = HT^{5/4} \sum_{\substack{M_j < n_j \le M_j',\, 1\le j\le 3\\ \alpha_3 \ne 0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|}.$$

Let $M = \max(M_1, M_2, M_3)$; then $T^{1/3-\varepsilon} \ll M \ll Y$. By Lemma 2.4 we get

$$\begin{aligned}
\Sigma_3 &\ll \frac{T^{7/4+\varepsilon}}{H(M_1M_2M_3)^{3/4}}\, \mathcal{A}(M_1, M_2, M_3; (M_1M_2M_3)^{1/2}T^{-1}) \\
&\ll \frac{T^{7/4+\varepsilon}}{H(M_1M_2M_3)^{3/4}}\, ((M_1M_2M_3)^{3/2}T^{-1}M^{-1/2} + (M_1M_2M_3)^{1/2}M^{-1/2}) \\
&\ll T^{3/4+\varepsilon}H^{-1}(M_1M_2M_3)^{3/4}M^{-1/2} + T^{7/4+\varepsilon}H^{-1}(M_1M_2M_3)^{-1/4}M^{-1/2} \\
&\ll T^{3/4+\varepsilon}Y^{7/4}H^{-1} + T^{7/4-1/6+\varepsilon}M^{-1/2} \\
&\ll T^{3/4+\varepsilon}Y^{7/4}H^{-1} + T^{17/12+\varepsilon}.
\end{aligned}$$

By Lemma 2.6 we have

$$\Sigma_4 \ll HT^{5/4+\varepsilon}Y^{1/4}.$$

Take $H = \max(Y^{3/4}T^{-1/4}, 100)$; we get

$$I_2 \ll YT^{1+\varepsilon} + T^{17/12+\varepsilon},$$

which combined with (2.15) and (2.16) gives

$$\int_T^{2T} S_{0,1}(t)\, dt \ll YT^{1+\varepsilon} + T^{17/12+\varepsilon}. \tag{2.22}$$

For $(i_1, i_2) = (1, 0), (1, 1)$, we can get the same estimates. This completes the proof of Lemma 2.8. ∎

3. Beginning of proof. Suppose $T > 100$ is a large real number. We shall evaluate the integral $\int_T^{2T} E^3(t)\, dt$. Let $y := T^{1/2}$. For any $T \le t \le 2T$, define

$$\mathcal{R}_1(t) := \frac{1}{\sqrt{2}} \sum_{n\le y} h(t, n) \cos(f(t, n)), \qquad \mathcal{R}_2(t) := E(t) - \mathcal{R}_1(t).$$

Define the following integrals:

$$\mathcal{I}_1(T) := \int_T^{2T} \mathcal{R}_1^3(t)\,dt, \tag{3.1}$$

$$\mathcal{I}_2(T) := \int_T^{2T} \mathcal{R}_1^2(t)\mathcal{R}_2(t)\,dt, \tag{3.2}$$

$$\mathcal{I}_3(T) := \int_T^{2T} \mathcal{R}_1(t)\mathcal{R}_2^2(t)\,dt, \tag{3.3}$$

$$\mathcal{I}_4(T) := \int_T^{2T} \mathcal{R}_2^3(t)\,dt. \tag{3.4}$$

We shall evaluate $\mathcal{I}_1(T)$ in Section 5 and estimate $\mathcal{I}_2(T), \mathcal{I}_3(T), \mathcal{I}_4(T)$ in Section 4 and Section 6.

4. Estimates of $\mathcal{I}_3(T)$ and $\mathcal{I}_4(T)$

4.1. *Higher-power moments of* $\mathcal{R}_1(t)$. In this subsection we study the higher-power moments of $\mathcal{R}_1(t)$. Since the proof is very similar to those of Theorems 13.8 and 13.9 of Ivić [5], we only mention the important points. From Huxley [3], we have

$$\mathcal{R}_1(t) \ll T^{72/227+\varepsilon}. \tag{4.1}$$

Suppose $T < t_1 < \cdots < t_N \le 2T$ are points which satisfy $|t_r - t_s| \ge V$ $(r \ne s \le N)$, $T^{1/4} \ll V \ll T^{72/227+\varepsilon}$, and $|\mathcal{R}_1(t_r)| \gg V$ for $r = 1, \ldots, N$. We shall give an upper bound of N.

Suppose $M \le y/2$. Take $\xi = \{\xi_n\}_{n=1}^{\infty}$ with $\xi_n = (-1)^n d(n) n^{-3/4}$ for $M < n \le 2M$ and zero otherwise, and let $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$ with

$$\varphi_{r,n} = n^{1/4} t^{-1/4} (t/2\pi n + 1/4)^{-1/4} g^{-1}(t,n) e(f(t,n))$$

for $M < n \le 2M$ and zero otherwise. Divide $[T, 2T]$ into subintervals of length not exceeding $T_0 \ge V$. Let N_0 denote the number of t_r's lying in an interval of length not exceeding T_0. Then

$$N \ll N_0(1 + T/T_0). \tag{4.2}$$

By (A.40) of Ivić [5] we get

$$\begin{aligned} N_0 V^2 &\ll T^{1/2} \log T \max_{M \le y/2} \sum_{r \le N_0} \Big| \sum_{M < n \le 2M} h(t,n) t^{-1/4} e(f(t,n)) \Big|^2 \\ &\ll T^{1/2} \log T \max_{M \le y/2} \max_{r \le N_0} \|\xi\|^2 \sum_{s \le N_0} |(\varphi_r, \varphi_s)|, \end{aligned} \tag{4.3}$$

where

$$\|\xi\|^2 := \sum_{M<n\le 2M} d^2(n)n^{-3/2} \ll M^{-1/2}\log^3 M,$$

$$\begin{aligned}(\varphi_r,\varphi_s) &:= \sum_{M<n\le 2M} n^{2/4}\left(\frac{t_r}{2\pi n}+\frac{1}{4}\right)^{-1/4}\left(\frac{t_s}{2\pi n}+\frac{1}{4}\right)^{-1/4} g^{-1}(t_r,n) \\ &\quad\times g^{-1}(t_s,n)(t_rt_s)^{-1/4}e(f(t_r,n)-f(t_s,n)) \\ &= \sum_{M<n\le 2M} G(n;r,s)e(F(n;r,s)),\end{aligned}$$

say.

It is easily seen that for any $r,s\le N_0$, $G(n;r,s)$ is a product of monotonic functions of n and $G(n;r,s)\ll 1$. The contribution of the terms with $r=s$ is

$$\ll T^{1/2}\log T \max_{M\le y/2} M^{1/2}\log^3 M \ll (Ty)^{1/2}\log^4 T. \tag{4.4}$$

By partial summation, the contribution of the terms with $r\ne s$ is

$$\begin{aligned}&\ll T^{1/2}\log T \max_{M\le y/2}\max_{r\le N_0}\frac{\log^3 M}{M^{1/2}}\sum_{s\le N_0,\, s\ne r}\Big|\sum_{M<n\le 2M} G(n;r,s)e(F(n;r,s))\Big| \\ &\ll T^{1/2}\log T \max_{M\le y/2}\max_{r\le N_0}\frac{\log^3 M}{M^{1/2}}\sum_{s\le N_0,\, s\ne r}\Big|\sum_{n\in I(r,s)} e(F(n;r,s))\Big|,\end{aligned} \tag{4.5}$$

where $I(r,s)$ is a subinterval of $[M,2M]$. It is easy to check that

$$|F^{(j)}(x;r,s)| \asymp |t_r^{1/2}-t_s^{1/2}|M^{1/2-j}, \qquad j=0,1,\ldots,6.$$

So the exponential sum $S=\sum_{n\in I(r,s)} e(F(n;r,s))$ can be estimated by the theory of exponent pairs. Using the first derivative test to estimate S for $|F^{(j)}(x;r,s)|\le 1/2$ and the exponent pair $(4/18, 11/18)$ to estimate S for $|F^{(j)}(x;r,s)|>1/2$, we get

$$\begin{aligned}T^{1/2}\log T \max_{M\le y/2}\max_{r\le N_0}\frac{\log^3 M}{M^{1/2}}\sum_{s\le N_0,\, s\ne r}\Big|\sum_{n\in I(r,s)} e(F(n;r,s))\Big| \\ \ll TV^{-1}\log^5 T + N_0T_0^{4/18}T^{7/18}\log^4 T,\end{aligned}$$

which combined with (4.3)–(4.5) gives

$$N_0V^2\log^{-5} T \ll (Ty)^{1/2} + TV^{-1} + N_0T_0^{4/18}T^{7/18}. \tag{4.6}$$

Choose $T_0=V^9T^{-7/4}\log^{-30} T$; then $T_0\gg V$ and (4.6) reduces to

$$N_0 \ll (Ty)^{1/2}V^{-2}\log^5 T + TV^{-3}\log^5 T,$$

which combined with (4.2) gives

$$N\log^{-35} T \ll (Ty)^{1/2}V^{-2} + TV^{-3} + T^{13/4}y^{1/2}V^{-11} + T^{15/4}V^{-12}. \tag{4.7}$$

Now we estimate the integral $\int_T^{2T} |\mathcal{R}_1(t)|^A\,dt$, where $A > 2$ is a fixed real number. Similarly to (13.70) of Ivić [5] we may write

$$\int_T^{2T} |\mathcal{R}_1(t)|^A\,dt \ll T^{(4+A)/4}\log T + \sum_V V \sum_{r\le N_V} |\mathcal{R}_1(t_r)|^A, \tag{4.8}$$

where $T^{1/4} \le V = 2^m \le T^{72/227+\varepsilon}$, $V < |\mathcal{R}_1(t_r)| \le 2V$ $(r = 1,\dots,N_V)$ and $|t_r - t_s| \ge V$ for $r \ne s \le N = N_V$. If $A < 10$, then by (4.1) and (4.7) we have

$$\begin{aligned} V\sum_{r\le N_V} |\mathcal{R}_1(t_r)|^A &\ll N_V V^{A+1} \\ &\ll (Ty)^{1/2}T^{72(A-1)/227+\varepsilon} + T^{1+72(A-2)/227+\varepsilon} \\ &\quad + T^{(3+A)/4}y^{1/2}\log^{40} T + T^{1+A/4}\log^{40} T \\ &\ll T^{1+A/4+\varepsilon} \end{aligned} \tag{4.9}$$

for any $2 \le A \le A_0 := 515/61$.

Thus for $2 \le A \le A_0$ we have

$$\int_T^{2T} |\mathcal{R}_1(t)|^A\,dt \ll T^{1+A/4+\varepsilon}. \tag{4.10}$$

4.2. *Higher-power moments of* $\mathcal{R}_2(t)$. We first consider the mean square of $\mathcal{R}_2(t)$. By Lemma 2.1 (take $N = T/\pi$) we have

$$\begin{aligned} \mathcal{R}_2(t) &= \mathcal{R}_2^*(t) + \Sigma_2(t) + O(\log^2 t), \\ \mathcal{R}_2^*(t) &:= \frac{1}{\sqrt{2}} \sum_{y<n\le T/\pi} h(t,n)\cos(f(t,n)). \end{aligned} \tag{4.11}$$

Hence we get

$$\int_T^{2T} \mathcal{R}_2^2(t)\,dt \ll \int_T^{2T} |\mathcal{R}_2^*(t)|^2\,dt + \int_T^{2T} |\Sigma_2(t)|^2\,dt + T\log^4 T. \tag{4.12}$$

We have the estimate

$$\int_T^{2T} |\Sigma_2(t)|^2\,dt \ll T\log^4 T, \tag{4.13}$$

which is (15.61) of Ivić [5].

For $m \ne n$, it is easy to check that $|f'(t,m) - f'(t,n)| \gg |\sqrt{n} - \sqrt{m}|/T^{1/2}$. Thus from (2.10) and Lemma 2.7 we have

$$\begin{aligned} \int_T^{2T} |\mathcal{R}_2^*(t)|^2\,dt &\ll \sum_{y<n\le T/\pi} \int_T^{2T} h(t,n)^2\,dt \\ &\quad + \sum_{y<m<n\le T/\pi} \left| \int_T^{2T} h(t,n)h(t,m)e(f(t,n) - f(t,m))\,dt \right| \end{aligned} \tag{4.14}$$

$$+\sum_{y<m,n\le T/\pi}\Big|\int_T^{2T} h(t,n)h(t,m)e(f(t,n)+f(t,m))\,dt\Big|$$
$$\ll T^{3/2}\sum_{y<n\le T/\pi}\frac{d^2(n)}{n^{3/2}}+T\sum_{m<n\le T/\pi}\frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n}-\sqrt{m})}$$
$$\ll T^{3/2}y^{-1/2}\log^3 T,$$

which combined with (4.12) and (4.13) gives

$$\int_T^{2T}\mathcal{R}_2^2(t)\,dt \ll T^{3/2}y^{-1/2}\log^3 T. \tag{4.15}$$

Ivić [5, Theorem 15.7] proved that

$$\int_1^T |E(t)|^A\,dt \ll T^{1+A/4+\varepsilon} \tag{4.16}$$

for $0<A<35/4$. From (4.10) and (4.16) we deduce that for any $2\le A\le A_0=515/61$,

$$\int_1^T |\mathcal{R}_2(t)|^A\,dt \ll \int_1^T |E(t)|^A\,dt+\int_1^T|\mathcal{R}_1(t)|^A\,dt \ll T^{1+A/4+\varepsilon}. \tag{4.17}$$

For any $2<A<A_0$, from (4.15), (4.17) and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T}|\mathcal{R}_2(t)|^A\,dt &= \int_T^{2T}|\mathcal{R}_2(t)|^{2(A_0-A)/(A_0-2)+A_0(A-2)/(A_0-2)}\,dt \\ &\ll \Big(\int_T^{2T}\mathcal{R}_2^2(t)\,dt\Big)^{(A_0-A)/(A_0-2)}\Big(\int_T^{2T}|\mathcal{R}_2(t)|^{A_0}\,dt\Big)^{(A-2)/(A_0-2)} \\ &\ll T^{1+A/4+\varepsilon}y^{-(A_0-A)/2(A_0-2)},\end{aligned} \tag{4.18}$$

which implies

$$\mathcal{I}_4(T)\ll T^{7/4+\varepsilon}y^{-(A_0-3)/2(A_0-2)}. \tag{4.19}$$

From (4.10), (4.18) and Hölder's inequality we get

$$\begin{aligned}\mathcal{I}_3(T) &\ll \int_T^{2T}|\mathcal{R}_1(t)\mathcal{R}_2^2(t)|\,dt \\ &\ll \Big(\int_T^{2T}|\mathcal{R}_1(t)|^{A_0}\,dt\Big)^{1/A_0}\Big(\int_T^{2T}|\mathcal{R}_2(t)|^{2A_0/(A_0-1)}\,dt\Big)^{(A_0-1)/A_0} \\ &\ll T^{7/4+\varepsilon}y^{-(A_0-3)/2(A_0-2)}.\end{aligned} \tag{4.20}$$

5. The evaluation of $\mathcal{I}_1(T)$. Let $y_0 := T^{1/3-\varepsilon}$. We write $\mathcal{R}_1(t) = \mathcal{R}_{11}(t) + \mathcal{R}_{12}(t)$, where

$$\mathcal{R}_{11}(t) := \frac{1}{\sqrt{2}} \sum_{n \le y_0} h(t,n)\cos(f(t,n)),$$

$$\mathcal{R}_{12}(t) := \frac{1}{\sqrt{2}} \sum_{y_0 < n \le y} h(t,n)\cos(f(t,n)).$$

5.1. *On the integral $\int_T^{2T} \mathcal{R}_{11}^3(t)\,dt$.* By the elementary formula

$$\cos a \cos b \cos c = \frac{1}{4} \sum_{(i_1,i_2)\in\{0,1\}^2} \cos(a + (-1)^{i_1} b + (-1)^{i_2} c), \tag{5.1}$$

we can write

$$\begin{aligned}
\mathcal{R}_{11}^3(t) &= \frac{1}{2^{3/2}} \sum_{n_1 \le y_0} \sum_{n_2 \le y_0} \sum_{n_3 \le y_0} h(t,n_1)h(t,n_2)h(t,n_3) \prod_{j=1}^{3} \cos(f(t,n_j)) \\
&= \frac{1}{2^{7/2}} \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{n_1 \le y_0} \sum_{n_2 \le y_0} \sum_{n_3 \le y_0} h(t,n_1)h(t,n_2)h(t,n_3) \\
&\quad \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)) \\
&= \frac{1}{2^{7/2}} (S_1(t) + S_2(t)),
\end{aligned}$$

where

$$\begin{aligned}
S_1(t) &:= \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{\substack{n_j \le y_0,\, 1\le j\le 3 \\ \alpha_3 = 0}} h(t,n_1)h(t,n_2)h(t,n_3) \\
&\qquad \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)), \\
S_2(t) &:= \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{\substack{n_j \le y_0,\, 1\le j\le 3 \\ \alpha_3 \ne 0}} h(t,n_1)h(t,n_2)h(t,n_3) \\
&\qquad \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)).
\end{aligned}$$

We first consider the contribution of $S_1(t)$. It is easy to see that $\alpha_3 = 0$ implies $(i_1,i_2) = (0,1)$ or $(1,0)$ or $(1,1)$. Let

$$S_1(t;i_1,i_2) := \sum_{\substack{n_j \le y_0,\, 1\le j\le 3 \\ \alpha_3 = 0}} h(t,n_1)h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;i_1,i_2)).$$

We consider the case $(i_1,i_2) = (0,1)$. Suppose $n_j \le y_0$ $(j = 1,2,3)$ is such that $\alpha_3 = 0$ for $(i_1,i_2) = (0,1)$, namely, $\sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3}$. From (2.11) we have

$$\begin{aligned}
&\cos(F(t;n_1,n_2,n_3;0,1)) \\
&\qquad = \cos\left(-\frac{\pi}{4} + O\left(\frac{n_3^{3/2}}{t^{1/2}}\right)\right) = 2^{-1/2} + O\left(\frac{n_3^{3/2}}{t^{1/2}}\right).
\end{aligned} \tag{5.2}$$

From (2.10), (5.2) and Lemmas 2.2 and 2.3 we get

$$(5.3)\quad \int_T^{2T} S_1(t;0,1)\,dt$$

$$= \sum_{\substack{n_1,n_2,n_3\le y_0\\ \sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}}} \int_T^{2T} h(t,n_1)h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;0,1))\,dt$$

$$= \frac{2^{9/4}}{\pi^{3/4}} \sum_{\substack{n_1,n_2,n_3\le y_0\\ \sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}}} \frac{(-1)^{n_1+n_2+n_3}d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}}$$

$$\times \int_T^{2T} t^{3/4}\left(1+O\left(\frac{n_3}{T}\right)\right)\left(2^{-1/2}+O\left(\frac{n_3^{3/2}}{T^{1/2}}\right)\right)dt$$

$$= \frac{2^{7/4}}{\pi^{3/4}} \sum_{\substack{n_1,n_2,n_3\le y_0\\ \sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}}} \frac{(-1)^{n_1+n_2+n_3}d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}} \int_T^{2T} t^{3/4}\left(1+O\left(\frac{n_3^{3/2}}{T^{1/2}}\right)\right)dt$$

$$= \frac{2^{7/4}}{\pi^{3/4}} \sum_{\substack{n_1,n_2,n_3\le y_0\\ \sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}}} \frac{(-1)^{n_1+n_2+n_3}d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}} \int_T^{2T} t^{3/4}\,dt+O(T^{5/4}H_1(y_0))$$

$$= \frac{2^{7/4}c_1}{\pi^{3/4}} \int_T^{2T} t^{3/4}\,dt + O(T^{7/4+\varepsilon}y_0^{-1} + T^{5/4+\varepsilon}y_0^{1/2})$$

$$= \frac{2^{7/4}c_1}{\pi^{3/4}} \int_T^{2T} t^{3/4}\,dt + O(T^{17/12+\varepsilon}).$$

We can get the same result for $S_1(t;1,0), S_1(t;1,1)$. Thus

$$(5.4)\qquad \int_T^{2T} S_1(t)\,dt = \frac{3\cdot 2^{7/4}c_1}{\pi^{3/4}} \int_T^{2T} t^{3/4}\,dt + O(T^{17/12+\varepsilon}).$$

Now we consider the contribution of $S_2(t)$. From Lemma 2.5 and (2.13) we get $|F'(t;n_1,n_2,n_3;i_1,i_2)| \gg |\alpha_3|/T^{1/2}$ if we notice $y_0 = T^{1/3-\varepsilon}$. By Lemmas 2.7 and 2.6 we have

$$(5.5)\quad \int_T^{2T} S_2(t)\,dt \ll T^{5/4} \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{\substack{n_1,n_2,n_3\le y_0\\ \alpha_3\ne 0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|}$$

$$= T^{5/4} \sum_{(i_1,i_2)\in\{0,1\}^2} H(y_0;i_1,i_2) \ll T^{5/4+\varepsilon}y_0^{1/4} \ll T^{4/3+\varepsilon}.$$

From (5.4) and (5.5) we get

$$\int_T^{2T} \mathcal{R}_{11}^3(t)\,dt = \frac{3c_1}{2^{7/4}\pi^{3/4}} \int_T^{2T} t^{3/4}\,dt + O(T^{17/12+\varepsilon}). \tag{5.6}$$

5.2. *On the integral* $\int_T^{2T} \mathcal{R}_{11}^2(t)\mathcal{R}_{12}(t)\,dt$. By (5.1) we can write

$$\begin{aligned} \mathcal{R}_{11}^2(t)\mathcal{R}_{12}(t) &= \frac{1}{2^{7/2}}\,(S_3(t) + S_4(t) + S_5(t)),\\ S_3(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{y_0<n_1\le y} &\sum_{\substack{n_2,n_3\le y_0\\ \alpha_3=0}} h(t,n_1)h(t,n_2)h(t,n_3)\\ &\times \cos(F(t;n_1,n_2,n_3;i_1,i_2)),\\ S_4(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{y_0<n_1\le 50y_0} &\sum_{\substack{n_2,n_3\le y_0\\ \alpha_3\ne 0}} h(t,n_1)h(t,n_2)h(t,n_3)\\ &\times \cos(F(t;n_1,n_2,n_3;i_1,i_2)),\\ S_5(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{50y_0<n_1\le y} &\sum_{\substack{n_2,n_3\le y_0\\ \alpha_3\ne 0}} h(t,n_1)h(t,n_2)h(t,n_3)\\ &\times \cos(F(t;n_1,n_2,n_3;i_1,i_2)). \end{aligned}$$

We first consider the contribution of $S_3(t)$. Since $n_2, n_3 \le y_0 < n_1 \le y$, the condition $\alpha_3 = 0$ implies $(i_1, i_2) = (1, 1)$ and $n_1 \le 4y_0$. So by (2.10) and Lemma 2.2 we get

$$\begin{aligned} \int_T^{2T} S_3(t)\,dt &\ll \sum_{\substack{\sqrt{n_2}+\sqrt{n_3}=\sqrt{n_1}\\ n_1>y_0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}} \int_T^{2T} t^{3/4}\,dt\\ &\ll T^{7/4}|c_1 - c_1(y_0)| \ll T^{7/4+\varepsilon} y_0^{-1} \ll T^{17/12+\varepsilon}. \end{aligned} \tag{5.7}$$

Concerning the contribution of $S_4(t)$, similarly to (5.5), by Lemmas 2.7 and 2.6 we get

$$\begin{aligned} \int_T^{2T} S_4(t)\,dt &\ll T^{5/4} \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{y_0<n_1\le 50y_0} \sum_{\substack{n_2,n_3\le y_0\\ \alpha_3\ne 0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}|\alpha_3|}\\ &\ll T^{5/4} \sum_{(i_1,i_2)\in\{0,1\}^2} H(50y_0; i_1, i_2) \ll T^{5/4+\varepsilon} y_0^{1/4} \ll T^{4/3+\varepsilon}. \end{aligned} \tag{5.8}$$

Now we consider the contribution of $S_5(t)$. Since $n_1 > 50y_0$, $n_2, n_3 \le y_0$, we have $|F'(t; n_1, n_2, n_3; i_1, i_2)| \gg n_1^{1/2} T^{-1/2}$. Thus from (2.10) and Lemma 2.7

we get

$$\int_T^{2T} S_5(t)\,dt \ll T^{5/4} \sum_{n_1>50y_0} \sum_{n_2,n_3\le y_0} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}n_1^{1/2}} \ll T^{5/4+\varepsilon}y_0^{1/4} \ll T^{4/3+\varepsilon}. \tag{5.9}$$

From (5.7)–(5.9) we deduce

$$\int_T^{2T} \mathcal{R}_{11}^2(t)\mathcal{R}_{12}(t)\,dt \ll T^{17/12+\varepsilon}. \tag{5.10}$$

5.3. *On the integrals* $\int_T^{2T} \mathcal{R}_{11}(t)\mathcal{R}_{12}^2(t)\,dt$ *and* $\int_T^{2T} \mathcal{R}_{12}^3(t)\,dt$. By (5.1) we can write

$$\mathcal{R}_{11}(t)\mathcal{R}_{12}^2(t) = \frac{1}{2^{7/2}}\,(S_6(t)+S_7(t)),$$

$$S_6(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{n_1\le y_0} \sum_{\substack{y_0<n_2,n_3\le y\\ \alpha_3=0}} h(t,n_1)h(t,n_2)h(t,n_3) \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)),$$

$$S_7(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{n_1\le y_0} \sum_{\substack{y_0<n_2,n_3\le y\\ \alpha_3\ne0}} h(t,n_1)h(t,n_2)h(t,n_3) \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)).$$

By (2.10) and Lemma 2.2 we have

$$\int_T^{2T} S_6(t)\,dt \ll T^{7/4} \sum_{\substack{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}\\ n_3>y_0}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}} \ll T^{7/4}|c_1-c_1(y_0)| \ll T^{17/12+\varepsilon}.$$

By Lemma 2.8 we get

$$\int_T^{2T} S_7(t)\,dt \ll T^{1+\varepsilon}y + T^{17/12+\varepsilon}.$$

Thus

$$\int_T^{2T} \mathcal{R}_{11}(t)\mathcal{R}_{12}^2(t)\,dt \ll T^{1+\varepsilon}y + T^{17/12+\varepsilon}. \tag{5.11}$$

Similarly,

$$\int_T^{2T} \mathcal{R}_{12}^3(t)\,dt \ll T^{1+\varepsilon}y + T^{17/12+\varepsilon}. \tag{5.12}$$

5.4. *The asymptotic formula for $\mathcal{I}_1(T)$.* From (5.6) and (5.10)–(5.12) and by writing

$$\mathcal{R}_1^3(t) = \mathcal{R}_{11}^3(t) + 3\mathcal{R}_{11}^2(t)\mathcal{R}_{12}(t) + 3\mathcal{R}_{11}(t)\mathcal{R}_{12}^2(t) + \mathcal{R}_{12}^3(t)$$

we get

$$\int_T^{2T} \mathcal{R}_1^3(t)\,dt = \frac{3c_1}{2^{7/4}\pi^{3/4}} \int_T^{2T} t^{3/4}\,dt + O(T^{1+\varepsilon}y + T^{17/12+\varepsilon}). \tag{5.13}$$

6. Estimate of $\mathcal{I}_2(T)$. We first estimate the integral $\int_T^{2T} \mathcal{R}_1^2(t)\mathcal{R}_2^*(t)\,dt$. By (5.1) again we can write

$$\mathcal{R}_1^2(t)\mathcal{R}_2^*(t) = \frac{1}{2^{7/2}}\,(S_8(t) + S_9(t) + S_{10}(t)),$$

$$S_8(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{y<n_1\le T/\pi} \sum_{\substack{n_2,n_3\le y \\ \alpha_3=0}} h(t,n_1)h(t,n_2)h(t,n_3) \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)),$$

$$S_9(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \sum_{y<n_1\le 50y} \sum_{\substack{y_0<\max(n_2,n_3)\le y \\ \alpha_3\ne 0}} h(t,n_1)h(t,n_2)h(t,n_3) \times \cos(F(t;n_1,n_2,n_3;i_1,i_2)),$$

$$S_{10}(t) := \sum_{(i_1,i_2)\in\{0,1\}^2} \Bigg(\sum_{y<n_1\le 50y} \sum_{\substack{\max(n_2,n_3)\le y_0 \\ \alpha_3\ne 0}} + \sum_{50y<n_1\le T/\pi} \sum_{\substack{n_2,n_3\le y \\ \alpha_3\ne 0}} \Bigg) h(t,n_1) \times h(t,n_2)h(t,n_3)\cos(F(t;n_1,n_2,n_3;i_1,i_2)).$$

We first consider the contribution of $S_8(t)$. Since $n_2, n_3 \le y < n_1 \le T/\pi$, the condition $\alpha_3 = 0$ implies $(i_1,i_2) = (1,1)$ and $n_1 \le 4y$. By (2.10) and Lemma 2.2 we get

$$\int_T^{2T} S_8(t)\,dt \ll T^{7/4} \sum_{\substack{y<n_1\le 4y,\ n_2,n_3\le y \\ \sqrt{n_1}=\sqrt{n_2}+\sqrt{n_3}}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}} \ll T^{7/4}|c_1 - c_1(y)| \ll T^{7/4+\varepsilon}y^{-1} \ll T^{4/3+\varepsilon}. \tag{6.1}$$

By Lemma 2.8 we have

$$\int_T^{2T} S_9(t)\,dt \ll T^{1+\varepsilon}y + T^{17/12+\varepsilon}. \tag{6.2}$$

Similarly to (5.9), from (2.10) and Lemma 2.7 we have

$$\int_T^{2T} S_{10}(t)\,dt \ll T^{5/4} \sum_{n_1>50y} \sum_{n_2,n_3\le y} \frac{d(n_1)d(n_2)d(n_3)}{(n_1n_2n_3)^{3/4}n_1^{1/2}} \ll T^{5/4+\varepsilon}y^{1/4} \ll T^{11/8+\varepsilon}. \tag{6.3}$$

From (6.1)–(6.3) we have

$$\int_T^{2T} \mathcal{R}_1^2(t)\mathcal{R}_2^*(t)\,dt \ll T^{1+\varepsilon}y + T^{17/12+\varepsilon}. \tag{6.4}$$

From (4.10), (4.13) and Cauchy's inequality we get

$$\int_T^{2T} |\mathcal{R}_1(t)|^2|\Sigma_2(t)|\,dt \ll \Big(\int_T^{2T} |\mathcal{R}_1(t)|^4\,dt\Big)^{1/2}\Big(\int_T^{2T} |\Sigma_2(t)|^2\,dt\Big)^{1/2} \ll T^{3/2+\varepsilon}, \tag{6.5}$$

which combined with (4.11) and (6.4) yields

$$\mathcal{I}_2(T) \ll \int_T^{2T} \mathcal{R}_1^2(t)\mathcal{R}_2(t)\,dt \ll T^{1+\varepsilon}y + T^{3/2+\varepsilon}. \tag{6.6}$$

7. Completion of proof. We write

$$E^3(t) = \mathcal{R}_1^3(t) + 3\mathcal{R}_1^2(t)\mathcal{R}_2(t) + 3\mathcal{R}_1(t)\mathcal{R}_2^2(t) + \mathcal{R}_2^3(t).$$

So from (4.19), (4.20), (5.13), (6.6) we get

$$\begin{aligned}\int_T^{2T} E^3(t)\,dt &= \mathcal{I}_1(T) + 3\mathcal{I}_2(T) + 3\mathcal{I}_3(T) + \mathcal{I}_4(T)\\ &= \frac{3c_1}{2^{7/4}\pi^{3/4}}\int_T^{2T} t^{3/4}\,dt\\ &\quad + O(T^{7/4+\varepsilon}y^{-(A_0-3)/2(A_0-2)} + T^{1+\varepsilon}y + T^{3/2+\varepsilon})\\ &= \frac{3c_1}{2^{7/4}\pi^{3/4}}\int_T^{2T} t^{3/4}\,dt + O(T^{7/4-83/393+\varepsilon}).\end{aligned} \tag{7.1}$$

Applying (7.1) repeatedly to the intervals $[T/2^{j+1}, T/2^j]$ ($j\ge 0$) and summing we get (1.7). ∎

References

[1] F. V. Atkinson, *The mean-value of the Riemann zeta function*, Acta Math. 81 (1949), 353–376.

[2] D. R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. 60 (1992), 389–415.

[3] M. N. Huxley, *Exponential sums and lattice points II*, Proc. London Math. Soc. 66 (1993), 279–301.

[4] A. Ivić, *On some problems involving the mean square of* $\zeta(1/2+it)$, Bull. Cl. Sci. Math. Nat. Sci. Math. 23 (1998), 71–76.

[5] —, *The Riemann Zeta-Function*, Wiley, 1985.

[6] M. Jutila, *On a formula of Atkinson*, in: Topics in Classical Number Theory, Vols. I, II (Budapest, 1981), Colloq. Math. Soc. János Bolyai 34, North-Holland, Amsterdam, 1984, 807–823.

[7] —, *Riemann's zeta-function and the divisor problem II*, Ark. Mat. 31 (1993), 61–70.

[8] T. Meurman, *On the mean square of the Riemann zeta-function*, Quart. J. Math. Oxford Ser. 38 (1987), 337–343.

[9] K. M. Tsang, *Higher-power moments of* $\Delta(x)$, $E(t)$ *and* $P(x)$, Proc. London Math. Soc. (3) 65 (1992), 65–84.

[10] W. G. Zhai, *On higher-power moments of* $\Delta(x)$, Acta Arith. 112 (2004), 367–395.

[11] —, *On higher-power moments of* $\Delta(x)$ *(II)*, ibid. 114 (2004), 35–54.

Department of Mathematics
Shandong Normal University
Jinan 250014, Shandong, P.R. China
E-mail: zhaiwg@hotmail.com

Received on 1.9.2003
and in revised form on 23.2.2004 (4610)