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A conjecture concerning the exponential diophantine equation $a^x + b^y = c^z$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of all integers, positive integers and rational numbers respectively. Let a, b, c be fixed coprime positive integers with $\min(a, b, c) > 1$. In 1933, Mahler [10] used his p-adic analogue of the method of Thue–Siegel to prove that the equation

$$(1) \qquad a^x + b^y = c^z, \quad x, y, z \in \mathbb{Z},$$

has only finitely many solutions (x, y, z). His method is ineffective. An effective result for solutions of (1) was given by Gel'fond [3]. In 1994, Terai [12] conjectured that if (1) has a solution $(x, y, z) = (p, q, r)$ with $\min(p, q, r) > 1$, then (1) has only one solution. In 1999, Cao [1] showed that Terai's conjecture is clearly false. He suggested that the condition $\max(a, b, c) > 7$ should be added to the hypotheses of the conjecture. He used the term "Terai–Jeśmanowicz conjecture" for the resulting statement. However, the Terai–Jeśmanowicz conjecture is also false. For example, if $a = 2$, $b = 2^n - 1$, $c = 2^n + 1$, where n is a positive integer with $n > 2$, then a, b, c satisfy $\max(a, b, c) > 7$ and $a^{n+2} + b^2 = c^2$, but (1) has two solutions $(x, y, z) = (1, 1, 1)$ and $(n + 2, 2, 2)$. This implies that there exist infinitely many counterexamples to the Terai–Jeśmanowicz conjecture. On the other hand, heuristics indicate that the following statements are true.

Conjecture. The equation (1) has at most one solution (x, y, z) with $\min(x, y, z) > 1$.

The above mentioned conjecture was first proposed by the author [6] for primes a, b and c. It was proved for some special cases. But, in general, the problem has not been solved yet.

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Most of the results concerning the above conjecture deal with the case that (1) has the solution $(x, y, z) = (2, 2, r)$, where r is an odd integer with $r > 1$ (see [1], [7]–[9], [12]–[16]). Then from (1) we get $2 \nmid c$ and $a \not\equiv b \pmod 2$. We may assume that $2 \nmid a$ and $2 \mid b$. Thus, a, b and c satisfy

$$a^2 + b^2 = c^r, \quad \gcd(a, b) = 1,\ 2 \nmid a,\ 2 \mid b,\ r > 1,\ 2 \nmid r. \tag{2}$$

In this connection, Terai [15] proved that if $a \geq 41b$, $a \equiv 3 \pmod 8$, $b \equiv 2 \pmod 4$ and $(b/a) = -1$, where $(*/*)$ denotes the Jacobi symbol, then (1) has only the solution $(x, y, z) = (2, 2, r)$. In this paper we prove the following general result:

Theorem. *Let a, b, c be positive integers satisfying* (2). *If* $a > b$, $a \equiv 3 \pmod 4$, $b \equiv 2 \pmod 4$ *and*

$$a/b > (e^{r/1856} - 1)^{-1/2}, \tag{3}$$

then (1) *has only the solution* $(x, y, z) = (2, 2, r)$.

Since $r \geq 3$, our theorem has the following immediate corollary.

Corollary. *Let a, b, c be positive integers satisfying* (2). *If* $a > b$, $a \equiv 3 \pmod 4$, $b \equiv 2 \pmod 4$ *and either* $a \geq 25b$ *or* $r \geq 1287$, *then* (1) *has only the solution* $(x, y, z) = (2, 2, r)$.

2. Preliminaries

Lemma 1 ([11, pp. 12–13]). *Every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^2, \quad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1,\ 2 \mid Y, \tag{4}$$

can be expressed as

$$X = A^2 - B^2, \quad Y = 2AB, \quad Z = A^2 + B^2, \tag{5}$$

where A, B are positive integers satisfying

$$A > B, \quad \gcd(A, B) = 1,\ 2 \mid AB. \tag{6}$$

Lemma 2 ([11, pp. 122–123]). *Let n be a positive integer with $n > 1$. Then every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^n, \quad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1,\ 2 \mid Y, \tag{7}$$

can be expressed as

$$\begin{aligned} X &= \left| \sum_{i=0}^{[n/2]} \binom{n}{2i} A^{n-2i} (-B^2)^i \right|, \\ Y &= B \left| \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} A^{n-2i-1} (-B^2)^i \right|, \\ Z &= A^2 + B^2, \end{aligned} \tag{8}$$

where A, B are positive integers satisfying $\gcd(A, B) = 1$ *and* $2 \mid AB$.

LEMMA 3 ([11, Theorem 4.1]). *The equation*

$$X^4 + Y^4 = Z^2, \quad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1, \tag{9}$$

has no solutions (X, Y, Z).

LEMMA 4 ([11, Theorem 4.2]). *The equation*

$$X^4 - Y^4 = Z^2, \quad X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1, \tag{10}$$

has no solutions (X, Y, Z).

LEMMA 5 ([2]). *Let* n *be a positive integer with* $n \geq 3$. *The equation*

$$X^n + Y^n = 2Z^n, \quad X, Y, Z \in \mathbb{N},\ \gcd(X, Y, Z) = 1,\ XYZ \neq 0 \text{ or } \pm 1, \tag{11}$$

has no solutions (X, Y, Z).

LEMMA 6 ([4, Theorems 6.7.1 and 6.7.4]). *For any positive integer* k *such that* $k > 1$ *and* $4 \nmid k$, *let*

$$V(k) = \prod_{p \mid k} (1 + \psi(p)),$$

where p *runs over distinct prime divisors of* k, *and*

$$\psi(p) = \begin{cases} 0 & \text{if } p = 2, \\ (-1)^{(p-1)/2} & \text{if } p \neq 2. \end{cases}$$

Then the equation

$$X^2 + Y^2 = k, \quad X, Y \in \mathbb{N},\ \gcd(X, Y) = 1, \tag{12}$$

has exactly $4V(k)$ *solutions* (X, Y).

LEMMA 7. *If* $a < 31b, c \equiv 5 \pmod 8$ *and* c *is not a prime power*, *then* $b > 58$.

Proof. Since $a < 31b$, we deduce from (2) that $926b^2 > c^r$. Hence, if $c^r > 926.58^2 = 3236168$, then $b > 58$. On the other hand, we see from (2) that every prime divisor p of c satisfies $p \equiv 1 \pmod 4$. Therefore, $c^r = 85^3$ is the unique integer such that $2 \nmid r$, $r \geq 3$, $c \equiv 5 \pmod 8$, $c^r < 3236168$ and c is not a prime power. By Lemma 6, the equation

$$A^2 + B^2 = 85, \quad A, B \in \mathbb{N},\ \gcd(A, B) = 1,\ 2 \mid B,$$

has exactly two solutions $(A, B) = (7, 6)$ and $(9, 2)$. Therefore, by Lemma 2, $(a, b) = (413, 666)$ and $(621, 478)$ are the only positive integers satisfying (2). Thus, the lemma is proved.

LEMMA 8 ([1, Theorem]). *If* $a \equiv 3 \pmod 4$, $c \equiv 5 \pmod 5$ *and* c *is a prime power*, *then* (1) *has only the solution* $(x, y, z) = (2, 2, r)$.

LEMMA 9. *If* $a > b$, $a \equiv 3 \pmod 4$, $b \equiv 2 \pmod 4$ *and* (x, y, z) *is a solution of* (1) *with* $(x, y, z) \neq (2, 2, r)$, *then* $2 \mid x$, $x \geq 6$, $y = 2$ *and* $2 \nmid z$.

Proof. Let (x,y,z) be a solution of (1) with $(x,y,z) \neq (2,2,r)$. By Lemma 2, we see from (2) that

$$\begin{aligned} a &= u\left|\sum_{i=0}^{(r-1)/2}\binom{r}{2i}u^{r-2i-1}(-v^2)^i\right|, \\ b &= v\left|\sum_{i=0}^{(r-1)/2}\binom{r}{2i+1}u^{r-2i-1}(-v^2)^i\right|, \\ c &= u^2+v^2, \quad u,v\in\mathbb{N},\ \gcd(u,v)=1,\ 2\,|\,v. \end{aligned} \tag{13}$$

Since $b \equiv 2 \pmod 4$, we see from (13) that $v \equiv 2 \pmod 4$ and $c \equiv 5 \pmod 8$. Therefore, by [1, Lemma 3], we get $2\,|\,y$. On the other hand, since $a \equiv 3 \pmod 4$, we deduce from (1) that $q^x \equiv 3^x \equiv c^z - b^y \equiv 1 \pmod 4$. This implies that $2\,|\,x$.

Since $(x,y,z) \neq (2,2,r)$, if $y=2$, then $x \geq 4$ and $z > r$. When $x=4$, we find from (1) and (2) that

$$a^2(a^2-1) = c^r(c^{z-r}-1). \tag{14}$$

Since $\gcd(a,c)=1$, by (14), we get $a^2-1 \equiv 0 \pmod{c^r}$. Hence we obtain $c^r = a^2+b^2 > a^2-1 \geq c^r$, which is a contradiction. So we have $x \geq 6$. Further, since $c^z \equiv a^x + b^2 \equiv 5 \pmod 8$, we get $2\nmid z$. Thus, if $y=2$, then $2\,|\,x$, $x\geq 6$ and $2\nmid z$.

If $y=4$, then $c^z \equiv a^x+b^4 \equiv 1 \pmod 8$, whence we get $2\,|\,z$. Hence, by Lemma 3, we see from (1) that $4\nmid x$. When $x=2$ or 6, we infer from (1) and (2) that either $b^2-1\equiv 0 \pmod{c^r}$ or $a^2+1 \equiv 0 \pmod{c^r}$. However, since $c^r = a^2+b^2 > \max(b^2-1, a^2+1)$ that is impossible. So we have $x \geq 10$. On the other hand, by Lemma 1, we find from (1) that

$$a^{x/2} = A^2-B^2, \quad b^2 = 2AB, \quad c^{z/2} = A^2+B^2, \tag{15}$$

where A,B are positive integers satisfying (6). Since $a>b$, we deduce from (15) that $A^2 > a^{x/2} \geq a^5 > b^5 = (2AB)^{5/2} > A^2$, a contradiction. So we have $y \neq 4$.

Similarly, if $y>4$, then we have $2\,|\,z$ and

$$a^{x/2} = A^2-B^2, \quad b^{y/2} = 2AB, \quad c^{z/2} = A^2+B^2. \tag{16}$$

Since $y/2 \geq 3$, we see from (16) that $4\,|\,AB$ and $c^{z/2} \equiv 1 \pmod 8$. This implies that $4\,|\,z$. Hence, by Lemma 4, we get $4\nmid y$. Since $(X,Y,Z) = (a^{x/2}, b^{y/2}, c^{z/4})$ is a solution of (7) for $n=4$, Lemma 2 yields

$$\begin{gathered} a^{x/2} = |X_1^4 - 6X_1^2Y_1^2 + Y_1^4|, \quad b^{y/2} = 4X_1Y_1(X_1^2-Y_1^2), \\ c^{z/4} = X_1^2+Y_1^2, \end{gathered} \tag{17}$$

where X_1, Y_1 are positive integers satisfying $X_1 > Y_1, \gcd(X_1,Y_1)=1$ and

$2\mid X_1Y_1$. From (17), we obtain

$$X_1+Y_1=b_1^{y/2},\qquad X_1-Y_1=b_2^{y/2},$$

$$X_1=\begin{cases} b_3^{y/2}, \\ b_4^{y/2}/4, \end{cases}\qquad Y_1=\begin{cases} b_4^{y/2}/4 & \text{if } 2\nmid X_1, \\ b_3^{y/2} & \text{if } 2\mid X_1, \end{cases} \tag{18}$$

where b_1, b_2, b_3, b_4 are positive integers satisfying $b_1b_2b_3b_4=b$, $2\nmid b_1b_2b_3$ and $2\mid b_4$. By (18), we get

$$2b_3^{y/2}=\begin{cases} b_1^{y/2}+b_2^{y/2} & \text{if } 2\nmid X_1, \\ b_1^{y/2}-b_2^{y/2} & \text{if } 2\mid X_1. \end{cases} \tag{19}$$

Since $y/2$ is an odd integer with $y/2\ge 3$, Lemma 5 shows that (19) is impossible. To sum up, the lemma is proved.

LEMMA 10. *Let a_1, a_2, b_1, b_2 be positive integers, and let $\Lambda=b_1\log a_1-b_2\log a_2$. If $a_1\ge 85$, $a_2\ge 553$ and $\Lambda\neq 0$, then*

$$\log|\Lambda|>-17.17(\log a_1)(\log a_2)(1.7735+B)^2, \tag{20}$$

where

$$B=\max\left(8.445, 0.2257+\log\left(\frac{b_1}{\log a_2}+\frac{b_2}{\log a_1}\right)\right). \tag{21}$$

Proof. For any real number ϱ with $\varrho>1$, by [5, Théorème 2], we have

$$\begin{aligned}(22)\quad &\log|\Lambda| \\ &\ge -\frac{16A_1A_2}{9\lambda^3}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^2\left(1+\frac{3}{2}\lambda^3(A_1^{-1}+A_2^{-1})\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1}\right. \\ &\quad +\sqrt[3]{2}\,\lambda^{3/2}\left(A_1A_2\left(B+\lambda+\frac{\lambda^2}{4B}\right)\right)^{-1/2}+\frac{9\lambda^3}{8A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1} \\ &\quad \left.+\frac{9\lambda^3}{16A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-2}\log\frac{A_1A_2(B+\lambda)^2}{\lambda^2}\right)+\frac{\lambda}{2}+\log\lambda-0.15,\end{aligned}$$

where $\lambda=\log\varrho$ and A_1, A_2, B satisfy $A_j\ge\max(2, 2\lambda, (\varrho+1)\log a_j)$ $(j=1,2)$,

$$B\ge\max\left(5\lambda, 1.56+\log\lambda+\log\left(\frac{b_1}{A_2}+\frac{b_2}{A_1}\right)\right). \tag{23}$$

We now choose $\varrho=e^{1.689}$ and B as in (21). Then we have $\lambda=1.689$, $A_j>6.414\log a_j$ $(j=1,2)$ and B satisfies (23). Since $B\ge 8.445$, we get $B+\lambda+\lambda^2/(4B)>B+1.7735\ge 10.2185$. Therefore, if $a_1\ge 85$ and $a_2\ge 553$, then

$$\frac{3}{2}\lambda^3(A_1^{-1}+A_2^{-1})\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1}<0.04264,$$

$$\sqrt[3]{2}\,\lambda^{3/2}\left(A_1A_2\left(B+\lambda+\frac{\lambda^2}{4B}\right)\right)^{-1/2} < 0.05745,$$
$$\frac{9\lambda^3}{8A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-1} < 0.00047,$$
$$\frac{9\lambda^3}{16A_1A_2}\left(B+\lambda+\frac{\lambda^2}{4B}\right)^{-2}\log\frac{A_1A_2(B+\lambda)^2}{\lambda^2} < 0.00025.$$

Since $\lambda/2+\log\lambda-0.15>0$, we find from (22) that (20) holds. Thus, the lemma is proved.

3. Proof of the Theorem. By Lemma 8, if c is a prime power, then the assertion of the Theorem holds. Therefore, we may assume that c is not a prime power. We observe that $c \equiv 5 \pmod 8$ if $b \equiv 2 \pmod 4$. As before, we see from (2) that every prime divisor p of c satisfies $p \equiv 1 \pmod 4$. Hence, we get $c \ge 85$.

Let (x,y,z) be a solution of (1) with $(x,y,z)\neq(2,2,r)$. By Lemma 9, then we have $2\,|\,x$, $x\ge 6$, $y=2$ and $2\nmid z$. Hence, (1) can be rewritten as

$$a^x+b^2=c^z,\qquad 2\,|\,x,\ x\ge 6,\ 2\nmid z. \tag{24}$$

From (24), we get

$$z\log c = x\log a+\theta, \tag{25}$$

where

$$0<\theta=\frac{2b^2}{a^x+c^z}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{b^2}{a^x+c^z}\right)^{2i}<\frac{3b^2}{2a^x}. \tag{26}$$

Let $\Lambda = z\log c - x\log a$. Since $a>b$ and $c^r\ge 85^3$, we deduce from (2) that $a>\sqrt{c^r/2}>553$. Therefore, by Lemma 10, we have

$$\log|\Lambda| > -17.17(\log c)(\log a)(1.7735+B)^2, \tag{27}$$

where

$$B=\max\left(8.445, 0.2257+\log\left(\frac{z}{\log a}+\frac{x}{\log c}\right)\right). \tag{28}$$

On the other hand, by (25) and (26), we get

$$\log(3b^2/2)-\log|\Lambda| > x\log a. \tag{29}$$

The combination of (27) and (29) yields

$$\frac{\log(3b^2/2)}{(\log a)(\log c)}+17.17(1.7735+B)^2>\frac{x}{\log c}. \tag{30}$$

If $8.445 \geq 0.2257 + \log(z/\log a + x/\log c)$, then

$$\frac{2x}{\log c} < \frac{z}{\log a} + \frac{x}{\log c} \leq e^{8.2193} < 3712, \tag{31}$$

whence we obtain

$$\frac{x}{\log c} < 1856. \tag{32}$$

Since $a > b$ and $c > 84$, if $8.445 < 0.2257 + \log(z/\log a + x/\log c)$, then from (28) and (30) we get

$$1 + 17.17\left(1.9992 + \log\left(\frac{2x}{\log c} + \theta\right)\right)^2 > \frac{x}{\log c}. \tag{33}$$

Since $\theta < 3b^2/(2a^x) < 3/(2a^{x-2}) \leq 3/(2a^4) < 10^{-10}$ by (26) and $a > 553$, we conclude from (33) that (32) also holds.

Let $t = a/b$. We find from (2) that

$$r \log c = 2\log a + \log\left(1 + \frac{1}{t^2}\right). \tag{34}$$

By (25) and (34), we obtain

$$0 < (rx - 2z)\log c = x\log\left(1 + \frac{1}{t^2}\right) - 2\theta < x\log\left(1 + \frac{1}{t^2}\right). \tag{35}$$

Since $2 \mid x$, we have $rx - 2z \geq 2$, and by (35), we get

$$\frac{x}{\log c} > \frac{2}{\log(1 + t^{-2})}. \tag{36}$$

The combination of (32) and (36) yields

$$928\log\left(1 + \frac{1}{t^2}\right) > 1, \tag{37}$$

whence we conclude that

$$t < 31. \tag{38}$$

On the other hand, we see from (2) and (24) that

$$\begin{aligned}(a^2 + b^2)^{x/2} &- (a^x + b^2) \\ &= b^2\left(\sum_{j=1}^{x/2}\binom{x/2}{j}a^{x-2j}b^{2(j-1)} - 1\right) = c^z(c^{rx/2-z} - 1).\end{aligned} \tag{39}$$

Since $\gcd(b, c) = 1$, we deduce from (39) that

$$\sum_{j=1}^{x/2}\binom{x/2}{j}a^{x-2j}b^{2(j-1)} \equiv 1 \pmod{c^z}. \tag{40}$$

This implies that

$$\sum_{j=1}^{x/2} \binom{x/2}{j} a^{x-2j} b^{2(j-1)} > c^z > a^x. \tag{41}$$

Since $a > b$, we find from (2) and (41) that

$$t^2\left(1+\frac{1}{t^2}\right)^{x/2} > t^2\left(\left(1+\frac{1}{t^2}\right)^{x/2} - 1\right) > a^2 > \frac{c^r}{2}, \tag{42}$$

whence we obtain

$$\frac{x}{\log c} > \left(2r - \frac{2\log(2t^2)}{\log c}\right) \Big/ \log\left(1+\frac{1}{t^2}\right). \tag{43}$$

If $r \le (2\log(2t^2))/\log c$, then we have

$$\frac{4a^4}{b^4} = 4t^4 \ge c^r = a^2 + b^2 > a^2, \tag{44}$$

whence $a^2 > b^4/4$ and $2t > b$. Since $b \equiv 2 \pmod 4$, (38) implies that $b \le 58$. However, by Lemma 7, this is impossible. So we have $r > (2\log(2t^2))/\log c$ and

$$\frac{x}{\log c} > \frac{r}{\log(1+t^{-2})}, \tag{45}$$

by (43). The combination of (32) and (45) shows that (3) is false. Thus, the Theorem is proved.

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