Sums of distances to the nearest integer
and the discrepancy of digital nets

by

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1. Introduction. The concept of digital nets provides at the moment
the most efficient method to generate point sets with small star-discrepancy
$D^*_N$. For a set of points $x_0, \ldots, x_{N-1}$ in $[0,1)^d$ the star-discrepancy of the
point set is defined by

$$D^*_N = \sup_B \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where the supremum is taken over all subintervals $B$ of $[0,1)^d$ of the form
$B = \prod_{i=1}^d [0, b_i)$, $0 < b_i \leq 1$, $A_N(B)$ denotes the number of $i$ with $x_i \in B$
and $\lambda$ is the Lebesgue measure.

It is known that for any set of $N$ points in $[0,1]^2$ one has

$$\frac{N D^*_N}{\log N} \geq 0.06$$

(see for example [1]).

A digital $(0, s, 2)$-net in base 2 is a point set of $N = 2^s$ points $x_0, \ldots, x_{N-1}$
in $[0,1)^2$ which is generated as follows. Choose two $s \times s$-matrices $C_1, C_2$ over
$\mathbb{Z}_2$ with the following property: For every integer $k$, $0 \leq k \leq s$, the system
of the first $k$ rows of $C_1$ together with the first $s - k$ rows of $C_2$ is linearly
independent over $\mathbb{Z}_2$. Then to construct $x_n := (x_n^{(1)}, x_n^{(2)})$ for $0 \leq n \leq 2^s - 1$,
represent $n$ in base 2:

$$n = n_{s-1}2^{s-1} + \ldots + n_1 2 + n_0,$$

multiply $C_i$ with the vector of digits:

$$C_i(n_0, \ldots, n_{s-1})^T =: (y_1^{(i)}, \ldots, y_s^{(i)})^T \in \mathbb{Z}_2^s$$
and set
\[ x_n^{(i)} := \sum_{j=1}^{s} \frac{y_j^{(i)}}{2^j}. \]

It was shown by Niederreiter [8] that for the star-discrepancy of any digital \((0, s, 2)\)-net in base 2 we have
\[ ND_N^* \leq \frac{1}{2} s + \frac{3}{2}, \]
hence
\[ \limsup_{N \to \infty} \max \frac{ND_N^*}{\log N} \leq \frac{1}{2 \log 2} = 0.7213 \ldots \]
where the maximum is taken over all digital \((0, s, 2)\)-nets in base 2 with \(N = 2^s\) elements.

The simplest digital \((0, s, 2)\)-net in base 2 is provided by choosing
\[
C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\text{ and } C_2 = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]
This gives the well-known Hammersley point set in base 2.

The star-discrepancy of this very special digital \((0, s, 2)\)-net was studied by Halton and Zaremba [4], de Clerck [2] and Entacher [3]. The first two papers are very technical and very hard to read. Indeed in [4] an essential part of the proof (determining the extremal intervals) is not carried out in detail. [3] uses a new approach but also essentially relies on results from [4].

In this paper we study much more generally the star-discrepancy of digital \((0, s, 2)\)-nets in base 2.

In Section 2 (see Theorem 1) we give a compact explicit formula for the discrepancy function of digital \((0, s, 2)\)-nets in base 2. Our approach is via Walsh series analysis.

It turns out that this explicit formula is based on sums of distances to the nearest integer (\(\|x\| := \min(x - \lfloor x \rfloor, 1 - (x - \lfloor x \rfloor))\) of the form
\[ \sum_{u=0}^{s-1} \|2^u \beta\| \varepsilon_u \]
with a real \(\beta\) and certain integer sequences \(\varepsilon_u \in \{-1, 0, 1\} \).
value of the “discrete discrepancy” and of the star-discrepancy of this point set (Theorem 4). Further we show that it is the “worst distributed” digital $(0, s, 2)$-net in base 2 with respect to star-discrepancy and we will get that for every digital $(0, s, 2)$-net in base 2 we have the (essentially) best possible bound
\begin{equation}
ND_N^* \leq \frac{1}{3} s + \frac{19}{9},
\end{equation}
and that
\begin{equation}
\lim \max_{N \to \infty} \frac{ND_N^*}{\log N} = \frac{1}{3 \log 2} = 0.4808 \ldots
\end{equation}
(the maximum is taken over all digital $(0, s, 2)$-nets in base 2 with $N = 2^s$ elements) with equality for the Hammersley point sets, thereby improving the bounds (1) and (2) of Niederreiter (Theorem 5).

Numerical investigations suggest that the minimal value for
\begin{equation}\lim \sup_{N \to \infty} \frac{ND_N^*}{\log N}\end{equation}
over all digital $(0, s, 2)$-nets in base 2 is attained for the net generated by the matrices
\begin{equation}
C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
C_2 = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\end{equation}
In Section 5 we give bounds for the star-discrepancy of this net and we show (Theorem 6) that for these nets
\begin{equation}
\frac{ND_N^*}{\log N} \geq \frac{1}{5 \log 2} = 0.2885 \ldots
\end{equation}
holds for all $N$ and that
\begin{equation}\lim \sup_{N \to \infty} \frac{ND_N^*}{\log N} \leq 0.32654 \ldots,
\end{equation}
thereby answering a question of Entacher in [3, Section 4].

2. The discrepancy function of digital $(0, s, 2)$-nets. For $0 \leq \alpha, \beta \leq 1$ we consider the discrepancy function
\begin{equation}
\Delta(\alpha, \beta) := A_N([0, \alpha] \times [0, \beta]) - N \alpha \beta
\end{equation}
for digital $(0, s, 2)$-nets $x_0, \ldots, x_{2^s-1}$ in base 2 (i.e. $N = 2^s$).

Since the generating matrices $C_1, C_2$ of a $(0, s, 2)$-net must be regular, and since multiplying $C_1, C_2$ by a regular matrix $A$ does not change the point set (only its order) we may assume in all the following that
We assume first that $\alpha$ and $\beta$ are “s-bit”, i.e. 
\[
\alpha = \frac{a_1}{2} + \ldots + \frac{a_s}{2^s}, \quad \beta = \frac{b_1}{2} + \ldots + \frac{b_s}{2^s}.
\]
For any $s$-bit number $\delta = d_1/2 + \ldots + d_s/2^s$ we write 
\[
\vec{\delta} := \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix},
\]
and for a non-negative integer $k = k_{s-1}2^{s-1} + \ldots + k_12 + k_0$ we write 
\[
\vec{k} := \begin{pmatrix} k_0 \\ \vdots \\ k_{s-1} \end{pmatrix}.
\]
We need some further notation: 
\[
\vec{\gamma} := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix} := C_2\vec{\alpha} + \vec{\beta}, \quad \vec{\gamma}(u) := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_u \end{pmatrix},
\]
\[
C'_2(u) := \begin{pmatrix} c_1^{s-u+1} & \cdots & c_u^{s-u+1} \\ \cdots & \cdots & \cdots \\ c_1^s & \cdots & c_u^s \end{pmatrix}^{-1}.
\]
($C'_2(u)$ exists since by the $(0, s, 2)$-net property the first $s - u$ rows of $C_1$ together with the first $u$ rows of $C_2$ must form a linearly independent system, hence the matrix 
\[
C_2(u) := \begin{pmatrix} c_1^{s-u+1} & \cdots & c_1^s \\ \cdots & \cdots & \cdots \\ c_u^{s-u+1} & \cdots & c_u^s \end{pmatrix}
\]
must be regular.) Note that $\gamma_u = (\vec{c}_u | \vec{\alpha}) + b_u$.

Further, for $0 \leq u \leq s - 1$ let 
\[
m(u) := \begin{cases} 
0 & \text{if } u = 0, \\
0 & \text{if } (\vec{\gamma}(u)|C'_2\vec{e}_i) = 1, \\
\max\{1 \leq j \leq u : (\vec{\gamma}(u)|C'_2\vec{e}_i) = 0; i = 1, \ldots, j\} & \text{otherwise}
\end{cases}
\]
(here $(\cdot|\cdot)$ denotes the usual inner product in $\mathbb{Z}_2^u$, $\vec{e}_i$ is the $i$th unit vector in $\mathbb{Z}_2^u$, and $C'_2 := C'_2(u)$).

Let $j(u) := u - m(u)$. Then we have
Theorem 1. For all $\alpha, \beta$ $s$-bit, for the discrepancy function $\Delta(\alpha, \beta)$ of the digital $(0, s, 2)$-net in base 2 generated by

$$C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}$$

and $C_2$ we have

$$\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\| (-1)^{(c_{u+1}^s)\overline{a}} (-1)^{(\overline{\gamma}(u))_{C_2}(u)(c_{u+1}^s-1,\ldots,c_{u+1}^s)}(\varepsilon_u)$$

(here for $u = 0$ we set $(\overline{\gamma}(u))_{C_2}(u)(c_{u+1}^s-1,\ldots,c_{u+1}^s) = 0$ and $a_{s+1} := 0$).

Before we prove this result we give some remarks and examples.

Remark 1. Note that $\Delta(\alpha, \beta)$ hence is of the form $\sum_{u=0}^{s-1} \|2^u \beta\| \varepsilon_u$ with some $\varepsilon_u \in \{-1, 0, 1\}$.

Remark 2. Let $0 \leq \alpha, \beta \leq 1$ now be arbitrary (not necessarily $s$-bit). Since all the points of the digital net have coordinates $x_n(i)$ of the form $a/2^s$ for some $a \in \{0, 1, \ldots, 2^s - 1\}$, we then have

$$\Delta(\alpha, \beta) = \Delta(\alpha(s), \beta(s)) + 2^s(\alpha(s)\beta(s) - \alpha\beta)$$

where $\alpha(s)$ (resp. $\beta(s)$) is the smallest $s$-bit number larger than or equal to $\alpha$ (resp. $\beta$).

Example 1. Let $C_2$ be of triangular form

$$C_2 = \begin{pmatrix}
c_1^1 & c_1^2 & \ldots & c_1^{s-1} & 1 \\
c_2^1 & c_2^2 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{s-1}^1 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}$$

Then

$$C_2'(u) = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & d_2^u \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & d_2^u & \ldots & d_{u-1}^u & d_u^u \\
\end{pmatrix}$$

with certain $d_i^u \in \mathbb{Z}_2$. Hence

$$C_2'(u)\overline{\varepsilon}_i = (0, \ldots, 0, 1, d_{u+2-i}^i, \ldots, d_u^i)^T,$$
and
\[(\gamma(u)|C_2^i(u)e_i) = \gamma_{u+1-i} + \gamma_{u+2-i}d_{u+2-i} + \ldots + \gamma_u d_u.\]
Therefore
\[\max\{1 \leq j \leq u : (\gamma(u)|C_2^i(u)e_i) = 0; \ i = 1, \ldots, j\} = \max\{1 \leq j \leq u : \gamma_{u+1-i} = 0; \ i = 1, \ldots, j\},\]
hence \(\gamma_u = \ldots = \gamma_{u+1-m(u)} = 0, \ \gamma_{u-m(u)} = 1\), so that
\[j(u) = u - m(u) = \max\{j \leq u : \gamma_j = 1\} = \max\{j \leq u : (\bar{c}_j|\bar{\alpha}) \neq b_j\}.
\]
Respectively
\[j(u) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } (\bar{c}_j|\bar{\alpha}) = b_j \text{ for } j = 1, \ldots, u. \end{cases}\]
Further \((c^{s+u+1}_u, \ldots, c^s_u) = (0, \ldots, 0)\), and so for \(\alpha, \beta \ s\)-bit we have
\[\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} 2^u \beta \|(-1)^{\bar{c}_u+1|\bar{\alpha}} - (-1)^{a_{s-u} + a_{s+1-j(u)}}\|/2.
\]

Example 2. For the discrepancy function of the Hammersley point set, i.e. for the \((0, s, 2)\)-net generated by
\[
C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix},
\]
because of \((\bar{c}_j|\bar{\alpha}) = a_{s+1-j}\) we obtain (for \(\alpha, \beta \ s\)-bit)
\[
\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} 2^u \beta \|(1 - (-1)^{a_{s-u} + a_{s+1-j(u)}})/2
\]
\[
\sum_{u=0}^{s-1} 2^u \beta \|\ (a_{s-u} \oplus a_{s+1-j(u)})
\]
(where \(\oplus\) denotes addition modulo 2).

Example 3. For the discrepancy function of the \((0, s, 2)\)-net generated by
\[
C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}
\]
because of \((\tilde{c}_j|\tilde{\alpha}) = a_1 \oplus \ldots \oplus a_{s+1-j}\) we obtain (for \(\alpha, \beta\) s-bit)

\[
\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\| (\alpha_1^{a_1} \ldots a_{s-u}^{u} \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}).
\]

For the proof of the Theorem 1 we need two auxiliary results.

**Lemma 1.** Let \(z\) be of the form \(z = p/2^s, p \in \{0, \ldots, 2^s - 1\}\). Then for the characteristic function \(\chi_{[0,z)}\) of the interval \([0,z)\) we have

\[
\chi_{[0,z)}(x) = \sum_{k=0}^{2^s-1} c_k(z) \text{wal}_k(x),
\]

where \(\text{wal}_k\) denotes the \(k\)th Walsh function in base 2 (see Remark 3),

\[
c_k(z) = \begin{cases} 
    z & \text{if } k = 0, \\
    \text{wal}_k(z) \frac{1}{2^v(k)} \psi(2^v(k) x) & \text{if } k \neq 0,
\end{cases}
\]

\(\psi(x)\) is periodic with period 1 and

\[
\psi(x) = \begin{cases} 
    x & \text{if } 0 \leq x < 1/2, \\
    x - 1 & \text{if } 1/2 \leq x < 1,
\end{cases}
\]

and \(v(k) = r\) if \(2^r \leq k < 2^{r+1}\).

**Remark 3.** Recall that Walsh functions in base 2 can be defined as follows: For a non-negative integer \(k\) with base 2 representation \(k = k_m2^m + \ldots + k_12 + k_0\) and a real \(x\) with (canonical) base 2 representation \(x = x_1/2 + x_2/2^2 + \ldots\) we have

\[\text{wal}_k(x) = (-1)^{x_1k_0 + x_2k_1 + \ldots + x_m+1k_m} = (-1)^{\langle k | x \rangle}.
\]

**Proof of Lemma 1.** This is a simple calculation, a proof can be found for example in [6, Lemma 2].

**Lemma 2.** Let \(\psi\) be as in Lemma 1. Then

\[
\psi(2^{l+1} \beta) - \sum_{i=0}^{l} \psi(2^i \beta) = \{\beta\} - b_{l+2}.
\]

(Here \(\{\beta\} = \beta - [\beta]\).)

**Proof.** Let \(\{\beta\} = \sum_{j=1}^{\infty} b_j 2^{-j}\). Then

\[
\psi(2^i \beta) = \sum_{j=i+1}^{\infty} b_j 2^{i-j} - b_{i+1}
\]

\[
\psi(2^{l+1} \beta) - \sum_{i=0}^{l} \psi(2^i \beta) = \{\beta\} - b_{l+2}.
\]
and therefore
\[
\sum_{i=0}^{l} \psi(2^i \beta) = \sum_{i=0}^{l} \left( \left( \sum_{j=i+1}^{\infty} b_j 2^{i-j} \right) - b_{i+1} \right)
\]
\[
= \sum_{j=1}^{l+1} b_j 2^{-j} \sum_{i=0}^{j-1} 2^i + \sum_{j=l+2}^{\infty} b_j 2^{-j} \sum_{i=0}^{l} 2^i - \sum_{i=0}^{l} b_{i+1}
\]
\[
= \sum_{j=l+2}^{\infty} b_j 2^{(l+1)-j} - \sum_{j=1}^{\infty} b_j 2^{-j} = \psi(2^{l+1} \beta) - \{\beta\} + b_{l+2}. \blacksquare
\]

**Proof of Theorem 1.** Let \(I := [0, \alpha) \times [0, \beta)\). Then for \(y = (y^{(1)}, y^{(2)}) \in [0, 1)^2\) by Lemma 1 we have
\[
\chi_I(y) - \lambda(I) = \chi_{[0, \alpha)}(y^{(1)}) \chi_{[0, \beta)}(y^{(2)}) - \alpha \beta
\]
\[
= \sum_{k,l=0}^{2^s-1} c_k(\alpha) c_l(\beta) \text{wal}_k(y^{(1)}) \text{wal}_l(y^{(2)})
\]
\[
= \alpha \sum_{l=1}^{2^s-1} \text{wal}_l(\beta) \frac{1}{2^{v(l)}} \psi(2^{v(l)} \beta) \text{wal}_l(y^{(2)})
\]
\[
+ \beta \sum_{k=1}^{2^s-1} \text{wal}_k(\alpha) \frac{1}{2^{v(k)}} \psi(2^{v(k)} \alpha) \text{wal}_k(y^{(1)})
\]
\[
+ \sum_{k,l=1}^{2^s-1} \text{wal}_k(\alpha) \text{wal}_l(\beta) \frac{1}{2^{v(k)+v(l)}} \psi(2^{v(k)} \alpha) \psi(2^{v(l)} \beta)
\]
\[
\times \text{wal}_k(y^{(1)}) \text{wal}_l(y^{(2)}).
\]
Hence
\[
\Delta(\alpha, \beta) = \alpha \sum_{l=1}^{2^s-1} \text{wal}_l(\beta) \frac{1}{2^{v(l)}} \psi(2^{v(l)} \beta) \sum_{i=0}^{2^s-1} \text{wal}_l(y_i)
\]
\[
+ \beta \sum_{k=1}^{2^s-1} \text{wal}_k(\alpha) \frac{1}{2^{v(k)}} \psi(2^{v(k)} \alpha) \sum_{i=0}^{2^s-1} \text{wal}_k(x_i)
\]
\[
+ \sum_{k,l=1}^{2^s-1} \text{wal}_k(\alpha) \text{wal}_l(\beta) \frac{\psi(2^{v(k)} \alpha) \psi(2^{v(l)} \beta)}{2^{v(k)+v(l)}} \sum_{i=0}^{2^s-1} \text{wal}_k(x_i) \text{wal}_l(y_i).
\]
(Here the net consists of the points \(x_i, i = 0, \ldots, 2^s - 1\), with \(x_i := (x_i, y_i)\).)

Since \(x_i, i = 0, \ldots, 2^s - 1\), is a digital \((0, s, 2)\)-net, for all \(0 < k, l < 2^s\) we have
\[ \sum_{i=0}^{2^s-1} \text{wal}_k(x_i) = \sum_{i=0}^{2^s-1} \text{wal}_l(y_i) = 0 \]

(see for example [5, Lemma 2]).

We now consider \( \sum_{i=0}^{2^s-1} \text{wal}_k(x_i)\text{wal}_l(y_i) \) with \( x_i := x_i^{(1)}/2 + \ldots + x_i^{(s)}/2^s \) and \( y_i := y_i^{(1)}/2 + \ldots + y_i^{(s)}/2^s \). We identify \( (x_i, y_i) \) with
\[
(x_i^{(1)}, \ldots, x_i^{(s)}, y_i^{(1)}, \ldots, y_i^{(s)}) \in (\mathbb{Z}_2)^{2s}
\]
and we define
\[
(x_i, y_i) \oplus (x_i', y_i') := (x_i^{(1)} + x_i'^{(1)}, \ldots, y_i^{(s)} + y_i'^{(s)}).
\]

Further \( \text{wal}_{k,l}(x_i, y_i) := \text{wal}_k(x_i)\text{wal}_l(y_i) \), hence
\[
\text{wal}_{k,l}((x_i, y_i) \oplus (x_i', y_i')) = \text{wal}_{k,l}(x_i, y_i)\text{wal}_{k,l}(x_i', y_i'),
\]
i.e. \( \text{wal}_{k,l} \) is a character on \( ((\mathbb{Z}_2)^{2s}, \oplus) \).

The digital net \( x_0, \ldots, x_{2^s-1} \) is a subgroup of \( ((\mathbb{Z}_2)^{2s}, \oplus) \), hence
\[
\sum_{i=0}^{2^s-1} \text{wal}_k(x_i)\text{wal}_l(y_i) = \begin{cases} 2^s & \text{if } \text{wal}_{k,l}(x_i, y_i) = 1 \text{ for all } i = 0, \ldots, 2^s - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

(For more details see [5] or [7].)

Now \( \text{wal}_{k,l}(x_i, y_i) = (-1)^{(\bar{k} | x_i) + (\bar{l} | y_i)} = 1 \) for all \( i = 0, \ldots, 2^s - 1 \) iff
\[
(\bar{k} | x_i) = (\bar{l} | y_i) \quad \text{for all } i = 0, \ldots, 2^s - 1,
\]
(by the definition of the net) this means
\[
(\bar{k} | i) = (\bar{l} | C_2 i) \quad \text{for all } i = 0, \ldots, 2^s - 1,
\]
and this is satisfied if and only if
\[
\bar{k} = C_2^T \bar{l} =: \bar{k}(l).
\]

Further
\[
\text{wal}_{k(l)}(\alpha)\text{wal}_{l}(\beta) = (-1)^{(\bar{k}(l) | \bar{\alpha}) + (\bar{l} | \bar{\beta})} = (-1)^{\bar{l}(\bar{C}_2 \bar{\alpha} + \bar{\beta})} = \text{wal}_l(\gamma)
\]
(see notations).

So
\[
\Delta(\alpha, \beta) = 2^s \sum_{u=0}^{s-1} \psi(2^u \beta) \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0 \gamma_1 + \ldots + l_{u-1} \gamma_u + \gamma_{u+1}} \frac{\psi(2^v(\bar{k}(l)) \alpha)}{2^v(\bar{k}(l))}
\]
\[
= 2^s \sum_{u=0}^{s-1} \|2^u \beta\|(-1)^{(\bar{c}_{u+1} | \bar{\alpha})}
\]
\[
\times \frac{1}{2^u} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0 \gamma_1 + \ldots + l_{u-1} \gamma_u} \frac{\psi(2^v(\bar{k}(l)) \alpha)}{2^v(\bar{k}(l))}
\]
(here \( l := l_0 + l_1 2 + \ldots + l_u 2^u \); note that \((-1)^{\gamma_{u+1}} = (-1)^{\tilde{c}_{u+1} |\alpha|} (-1)^{b_{u+1}}\) and \(\psi(2^u \beta)(-1)^{b_{u+1}} = ||2^u \beta||\)).

We now consider

\[
\Sigma_1 := \frac{1}{2^u} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0 \gamma_1 + \ldots + l_{u-1} \gamma_u} \frac{\psi(2^u (k(l)) \alpha)}{2^u (k(l))}.
\]

For \(2^u \leq l < 2^{u+1}\), the condition \(v(k(l)) = w\) means that there are \(k_0, \ldots, k_{w-1} \in \mathbb{Z}_2\) such that

\[
C_2^T \tilde{l} = (k_0, \ldots, k_{w-1}, 1, 0, \ldots, 0)^T,
\]

that is,

\[(5) \quad \tilde{c}_1 l_0 + \ldots + \tilde{c}_u l_{u-1} + \tilde{c}_{u+1} = k_0 \tilde{e}_1 + \ldots + k_{w-1} \tilde{e}_w + \tilde{e}_{w+1}\]

where \(\tilde{e}_i\) is the \(i\)th unit vector in \(\mathbb{Z}_2^s\).

Since \(\tilde{c}_1, \ldots, \tilde{c}_{u+1}, \tilde{e}_1, \ldots, \tilde{e}_{w+1}\) by the \((0,s,2)\)-net property are linearly independent as long as \((u+1) + (w+1) \leq s\) we must have \(u + w \geq s - 1\). Hence

\[
\Sigma_1 = \sum_{w=s-1-u}^{s-1} \frac{\psi(2^w \alpha)}{2^{u+w}} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0 \gamma_1 + \ldots + l_{u-1} \gamma_u}.
\]

In the following we are concerned with evaluating the last sum in the above expression which equals

\[
\Sigma_2 := \sum_{l=0}^{2^u-1} \text{wal}(\gamma) = \sum_{l=0}^{2^u-1} \text{wal}_{C_2^l}(\gamma)
\]

(here \(C_2^l\) stands for \(C_2^l(u)\); see notation). Now \(v(k(C_2^l + 2^u)) = w\) means

\[
C_2^T \begin{pmatrix}
C_2^l
1
0
\vdots
0
\end{pmatrix} = \begin{pmatrix}
k_0
\vdots
k_{w-1}
1
0
\vdots
0
\end{pmatrix}
\]

for some \(k_i \in \mathbb{Z}_2\). This is equivalent to
Assume that \( s \) is \( k \)-dependent, and hence we can find that there are \( s \) solution \( l_0, \ldots, l_{u-1}, k_0, \ldots, k_{w-1} \) in \( \mathbb{Z}_2 \) such that
\[
\bar{c}_1 l_0 + \ldots + \bar{c}_u l_{u-1} + \bar{c}_{u+1} l_u + \bar{c}_1 k_0 + \ldots + \bar{c}_w k_{w-1} + \bar{c}_{w+1} = 0. 
\]
Since \( s = u + w + 1 \) the vectors \( \bar{c}_1, \ldots, \bar{c}_{u+1}, \bar{c}_1, \ldots, \bar{c}_{w+1} \) are linearly dependent, and hence we can find \( l_0, \ldots, l_{u-1}, l_u, k_0, \ldots, k_{w-1}, k_w \) in \( \mathbb{Z}_2 \) not all zero such that
\[
\bar{c}_1 l_0 + \ldots + \bar{c}_u l_{u-1} + \bar{c}_{u+1} l_u + \bar{c}_1 k_0 + \ldots + \bar{c}_w k_{w-1} + \bar{c}_{w+1} k_w = 0. 
\]
Assume that \( l_u = 0 \). Then \( \bar{c}_1, \ldots, \bar{c}_u, \bar{c}_1, \ldots, \bar{c}_{w+1} \) are linearly dependent. But this contradicts the \((0, s, 2)\)-net property since \( \bar{c}_1, \ldots, \bar{c}_u \) are the first \( u \) rows of the matrix \( C_2 \) and \( \bar{c}_1, \ldots, \bar{c}_{w+1} \) are the first \( w + 1 \) rows of the matrix \( C_1 \) and \( u + w + 1 = s \). Hence \( l_u = 1 \). In the same way one can show that \( k_w = 1 \). This shows that system (5), and hence also (6), has a solution.

Now the unique solution \( \bar{l} \) of (6) is given by
\[
\bar{l} = (c_{u+1}^{s-u+1}, \ldots, c_{u+1}^s)^T.
\]
If \( s - u \leq w \), then the \( 2^{u+w-s} \) solutions therefore are given by
\[
\bar{l} = (l_0, \ldots, l_{u+w-(s+1)}, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \ldots, c_{u+1}^s)^T
\]
with \( l_0, \ldots, l_{u+w-(s+1)} \) arbitrary in \( \mathbb{Z}_2 \).

Hence for \( w \geq s - u \) we have
\[
\Sigma_2 = \sum_{l_0, \ldots, l_{u+w-(s+1)} \in \mathbb{Z}_2} (-1)^{\gamma(u)} C_2^T(l_0, \ldots, l_{u+w-(s+1)}, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \ldots, c_{u+1}^s)^T
\]
\[
= (-1)^{\bar{c}_2^T \gamma(u)} (0, \ldots, 0, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \ldots, c_{u+1}^s)^T \sum_{l=0}^{2^{u+w-s}-1} \text{wal}_l(C_2^T \gamma(u)).
\]
The last sum is a sum over all characters of \( ((\mathbb{Z}_2)^{u+w-s}, \oplus) \), and is therefore 
2^{u+w-s} if \( (C'_2 \bar{\gamma}(u)|\bar{e}_i^j) = 0 \) for all \( i = 1, \ldots, u+w-s \) (\( \bar{e}_i \) is the \( i \)th unit vector in \( \mathbb{Z}_2^2 \)) and it is 0 otherwise.

Further, if \( (C'_2 \bar{\gamma}(u)|\bar{e}_i^j) = 0 \) for all \( i = 1, \ldots, u+w-s \) (we will call this the condition \( *_u \)), then

\[
(C'_2 \bar{\gamma}(u)|((0, \ldots, 0, c_{u+1}^w \oplus 1, c_{u+1}^w, \ldots, c_{u+1}^s)^T) = (\bar{\gamma}(u)|C'_2(c_{u+1}^{s-u+1}, \ldots, c_{u+1}^s)^T) + (\bar{\gamma}(u)|C'_2 \bar{e}_{u+w-s+1})
\]

so that altogether we have

\[
\Sigma_1 = \frac{1}{2^s} \sum_{u=0}^{s-1} (\bar{\gamma}(u)|C'_2(c_{u+1}^{s-u+1}, \ldots, c_{u+1}^s)^T) f(u),
\]

where

\[
f(u) := 2\psi(2^{s-u-1} \alpha) + \left\{ \begin{array}{ll}
\sum_{w=s-u}^{s-1} \psi(2^w \alpha)(-1)^{(\bar{\gamma}(u)|C'_2 \bar{e}_{u+w-s+1})} & \text{if } *_u \text{ holds,} \\
0 & \text{otherwise,}
\end{array} \right.
\]

and therefore

\[
\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\|(1-\bar{e}_{u+1} | \bar{e}_i^j)(\bar{\gamma}(u)|C'_2(c_{u+1}^{s-u+1}, \ldots, c_{u+1}^s)^T) f(u).
\]

It remains to show that

\[
f(u) = \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}.
\]

By the definition of \( m(u) \) we have \( (\bar{\gamma}(u)|C'_2 \bar{e}_1) = \ldots = (\bar{\gamma}(u)|C'_2 \bar{e}_{m(u)}) = 0 \) and \( (\bar{\gamma}(u)|C'_2 \bar{e}_{m(u)+1}) = 1 \), hence \( *_u \) holds iff \( u+w-s \leq m(u) \). So finally

\[
f(u) = 2\psi(2^{s-u-1} \alpha) + \sum_{w=s-u}^{s-u+m(u)} \psi(2^w \alpha)(-1)^{(\bar{\gamma}(u)|C'_2 \bar{e}_{u+w-s+1})}
\]

\[
= 2\psi(2^{s-u-1} \alpha) + \sum_{w=s-u}^{s-u-2} \psi(2^w \alpha) - \psi(2^{s-u+m(u)} \alpha)
\]

\[
= \psi(2^{s-u-1} \alpha) - \sum_{w=0}^{s-u-2} \psi(2^w \alpha) + \sum_{w=0}^{s-u+m(u)-1} \psi(2^w \alpha) - \psi(2^{s-u+m(u)} \alpha)
\]

\[
= \alpha - a_{s-u} - \alpha + a_{s+1-(u-m(u))} = a_{s+1-(u-m(u))} - a_{s-u}
\]

\[
= \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}
\]

where we used Lemma 2 and \( j(u) = u - m(u) \). The result follows.
3. A spectrum result for sums of distances to the nearest integer. Here we study sums of the form $\sum_{u=0}^{s-1} \|2^u \beta\|$ for $\beta \in \mathbb{R}$, especially for $s$-bit $\beta$, and we derive results which are of independent interest and/or will be used in Section 4.

The essential technical tool is provided by

**Lemma 3.** Assume that $\beta = 0.\overline{b_1 b_2 \ldots}$ (this always means base 2 representation) has two equal consecutive digits $b_i b_{i+1}$ with $i \leq s - 1$ and let $i$ be minimal with this property, i.e.

$$
\beta = 0.01\ldots0100b_{i+2}\ldots \quad \text{or} \\
\beta = 0.10\ldots0100b_{i+2}\ldots \quad \text{or} \\
\beta = 0.01\ldots1011b_{i+2}\ldots \quad \text{or} \\
\beta = 0.10\ldots1011b_{i+2}\ldots
$$

Replace $\beta$ by

$$
\gamma = 0.10\ldots1010b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.01\ldots1010b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.10\ldots0101b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.01\ldots0101b_{i+2}\ldots
$$

Then

$$
\sum_{u=0}^{s-1} \|2^u \gamma\| = \sum_{u=0}^{s-1} \|2^u \beta\| + \begin{cases}
\frac{1}{3}(1 - (-1)^i/2^i)(1 - \tau) & \text{in the first two cases}, \\
\frac{1}{3}(1 - (-1)^i/2^i)\tau & \text{in the last two cases},
\end{cases}
$$

where $\tau := 0.b_{i+2}b_{i+3}\ldots$

**Remark 4.** In any case we have $\sum_{u=0}^{s-1} \|2^u \gamma\| \geq \sum_{u=0}^{s-1} \|2^u \beta\|$ with equality iff $\tau = 1$ in the first two cases and iff $\tau = 0$ in the last two cases.

**Proof of Lemma 3.** This is simple calculation. We just handle the first case here:

$$
\sum_{u=0}^{s-1} (\|2^u \gamma\| - \|2^u \beta\|)
= \|\gamma\| - \|2^i \beta\| + \left( \frac{\tau}{2^i} - \frac{\tau}{4} \right) - \left( \frac{\tau}{4} - \frac{\tau}{8} \right) \pm \ldots + \left( \frac{\tau}{2^{i+1}} - \frac{\tau}{2^{i+2}} \right)
= \left( \frac{1}{3}(1 + \frac{1}{2^i} - \frac{\tau}{2^{i+1}}) \right) - \frac{\tau}{2} + \frac{1}{6} (1 + \frac{1}{2^i}) \tau
= \frac{1}{3} \left( 1 + \frac{1}{2^i} \right) (1 - \tau).
$$

The other cases are calculated in the same way. ■

We immediately obtain a corollary which is useful in Section 4.
Corollary 1. Assume that $\beta = 0.1b_2b_3\ldots$ has two equal consecutive digits $b_ib_{i+1}$ with $2 \leq i \leq s-1$ and let $i$ be the minimal index with this property, i.e.

$$
\beta = 0.101\ldots0100b_{i+2}\ldots \quad \text{or} \\
\beta = 0.110\ldots0100b_{i+2}\ldots \quad \text{or} \\
\beta = 0.101\ldots1011b_{i+2}\ldots \quad \text{or} \\
\beta = 0.110\ldots1011b_{i+2}\ldots
$$

Replace $\beta$ by

$$
\gamma = 0.110\ldots1010b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.101\ldots1010b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.110\ldots0101b_{i+2}\ldots \quad \text{resp.} \\
\gamma = 0.101\ldots0101b_{i+2}\ldots
$$

Then

$$
\gamma + \sum_{u=0}^{s-1} \|2^u\gamma\| = \beta + \sum_{u=0}^{s-1} \|2^u\beta\| + \begin{cases} 
\frac{1}{3}(1 - (-1)^{i-1}/2^{i-1})(1 - \tau) & \text{in the first two cases}, \\
\frac{1}{3}(1 - (-1)^{i-1}/2^{i-1})\tau & \text{in the last two cases},
\end{cases}
$$

where $\tau := 0.b_{i+2}b_{i+3}\ldots$

Proof. This follows from $\beta + \|\beta\| = \gamma + \|\gamma\| = 1$, by applying Lemma 3 to $\beta' := 0.b_2b_3\ldots$.

We obtain

Theorem 2. Consider $\beta \in \mathbb{R}$ with the canonical base 2 representation (i.e. with infinitely many digits equal to zero). Then there exists

$$
\max_{\beta} \sum_{u=0}^{s-1} \|2^u\beta\| = \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s}
$$

and it is attained if and only if $\beta$ is of the form $\beta_0$ with

$$
\beta_0 = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{s+1}\right) \quad \text{or} \quad \beta_0 = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{s}\right).
$$

Remark 5. Note that

$$
\frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{s+1}\right) = \begin{cases} 
0.1010\ldots101 & \text{if } s \text{ is odd}, \\
0.1010\ldots011 & \text{if } s \text{ is even},
\end{cases}
$$

$$
\frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{s}\right) = \begin{cases} 
0.0101\ldots011 & \text{if } s \text{ is odd}, \\
0.0101\ldots101 & \text{if } s \text{ is even}.
\end{cases}
$$

Proof of Theorem 2. For any $\gamma = 0.c_1c_2\ldots csc_{s+1}\ldots$ with fixed $c_1,\ldots, c_s$ the sum $\sum_{u=0}^{s-1} \|2^u\gamma\|$ obviously becomes maximal if $c_s = 0$ and $c_{s+1} = \ldots$
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\[ c_{s+2} = \ldots = 1, \text{ or if } c_s = 1 \text{ and } c_{s+1} = c_{s+2} = \ldots = 0. \] Hence by Lemma 3 the supremum

\[ \sup_{\beta} \sum_{u=0}^{s-1} \|2^u \beta\| \]

can only be attained, respectively approached by

\[ \beta_1 = 0.1010 \ldots 10 \ldots 11 \ldots \quad \text{or} \quad (b_s \text{ is the last zero}) \]
\[ \beta_2 = 0.0101 \ldots 01 \quad \text{or} \]
\[ \beta_3 = 0.1010 \ldots 11 \]

(b \text{ is the last one}) if \( s \) is even, and by

\[ \beta_4 = 0.0101 \ldots 10 \ldots 11 \ldots \quad \text{or} \]
\[ \beta_5 = 0.1010 \ldots 01 \quad \text{or} \]
\[ \beta_6 = 0.0101 \ldots 11 \]

if \( s \) is odd.

Now we check easily that

\[ \sum_{u=0}^{s-1} \|2^u \beta_i\| = \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s} \]

for \( i = 1, \ldots, 6 \) and the result follows. \( \blacksquare \)

The next theorem gives the result which we call the “spectrum” result (see Remark 6).

**Theorem 3.** (a) The maximum

\[ \max_{\beta \text{ s-bit}} \sum_{u=0}^{s-1} \|2^u \beta\| = \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s} \]

is attained if and only if \( \beta \) is one of the \( \beta_0 \) from Theorem 2.

(b) We have

\[ \max_{\beta \text{ s-bit} \neq \beta_0} \sum_{u=0}^{s-1} \|2^u \beta\| = \frac{s}{3} + \frac{1}{36} - (-1)^s \frac{7}{9 \cdot 2^s} \]

and this second successive maximum is attained if and only if \( \beta \) is of the form \( \beta' \) with

\[ \beta' = \begin{cases} 0.01101010 \ldots 101 & \text{or} \\ 0.010101 \ldots 01101 & \text{or} \\ 0.10010101 \ldots 011 \end{cases} \]
if $s$ is odd and
\[
\beta' = \begin{cases} 
0.100101010 \ldots 101 & \text{or} \\
0.010101 \ldots 010011 & \text{or} \\
0.101010 \ldots 101101 & \text{or} \\
0.011010 \ldots 101101 & 
\end{cases}
\]
if $s$ is even.

**Remark 6.** Let
\[
\max_{\beta \text{ s-bit}} \sum_{u=0}^{s-1} \|2^u \beta\| =: \sum_{u=0}^{s-1} \|2^u \beta_0(s)\|.
\]

Then by Theorem 3 we have
\[
\lim_{s \to \infty} \left( \sum_{u=0}^{s-1} \|2^u \beta_0(s)\| - \max_{\beta \text{ s-bit}} \sum_{\beta \neq \beta_0(s)}^{s-1} \|2^u \beta\| \right) = \frac{1}{12}.
\]
So one may ask the further usual “spectrum questions”.

**Proof of Theorem 3.** (a) follows from Theorem 2.

Concerning part (b) it follows from Lemma 3 that it must be possible to reach one of the $\beta_0$ by applying a single transformation of Lemma 3 to $\beta'$.

For $s$ odd this means ($s$ even is handled quite analogously) that
\[
\beta' \to 0.1010 \ldots 101
\]
by the first or third transformation, i.e.
\[
\beta' = 0.0101 \ldots 0100101 \ldots 10101 \quad \text{or} \\
\beta' = 0.0101 \ldots 10110101 \ldots 10101,
\]
or that
\[
\beta' \to 0.0101 \ldots 011
\]
by the second or fourth transformation, i.e.
\[
\beta' = 0.1010 \ldots 010010 \ldots 1011 \quad \text{or} \\
\beta' = 0.1010 \ldots 101101 \ldots 1011.
\]

Further the double blocks $b_i b_{i+1}$ must be placed so that the “error term” in Lemma 3 becomes minimal. We carry this out for the two transformations yielding
\[
\beta' \to 0.1010 \ldots 101
\]
(the second case is treated quite analogously).

If
\[
\beta' = 0.0101 \ldots 01001010 \ldots 10101
\]
then the “error term” has the form
\[
\frac{1}{3} \left( 1 - \frac{(-1)^i}{2^i} \right) (1 - \tau) =: E(i)
\]
with

\[ \tau = 0.1010 \ldots 101 = \frac{2}{3} \left( 1 - \frac{1}{2^{s-i}} \right), \]

and \( i \) is odd. Hence

\[ E(i) = \frac{1}{9} \left( 1 + \frac{1}{2^i} \right) \left( 1 + \frac{1}{2^{s-1-i}} \right), \]

which becomes minimal for \( i = (s-1)/2 \), with value

\[ E = \frac{1}{9} \left( 1 + \frac{1}{2^{(s-1)/2}} \right)^2. \]

If \( \beta' = 0.0101 \ldots 101101 \ldots 10101 \) then

\[ E(i) = \frac{1}{3} \left( 1 - \frac{(-1)^i}{2^i} \right) \tau \]

with

\[ \tau = 0.0101 \ldots 10101 = \frac{1}{3} \left( 1 - \frac{1}{2^{s-i-1}} \right), \]

and \( i \) is even. Hence

\[ E(i) = \frac{1}{9} \left( 1 - \frac{1}{2^i} \right) \left( 1 - \frac{1}{2^{s-1-i}} \right), \]

which becomes minimal for \( i = 2 \) and for \( i = s-3 \) (note that \( i = s-1 \) would give one of the \( \beta_0 \) and \( E(i) = 0 \)), with value

\[ E = \frac{1}{12} \left( 1 - \frac{8}{2^s} \right), \]

which is smaller than the \( E \) above.

By also dealing with the second case we find that this is the minimal possible value for \( E \) and we have found the first two values of \( \beta' \). The third value for \( \beta' \) is found by treating the second case.

The minimal error term \( E \) also determines the value for

\[ \sum_{u=0}^{s-1} \|2^u \beta'\| = \sum_{u=0}^{s-1} \|2^u \beta_0\| - E = \frac{s}{3} + \frac{1}{9} + \frac{1}{9 \cdot 2^s} - \frac{12}{12} \left( 1 - \frac{8}{2^s} \right) \]

\[ = \frac{s}{3} + \frac{1}{36} + \frac{7}{9 \cdot 2^s}. \]

The case of \( s \) even is dealt with quite analogously. □

We again obtain a corollary:
Corollary 2. The maximum
\[
\max_{\beta \text{ s-bit}} \left( \beta + \sum_{u=0}^{s-1} \|2^u \beta\| \right) = \frac{s}{3} + \frac{7}{9} + (-1)^s \frac{1}{9} \cdot 2^{s-1}
\]
is attained if and only if \(\beta\) is of the form
\[
\beta_0 = \frac{2}{3} \left( 1 - \left( -\frac{1}{2} \right)^{s+1} \right) \quad \text{or} \quad \beta_0 = \frac{5}{6} - \frac{1}{3} \left( -\frac{1}{2} \right)^s.
\]

Remark 7. Note that here
\[
\beta_0 = 0.110101 \ldots 101 \quad \text{or} \quad \beta_0 = 0.101010 \ldots 011
\]
if \(s\) is even and
\[
\beta_0 = 0.101010 \ldots 101 \quad \text{or} \quad \beta_0 = 0.110101 \ldots 011
\]
if \(s\) is odd.

Proof of Corollary 2. If \(\beta < 1/2\) then we replace \(\beta\) by \(\beta + 1/2\) and we obtain a larger value for the sum in question. So we can assume \(\beta = 0.1b_2 b_3 \ldots b_s\), and we note that \(\beta + \|\beta\| = 1\) always. So we have to maximize \(\sum_{u=0}^{s-2} \|2^u (2\beta)\|\). By Theorem 3(a) the result follows.

For later use (proof of Theorem 4(a)) we need a further type of “spectrum” result, namely Lemma 5. To prove it, we will use Lemma 4.

Lemma 4. Let \(0 \leq \kappa < 1\). Then

(a) The maximum
\[
\max_{\beta \text{ s-bit}} \left( \kappa \beta + \sum_{u=0}^{s-1} \|2^u \beta\| \right) =: \Sigma_s^{\kappa}
\]
is attained by
\[
\beta = \begin{cases} 
0.1010 \ldots 101 & \text{for } s \text{ even,} \\
0.1010 \ldots 101 & \text{for } s \text{ odd.}
\end{cases}
\]

(b) The maximum
\[
\max_{\beta \text{ s-bit}} \left( -\kappa \beta + \sum_{u=0}^{s-1} \|2^u \beta\| \right) =: \Sigma_s^{-\kappa}
\]
is attained by
\[
\beta = \begin{cases} 
0.0101 \ldots 0101 & \text{for } s \text{ even,} \\
0.0101 \ldots 011 & \text{for } s \text{ odd.}
\end{cases}
\]

Proof. (a) We must have \(b_1 = 1\), otherwise \(1 - \beta\) gives a larger value than \(\beta\). We proceed by induction on \(s\). For \(s = 1, 2, 3\) the assertion is easily
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checked. Now (since \( b_1 = 1 \))

\[
\Sigma_{s+2}^\kappa = \max_{\beta \text{ s-bit}} \left( \kappa \beta + \sum_{u=0}^{s+1} \|2^u \beta\| \right)
\]

\[
= \max_{\beta' \text{ s+1-bit}} \left( \frac{\kappa + 1}{2} + \beta' \left( \frac{\kappa - 1}{2} \right) + \sum_{u=0}^{s} \|2^u \beta'\| \right).
\]

Now \((\kappa - 1)/2 < 0\), so \(b_1'\) must be zero, otherwise \(1 - \beta'\) would give a larger value. Hence \(\beta' = \beta''/2\) with \(\beta''\) s-bit, and therefore

\[
\Sigma_{s+2}^\kappa = \frac{\kappa + 1}{2} + \max_{\beta'' \text{ s-bit}} \left( \beta'' \left( \frac{\kappa + 1}{4} \right) + \sum_{u=0}^{s-1} \|2^u \beta''\| \right).
\]

By the induction hypothesis the result follows.

(b) Set \(\gamma = 1 - \beta\). Then

\[-\gamma \kappa + \sum_{u=0}^{s-1} \|2^u \gamma\| = -\kappa + \kappa \beta + \sum_{u=0}^{s-1} \|2^u \beta\|\]

and by part (a) the result follows. ■

The next lemma is of independent interest. Note for example that \(1/4\) is the "average value" for \(\|x\|\).

**Lemma 5.** \(\max_{\beta \text{ s-bit}} \sum_{0 \leq u_0 \leq s-1}^{s-1} \|2^u \beta\| = \max_{\beta \text{ s-bit}} \sum_{u=0}^{s-1} \|2^u \beta\| - \frac{1}{4}\).

**Proof.** For \(u_0\) fixed let

\[
\Sigma_{u_0}(\beta) := \sum_{u=0}^{s-1} \|2^u \beta\| \quad \text{and} \quad \Sigma_{u_0}(\beta_0) := \max_{\beta \text{ s-bit}} \Sigma_{u_0}(\beta).
\]

By Lemma 3, \(\beta_0\) must be of the form

\(\beta_0 = 0.0101 \ldots b_{u_0+1}b_{u_0+2} \ldots b_s\) or \(\beta_0 = 0.1010 \ldots b_{u_0+1}b_{u_0+2} \ldots b_s\).

Let

\(\beta_0 := 0. b_1 \ldots b_{u_0+1} \) and \(\tilde{\beta}_0 := 0. b_{u_0+2} \ldots b_s\).

Then

\[
\Sigma_{u_0}(\beta_0) = \sum_{u=0}^{u_0-1} \|2^u \beta_0\| + \kappa \tilde{\beta}_0 + \sum_{u=u_0+1}^{s-1} \|2^u \beta_0\|
\]

\[
= \sum_{u=0}^{u_0-1} \|2^u \beta_0\| + \kappa \tilde{\beta}_0 + \sum_{u=0}^{s-u_0-2} \|2^u \tilde{\beta}_0\|
\]

with \(\kappa = \sum_{i=1}^{u_0} (-1)^{b_i}/2^{u_0+2-i}\). If \(b_{u_0} = 0\) then \(\kappa > 0\), if \(b_{u_0} = 1\) then \(\kappa < 0\).
So by Lemma 3 (see also Theorem 3) the form of $\tilde{\beta}_0$, and by Lemma 4 and by $b_{u_0}$ the form of $\tilde{\beta}_0$ is determined (note that the form of $b_{u_0+1}$ must be different from $b_{u_0}$ and hence is 0 in any case).

We have

$$\tilde{\beta}_0 = \frac{1}{3} \left( 1 - \frac{(-1)^{s-u_0-1}}{2^{s-u_0-1}} \right)$$

and

$$\kappa = -\frac{1}{6} \left( 1 - \frac{(-1)^{u_0}}{2^{u_0}} \right) \quad \text{or} \quad \kappa = -\frac{1}{3} \left( 1 + \frac{(-1)^{u_0}}{2^{u_0+1}} \right)$$

giving to which value for $\tilde{\beta}_0$ is chosen from Theorem 3.

Since we want to maximize

$$\Sigma_{u_0}(\beta_0) = \sum_{u=0}^{u_0-1} \|2^u \tilde{\beta}_0\| + \kappa \tilde{\beta}_0 + \sum_{u=0}^{s-u_0-2} \|2^u \tilde{\beta}_0\|,$$

only the larger first value for $\kappa$ is of relevance. Inserting it yields

$$\max_{\beta \text{ s-bit}} \left( \sum_{u=0}^{s-1} \|2^u \beta\| - \Sigma_{u_0}(\beta_0) \right)$$

$$\quad = \frac{1}{18} \left( 5 + \frac{(-1)^{u_0}}{2^{u_0}} + \frac{(-1)^{s-u_0-1}}{2^{s-u_0-1}} + \frac{(-1)^{s-1}}{2^{s-2}} \right),$$

which attains its minimal value 1/4 for $u_0 = s-2$ if $s$ is odd, and for $u_0 = 1$ if $s$ is even. ■

4. The discrepancy of the Hammersley net and an improved upper bound for the discrepancy of digital $(0, s, 2)$-nets. In Theorem 1 for $\alpha, \beta$ s-bit we have given an explicit formula for the discrepancy function

$$\Delta(\alpha, \beta) = A_{2^s}([0, \alpha) \times [0, \beta)) - 2^s \alpha \beta$$

of a digital $(0, s, 2)$-net in base 2.

Take now arbitrary $\alpha', \beta'$ with

$$\alpha - \frac{1}{2^s} < \alpha' \leq \alpha \quad \text{and} \quad \beta - \frac{1}{2^s} < \beta' \leq \beta.$$  

Then (since all coordinates of the points of a digital net are s-bit) we have

$$\Delta(\alpha', \beta') = \Delta(\alpha, \beta) - 2^s (\alpha' \beta' - \alpha \beta),$$

hence for the star-discrepancy $D_N^*$ of the net we have

$$\left| D_N^* - \frac{1}{N} \max_{\alpha, \beta \text{ s-bit}} \Delta(\alpha, \beta) \right| < \frac{2}{N} - \frac{1}{N^2}$$

(note that $N = 2^s$).
We will call
\[
\frac{1}{N} \max_{\alpha, \beta \text{ s-bit}} \Delta(\alpha, \beta) =: D_N^d
\]
the *discrete discrepancy* of the net. $D_N^d$ differs from $D_N^*$ at most by the almost negligible quantity $2/N$ and seems for nets to be the more natural measure for the irregularities of distribution.

For a sequence of digital $(0, s, 2)$-nets, $s = 1, 2, \ldots, N = 2^s$, we have
\[
\limsup_{N \to \infty} \frac{N D_N^*}{\log N} \leq \limsup_{N \to \infty} \frac{N D_N^d}{\log N}
\]
(the same holds for lim inf and for lim if it exists).

But if we want to obtain “exact results” the quantity $D_N^d$ in spite of the minimal difference is much easier to handle than $D_N^*$.

This is clearly illustrated by the proof of the following theorem, in which we give the exact value of $D_N^d$ and of $D_N^*$ for the Hammersley net and the exact places where they are attained. For $D_N^d$ we moreover give the “second successive maxima” and the exact places where they are attained. The proof for $D_N^d$ is much shorter than the one for $D_N^*$.

In [4] Halton and Zaremba claim that they give the exact value of $D_N^*$, but they only give a vague hint on how to prove the extremality of the extremal intervals. Entacher [3] uses their result.

**Theorem 4.** (a) For the discrete discrepancy $D_N^d$ of the Hammersley net with $N = 2^s$ points we have
\[
ND_N^d = \max_{\alpha, \beta \text{ s-bit}} \Delta(\alpha, \beta) = \frac{s}{3} + \frac{1}{9} - \frac{(-1)^s}{9 \cdot 2^s}
\]
and the maximum will be attained if and only if $\alpha, \beta$ are of the form $\alpha_0, \beta_0$ with:

- for $s$ odd,
  \[
  \alpha_0 = 0.0101 \ldots 101, \quad \beta_0 = 0.1010 \ldots 0101
  \]
or
  \[
  \alpha_0 = 0.1010 \ldots 0101, \quad \beta_0 = 0.0110 \ldots 1011,
  \]
- for $s$ even,
  \[
  \alpha_0 = \beta_0 = 0.1010 \ldots 1011 \quad \text{or} \quad \alpha_0 = \beta_0 = 0.0101 \ldots 0101.
  \]

The second successive maximum for $\Delta(\alpha, \beta)$ (\(\alpha, \beta\) s-bit) is given by
\[
\max_{\alpha, \beta \text{ s-bit}} \Delta(\alpha, \beta) = \frac{s}{3} + \frac{1}{36} - \frac{(-1)^s}{9 \cdot 2^s}
\]
and the places where this is attained can easily be obtained from the proof and from Theorem 3(b).
(b) For the star-discrepancy $D_N^*$ of the Hammersley net with $N = 2^s$ points we have

$$ND_N^* = \frac{s}{3} + \frac{13}{9} - (-1)^s \frac{4}{9 \cdot 2^s}$$

and the maximum is attained if and only if $\alpha, \beta$ are of the form $\alpha_0, \beta_0$ with:

- for $s$ odd,
  $$\alpha_0 = 0.1010 \ldots 10111, \quad \beta_0 = 0.1101 \ldots 01011$$
  or
  $$\alpha_0 = 0.1101 \ldots 01011, \quad \beta_0 = 0.1010 \ldots 10111,$$

- for $s$ even,
  $$\alpha_0 = \beta_0 = 0.1010 \ldots 01011 \quad \text{or} \quad \alpha_0 = \beta_0 = 0.1101 \ldots 10111$$

for $s \geq 4$. For $s \leq 3$ the extremal values $(\alpha_0, \beta_0)$ are $(1/2, 1/2)$ $(s = 1)$, $(3/4, 3/4)$ $(s = 2)$ and $(7/8, 7/8)$ $(s = 3)$.

Let us first draw a further consequence from the result and let us defer the proof of Theorem 4 to the end of this section.

As an almost immediate consequence we get the following bound for the discrepancy of digital $(0, s, 2)$-nets in base 2, which improves the bounds (1) and (2).

**Theorem 5.** For the star-discrepancy $D_N^*$ of a digital $(0, s, 2)$-net in base 2 we have

$$ND_N^* \leq \frac{s}{3} + \frac{19}{9}.$$

This bound is (by Theorem 4(b)) up to the summand $19/9$ (which could be improved to $15/9$) best possible.

In particular,

$$\lim_{N \to \infty} \max \frac{ND_N^*}{\log N} = \frac{1}{3 \log 2} = 0.4808\ldots$$

where the maximum is taken over all digital $(0, s, 2)$-nets in base 2.

The value $1/(3 \log 2)$ is attained for example for the sequence of Hammersley nets.

**Proof.** We have

$$D_N^* \leq D_N^d + \frac{2}{N} - \frac{1}{N^2},$$

hence by Theorems 1 and 3,

$$ND_N^* \leq 2 + \max_{\beta \text{ s-bit}} \sum_{u=0}^{s-1} \|2^u \beta\| - \frac{1}{2^s} \leq \frac{s}{3} + \frac{19}{9}.$$
From this and from Theorem 4,
\[ \lim_{N \to \infty} \max_N \frac{ND_N^*}{\log N} = \frac{1}{3 \log 2}. \]

For the proof of part (b) of Theorem 4 we need some notation:

REMARK 8. For
\[ \alpha = 0.a_1 \ldots a_t \ldots a_s, \quad \beta = 0.b_1 \ldots b_{s-t} \ldots b_s \]
we define
\[ \alpha_t := 0.a_1 \ldots a_t, \quad \beta_t := 0.b_{s+1-t} \ldots b_s, \]
\[ \bar{\alpha}_t := 0.a_{t+1} \ldots a_s, \quad \bar{\beta}_t := 0.b_1 \ldots b_{s-t}. \]

Further, set
\[ \Sigma_s(\alpha, \beta) := \sum_{u=0}^{s-1} \|2^u \beta\| \sigma(u) \quad \text{ with } \sigma(u) := a_{s-u} \oplus a_{s+1-j(u)}. \]

In \( \sigma(u) \) we usually set \( a_{s+1-j(u)} = 0 \) as long as \( j(u) = 0 \). If in this case we alternatively set \( a_{s+1-j(u)} := 1 \) then we denote the corresponding sum by \( \Sigma^1_s(\alpha, \beta) \).

Further we define
\[ T_s(\alpha, \beta) := \alpha + \beta + \Sigma_s(\alpha, \beta). \]

For \( \kappa, \tau \in \mathbb{R} \) we more generally define
\[ T^\tau_\kappa(\alpha, \beta) := \tau \alpha + \kappa \beta + \Sigma_s(\alpha, \beta). \]

Now
\[ T_s(\alpha, \beta) = \alpha + \beta + \Sigma_s(\alpha, \beta) \]
\[ = \alpha + \beta + \Sigma_{s-t}(\bar{\alpha}_t, \bar{\beta}_t) + \beta_t \sum_{u=0}^{s-t-1} \left( \frac{-1}{2^{s-t-u}} \right) \sigma(u) + \bar{T}_t(\alpha_t, \beta_t). \]

Here \( \bar{T}_t(\alpha_t, \beta_t) \) is either \( \Sigma_t(\alpha_t, \beta_t) \) or \( \Sigma^1_t(\alpha_t, \beta_t) \).

Since \( \alpha = \alpha_t + \frac{1}{2^t} \bar{\alpha}_t \) and \( \beta = \bar{\beta}_t + \frac{1}{2^t} \beta_t \) we get
\[ T_s(\alpha, \beta) = T^\tau_\kappa_{s-t}(\bar{\alpha}_t, \bar{\beta}_t) + \bar{T}_t^{\tau_\kappa_1}(\alpha_t, \beta_t), \]

where \( \bar{T} \) is defined via \( \bar{T} \) instead of \( \Sigma \), and \( \tau = 1/2^t \), and
\[ \kappa_t = \frac{1}{2^{s-t}} + \sum_{u=0}^{s-t-1} \left( \frac{-1}{2^{s-t-u}} \right) \sigma(u). \]

Here it is important to note that \( \kappa \) only depends on the form of \( \bar{\alpha}_t \) and \( \bar{\beta}_t \).

Let us consider for example \( t = 6 \). Then it is an easy task to show with the help of MATHEMATICA that for all \( d \in \{0, \ldots, 2^6 - 1\} \) we have
\[ |\max_{\alpha_6, \beta_6} T^1_6 d/2^6(\alpha_6, \beta_6) - \max_{\alpha_6, \beta_6} \bar{T}^1_6 d/2^6(\alpha_6, \beta_6)| \leq 1/2^6. \]
Hence for all $\kappa$
\[ |\max_{\alpha_6, \beta_6} T_6^{1,\kappa}(\alpha_6, \beta_6) - \max_{\alpha_6, \beta_6} \tilde{T}_6^{1,\kappa}(\alpha_6, \beta_6)| < 1/2^5. \]

Further we need the following lemma:

**Lemma 6.** If
\[ T_s(\alpha_0, \beta_0) = \max_{\alpha, \beta, s-bit} T_s(\alpha, \beta), \]
then $\beta_0$ has at most three consecutive equal digits $b_ib_{i+1}b_{i+2}, i \geq 2$, in its base 2 representation.

**Proof.** First we note that the first digit of $\beta_0$ must be one, otherwise replacing $\beta_0$ by $\beta_0 + 1/2$ and choosing a suitable $\alpha_0$ gives a larger value $T$.

Then we note that, as is easily calculated, the special choice
\[ \alpha' = 0.101\ldots011, \quad \beta' = 0.101\ldots011 \]
if $s$ is even and
\[ \alpha' = 0.1101\ldots011, \quad \beta' = 0.1010\ldots011 \]
if $s$ is odd gives the value
\[ T_s(\alpha', \beta') = \frac{s}{3} + \frac{13}{9} + \frac{1}{2s} - (-1)^s \cdot \frac{4}{9} \cdot \frac{1}{2s}. \]

Assume now on the contrary that $\beta_0$ has at least four equal digits $b_ib_{i+1}b_{i+2}b_{i+3}, i \geq 2$, in its base 2 representation. Assume these are ones (the other case is handled in the same way). Then
\[ T_s(\alpha_0, \beta_0) \leq 1 + \beta_0 + \sum_{u=0}^{s-1} ||2^u\beta_0||. \]

Now we can apply some of the transformations from Corollary 1 to $\beta_0$ until $b_ib_{i+1}$ is the first block of equal digits (with $i \geq 2$). Therefore
\[ \beta_0 + \sum_{u=0}^{s-1} ||2^u\beta_0|| \]
will not decrease. Now we can apply two times one of the last two transformations from Corollary 1 to $b_ib_{i+1}$ and then to $b_{i+1}b_{i+2}$. Note that $\tau \geq 3/4$ in the first application and $\tau \geq 1/2$ in the second. Therefore
\[ \beta_0 + \sum_{u=0}^{s-1} ||2^u\beta_0|| \]
increases at least by
\[
\frac{1}{3} \cdot \frac{3}{4} \left( 1 - \frac{(-1)^{i-1}}{2^{i-1}} \right) + \frac{1}{3} \cdot \frac{1}{2} \left( 1 - \frac{(-1)^i}{2^i} \right) = \frac{5}{12} + \frac{(-1)^i}{3 \cdot 2^i} \geq \frac{3}{8}.
\]
Hence we have, by the remark at the beginning of this proof and by Corollary 2,
Sums of distances to the nearest integer

\[
\frac{s}{3} + \frac{13}{9} + \frac{1}{2^s} - (-1)^s \left( \frac{4}{9} \cdot \frac{1}{2^s} \right) \leq T_s(\alpha_0, \beta_0) \\
\leq 1 + \max_{\beta \text{ s-bit}} \left( \beta + \sum_{u=0}^{s-1} \|2^u \beta\| \right) - \frac{3}{8} \\
= \frac{5}{8} + \frac{s}{3} + \frac{7}{9} + (-1)^s \frac{2}{9 \cdot 2^s},
\]

hence

\[
\frac{1}{24} + \frac{1}{2^s} \left( 1 - \frac{2}{3}(-1)^s \right) \leq 0,
\]
a contradiction. ■

Remark 9. It is easy to show with the help of a C++ program that the assertion of Theorem 4(b) holds for \( s \leq 11 \).

In fact it is not difficult to prove (with the help of Lemmas 5 and 6) that the extremal values \( \alpha_0, \beta_0 \) from Theorem 4(b) must have the property that \( a_{s-u} \oplus a_{s+1-j(u)} = 1 \) for all \( u = 0, \ldots, s - 1 \). Hence for every \( \beta_0 \) there is only one possible \( \alpha_0 \). So it was easily possible to carry out the numerical calculation with Mathematica.

Proof of Theorem 4. (a) We use Example 2. For a given \( \beta \) the value

\[
\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\|(a_{s-u} \oplus a_{s+1-j(u)})
\]
always becomes maximal if \( \alpha \) is chosen such that \( a_{s-u} \oplus a_{s+1-j(u)} = 1 \) for all \( u \). Hence \( D_N^1 \) is attained for the \( \beta \) maximizing

\[
\sum_{u=0}^{s-1} \|2^u \beta\|
\]
(those are provided by Theorem 3) and the corresponding \( \alpha \). This gives the values claimed in the result.

For the second successive maximum there are principally two possible cases: either \( a_{s-u} \oplus a_{s+1-j(u)} = 1 \) for all \( u \), and then \( \beta \) must be of the form from Theorem 3(b), or \( a_{s-u} \oplus a_{s+1-j(u)} = 0 \) for some \( u \). But comparing Theorem 3 and Lemma 5 shows that only the first case can give the second successive maximum.

(b) For \( \alpha, \beta \) \( s \)-bit \( \Delta(\alpha, \beta) \) always is positive by Example 2. Hence \( D_N^* \) will certainly be attained for intervals of the form

\[
[0, \alpha - 1/2^s] \times [0, \beta - 1/2^s]
\]
with \( \alpha, \beta \) \( s \)-bit, and therefore

\[
ND_N^* = \max_{\alpha, \beta \text{ s-bit}} (\Delta(\alpha, \beta) + \alpha + \beta) - 1/2^s
\]
(see Remark 2). By Remark 9 it suffices to assume that \( s \geq 12 \). Let \( \alpha^{(0)}, \beta^{(0)} \) be such that

\[ T_s(\alpha^{(0)}, \beta^{(0)}) = \max_{\alpha, \beta \text{ s-bit}} T_s(\alpha, \beta). \]

By Lemma 6, \( \beta^{(0)} \) has at most three consecutive equal digits (after the first place) and the first digit \( b_1 \) of \( \beta^{(0)} \) is 1. Assume there is a \( u \leq s - 12 \) such that \( \sigma(u) = 0 \) (see Remark 8 for the notations here and in the following), and let \( u_0 \) be maximal with this property. Then change \( a_{s-u_0}, \ldots, a_7 \) so that \( \sigma(u_0) \) becomes 1 and \( \sigma(u_0+1), \ldots, \sigma(s-7) \) remain unchanged. Thereby \( \kappa_6 \) changes at most by \( 1/2^{s-6-u_0} \leq 1/2^6 \). Finally choose \( a_6, \ldots, a_1 \) and \( b_{s-5}, \ldots, b_s \) so that \( \tilde{T}^{1,\kappa_6}(\alpha'_6, \beta'_6) \) becomes maximal for the new values \( \alpha', \beta' \). Then (see Remark 8),

\[ T_s(\alpha', \beta') = T_{s-6}^{1,1}(-\alpha'_6, -\beta'_6) + \tilde{T}^{1,\kappa_6}(\alpha'_6, \beta'_6) \]

(note that we obtain a new summand of value at least 1/4, but \( \alpha \) may decrease to almost zero)

\[ \geq T_{s-6}^{1,1}(\alpha^{(0)}_6, \beta^{(0)}_6) + \frac{1}{4} - \frac{1}{2^6} + \tilde{T}^{1,\kappa_6}(\alpha'_6, \beta'_6) \]

(by the numerical result in Remark 8; note that the tilde on \( \tilde{T} \) is here related to \( \alpha', \beta' \) and in the following line to \( \alpha^{(0)}, \beta^{(0)} \))

\[ \geq T_{s-6}^{1,1}(\alpha^{(0)}_6, \beta^{(0)}_6) + \frac{1}{4} - \frac{1}{2^6} + \tilde{T}^{1,\kappa_6}(\alpha^{(0)}_6, \beta^{(0)}_6) \]

\[ > T_s(\alpha^{(0)}, \beta^{(0)}) + \frac{1}{2^4} - \frac{4}{2^6} \]

\[ = T_s(\alpha^{(0)}, \beta^{(0)}), \]

a contradiction. Hence

\[ T_s(\alpha^{(0)}, \beta^{(0)}) = \beta^{(0)} + \sum_{u=0}^{s-12} \|2^u \beta^{(0)}\| \]

\[ + \frac{1}{2^1} \alpha^{(0)}_{11} + \beta^{(0)}_{11} + \tilde{\Sigma}_{11}(\alpha^{(0)}_{11}, \beta^{(0)}_{11}). \]

Therefore by Corollary 1, \( b^{(0)}_{s-11} \) and \( a^{(0)}_{s-12} \) must be of the form (we concentrate on “\( s \) odd”, “\( s \) even” being carried out quite analogously)

\[ \beta^{(0)}_{11} = 0.110101 \ldots 01, \quad \alpha^{(0)}_{11} = 0.0101 \ldots 0111 \]

or

\[ \beta^{(0)}_{11} = 0.1010 \ldots 011, \quad \alpha^{(0)}_{11} = 0.0101 \ldots 011. \]

So it remains to maximize \( \tilde{T}^{1,\kappa}(\alpha_{11}, \beta_{11}) \).
In the first case we have
\[ |\kappa + \frac{1}{3} \left( 1 - \frac{1}{2^{12}} \right) | < \frac{1}{2^{13}}, \]
in the second case we have
\[ |\kappa - \frac{1}{3} \left( 1 - \frac{1}{2^{12}} \right) | < \frac{1}{2^{13}}, \]
so it suffices to maximize
\[ \tilde{T}_{11}^{1, 1/2^{12}}(\alpha_{11}, \beta_{11}) \] respectively \[ \tilde{T}_{11}^{\frac{1}{2} (1-1/2^{12})}(\alpha_{11}, \beta_{11}). \]
This is easily done with a Mathematica program and the result follows. ■

**5. A class of nets with smaller star-discrepancy.** We have seen in Theorem 5 that the Hammersley net essentially is the “worst” distributed digital \((0, s, 2)\)-net in base 2.

We will show here that the star-discrepancy of the nets generated by

\[
C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

and

\[
C_2 = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

is essentially smaller. Indeed it seems, by numerical experiments carried out by Entacher, that these nets are the essentially best distributed digital \((0, s, 2)\)-nets in base 2. We have

**Theorem 6.** For the star-discrepancy \(D^*_N\) of the digital net in base 2 generated by

\[
C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

and

\[
C_2 = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

we have

\[ \frac{ND^*_N}{s} \geq 0.2 \]

for all \(N \) \((N = 2^s)\) and

\[ \limsup_{N \to \infty} \frac{ND^*_N}{s} \leq 0.226341 \ldots \]

**Remark 10.** Hence for these nets we have

\[ 0.2885 \ldots = \frac{1}{5 \log 2} \leq \liminf_{N \to \infty} \frac{ND^*_N}{\log N} \leq \limsup_{N \to \infty} \frac{ND^*_N}{\log N} \leq 0.32654 \ldots \]
Indeed we conjecture that
\[ \lim_{N \to \infty} \frac{ND_N^*}{\log N} = \frac{1}{5 \log 2}, \]
and that this is the best possible value at all, i.e.
\[ \lim_{N \to \infty} \min \frac{ND_N^*}{\log N} = \frac{1}{5 \log 2}, \]
where the minimum is taken over all digital \((0, 2)-nets\) in base 2.

**Proof of Theorem 6.** We will show that the lower bound even holds for
\[ \max_{\alpha, \beta \text{-} s \text{-bit}} \Delta(\alpha, \beta), \]
and also for the upper bound it suffices to consider \(\Delta(\alpha, \beta)\) for \(\alpha, \beta\) \(s\)-bit.

Recall from Example 3 that for \(\alpha, \beta\) \(s\)-bit we have
\[ \Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\| (1)(-1)^{a_1+\ldots+a_s-u} (-1)^{a_{s-u} - (1)^{a_{s+1-j(u)}}} \]
\[ = \sum_{u=0}^{s-1} \|2^u \beta\| (1)(-1)^{a_1+\ldots+a_{s-u}-1} (a_{s-u} \oplus a_{s+1-j(u)}), \]
where
\[ j(u) := \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } a_1 + \ldots + a_{s+1-j} = b_j \text{ for } j = 1, \ldots, u, \\ \max\{j \leq u : a_1 + \ldots + a_{s+1-j} \neq b_j\} & \text{otherwise}. \end{cases} \]

We set \(\tilde{a}_i := a_1 + \ldots + a_{s+1-i}\) and \(\tilde{\alpha} := 0.\tilde{a}_1 \ldots \tilde{a}_s\). Then
\[ a_{s+1-i} = \tilde{a}_i \oplus \tilde{a}_{i+1}, \quad a_{s+1-j(u)} = \tilde{a}_{r(u)} \oplus \tilde{a}_{r(u)+1}, \]
where
\[ r(u) := \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } b_j = \tilde{a}_j \text{ for } j = 1, \ldots, u, \\ \max\{r \leq u : b_j \neq \tilde{a}_j\} & \text{otherwise}, \end{cases} \]
and where we have to set \(\tilde{a}_{r(u)} \oplus \tilde{a}_{r(u)+1} := 0\) if \(r(u) = 0\) and \(\tilde{a}_{s+1} := 0\). Then
\[ \Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u \beta\| g(u) =: \delta(\tilde{\alpha}, \beta), \]
where
\[ g(u) := (-1)^{\tilde{a}_{u+2}} (\tilde{a}_{u+1} \oplus \tilde{a}_{u+2} \oplus \tilde{a}_{r(u)} \oplus \tilde{a}_{r(u)+1}). \]

To obtain the lower bound consider
\[ \beta = 0.00100010001 \ldots b_s, \quad \tilde{\alpha} = 0.10001000100 \ldots a_s \]
with the exception that \( b_s = 1 \) instead of 0 if \( s = 4l + 1 \) or \( s = 4l + 2 \). Then

\[
g(u) = \begin{cases}  \ -1 & \text{if } u = 4l + 3, \\  \ 1 & \text{otherwise} \end{cases}
\]

with the only exception that \( g(s - 1) = 0 \) if \( s = 4l \). Then

\[
\beta = \sum_{i=0}^{[s/4] - 1} \frac{1}{2^{4i+3}} + \frac{b_s}{2^s},
\]

hence

\[
\| 2^u \beta \| = \sum_{i=\lceil u/4 - 1/2 \rceil}^{[s/4] - 1} \frac{1}{2^{4i+3-u}} + \frac{b_s}{2^{s-u}}
\]

for \( u \neq 4l + 2 \) and it is 1 minus this quantity if \( u = 4l + 2 \). So

\[
\delta(\tilde{\alpha}, \beta) = \sum_{i=0}^{[(s-5)/4]} (\| 2^i \beta \| + \| 2^{4i+1} \beta \| + \| 2^{4i+2} \beta \| - \| 2^{4i+3} \beta \|) + R,
\]

with

\[
R = \begin{cases}  \ 1/2 & \text{if } s = 4l + 1, \\  \ 3/4 & \text{if } s = 4l + 2, \\  \ 7/8 & \text{else.} \end{cases}
\]

Inserting for \( \| 2^u \beta \| \) and evaluating the resulting finite geometric series then yields

\[
\delta(\tilde{\alpha}, \beta) = \frac{4}{5} \left[ \frac{s - 1}{4} \right] + \frac{16^{[(s-1)/4]} - 1}{16^{[s/4]}} \cdot \begin{cases}  \ 2/25 + 7/8 & \text{if } s = 4l, \\  \ (-11/50) + 1/2 & \text{if } s = 4l + 1, \\  \ (-7/100) + 3/4 & \text{if } s = 4l + 2, \\  \ 1/200 + 7/8 & \text{if } s = 4l + 3. \end{cases}
\]

Now it is a simple task to check that in each of the four cases \( \delta(\tilde{\alpha}, \beta)/s \) is decreasing to 1/5, and so the lower bound follows.

To obtain the upper bound consider for given \( r \in \mathbb{N} \) the quantity

\[
\delta_r := \sup_{\alpha, \beta} \sum_{u=0}^{r-1} g(u) \| 2^u \beta \|,
\]

where the supremum is taken over all \( \beta \in [0, 1) \) and over all \( r + 1 \)-bit \( \alpha = 0.a_1 \ldots a_{r+1} \). (Note that this means that \( a_{r+1} \) is not automatically set to 0 as is done for \( r \)-bit \( \alpha \).

This supremum is obviously attained (respectively approached) in the following form: let \( u_0 \) be the largest index such that \( g(u_0) \neq 0 \); then \( g(u_0) = 1 \). Further the supremum is attained for some \( \beta \) with \( b_{r+1} = b_{r+2} = \ldots = 0 \) if \( b_{u_0+1} = 1 \) and it is approached by \( \beta \) with \( b_{r+1} = b_{r+2} = \ldots = 1 \) if \( b_{u_0+1} = 0 \).
So it can be shown for example with MATHEMATICA that

\[ \delta_{11} = \frac{5099}{2048} = 2.48975 \ldots, \]

and this value is attained with \( b_{u0+1} = 1 \).

Now for \( s \) with \( s = 11q + w, 0 \leq w \leq 10 \), for all \( \alpha, \beta \) we have \( \delta(\alpha, \beta) \leq q\delta_{11} + w \), hence

\[ \frac{\delta(\alpha, \beta)}{s} \leq \frac{1}{s} \left\lfloor \frac{s}{11} \right\rfloor \cdot 2.48975 \ldots + \frac{10}{s}, \]

which tends to 0.226341 \ldots as \( s \to \infty \), and the result follows.

References