Artin $L$-functions and modular forms associated to quasi-cyclotomic fields

by

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1. Introduction. A quadratic extension of a cyclotomic field, which is non-abelian Galois over the rational number field $\mathbb{Q}$, is called a quasi-cyclotomic field. All quasi-cyclotomic fields are described explicitly in [8] following the work in [1] and [3]. Actually for any cyclotomic field $\mathbb{Q}(\zeta_n)$ we construct a canonical $\mathbb{Z}/2\mathbb{Z}$-basis of the quotient space of $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q} \text{ is Galois}\}$ modulo the subspace $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q} \text{ is abelian}\}$. The minimal quasi-cyclotomic field containing the square root of a special element of the basis is called a primary quasi-cyclotomic field. L. S. Yin and C. Zhang [7] have studied the arithmetic of any quasi-cyclotomic field. In this paper we determine all irreducible representations of primary quasi-cyclotomic fields. Our methods enable one to determine the irreducible representations of an arbitrary quasi-cyclotomic field. We also compute the Artin conductors of the representations and the Artin $L$-functions for a class of quasi-cyclotomic fields. They correspond to a series of normalized newforms of weight one by Deligne–Serre’s theorem [6, Th. 2]. We describe these modular forms explicitly.

First we recall the construction of primary quasi-cyclotomic fields. Let $S$ be the set consisting of $-1$ and all prime numbers. For $p \in S$, we put $\bar{p} = 4, 8, p$ and set $p^* = -1, 2, (-1)^{(p-1)/2}p$ if $p = -1, 2$ and an odd prime number, respectively. For prime numbers $p < q$, we define

$$v_{pq} = \prod_{i=0}^{(p-1)/2} \prod_{j=0}^{(q-1)/2} \frac{\sin \frac{iq+i\pi}{pq}}{\sin \frac{jp+j\pi}{pq}} \quad ((i, j) \neq (0, 0), \ p > 2)$$

and

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We also define \( \sqrt{G} \) by generators and relations. An element \( G \) has the form \( \sqrt{pq} \), and let \( K = \mathbb{Q}(\sqrt{pq}) \) be the cyclotomic field of conductor \( \sqrt{pq} \) and let \( \tilde{K} = K(\sqrt{u_{pq}}) \). Then \( \tilde{K} \) is the smallest quasi-cyclotomic fields containing \( \sqrt{u_{pq}} \). We call these fields \( \tilde{K} \) primary quasi-cyclotomic fields. Let \( G = \text{Gal}(\mathbb{K}/\mathbb{Q}) \) and \( \tilde{G} = \text{Gal}(\tilde{K}/\mathbb{Q}) \). We always denote by \( \varepsilon \) the unique non-trivial element of \( \text{Gal}(\tilde{K}/\mathbb{K}) \). If \((p, q) = (1, 2)\), then the group \( G \) is generated by two elements \( \sigma_{-1} \) and \( \sigma_2 \), where \( \sigma_{-1}(\zeta_8) = \zeta_8^{-1} \) and \( \sigma_2(\zeta_8) = \zeta_8^5 \). If \( p = 1 \) and \( q \neq 2 \), or if \( p > 2 \), then \( G \) is generated by two elements \( \sigma_p \) and \( \sigma_q \), where \( \sigma_p(\zeta_q) = \zeta_8^q \), \( \sigma_p(\zeta_q) = \zeta_q \), and \( \sigma_q(\zeta_p) = \zeta_q \), \( \sigma_q(\zeta_p) = \zeta_q \), with \( a, b \) being generators of \((\mathbb{Z}/p\mathbb{Z})^* \) and \((\mathbb{Z}/q\mathbb{Z})^* \) respectively.

Next we describe the group \( \tilde{G} \) by generators and relations. An element \( \sigma \in G \) has two lifts in \( \tilde{G} \). By \[6\] Sect. 3 the action of the two lifts on \( \sqrt{u_{pq}} \) has the form \( \pm \alpha \sqrt{u_{pq}} \) or \( \pm \alpha \sqrt{u_{pq}}/\sqrt{-1} \) with \( \alpha > 0 \). We fix the lift \( \tilde{\sigma} \) of \( \sigma \) to be the one with a positive sign. Then the other lift of \( \sigma \) is \( \tilde{\sigma} \varepsilon \). The group \( \tilde{G} \) is generated by \( \varepsilon, \tilde{\sigma}_p \) and \( \tilde{\sigma}_q \) (and \( \tilde{\sigma}_{-1} \) if \( p = 2 \)). Clearly \( \varepsilon \) commutes with the other generators. In addition, we have \( \tilde{\sigma}_p \tilde{\sigma}_q = \tilde{\sigma}_q \tilde{\sigma}_p \varepsilon \) (and \( \tilde{\sigma}_{-1} \) commutes with \( \tilde{\sigma}_2 \) and \( \tilde{\sigma}_q \) if \( p = 2 \)). For an element \( g \) of a group, we denote by \( |g| \) the order of \( g \) in the group. Let \( \log_{-1} : \{ \pm 1 \} \to \mathbb{Z}/2\mathbb{Z} \) be the unique isomorphism. For an odd prime \( p \) and an integer \( a \) with \( p \nmid a \), let \( \left( \frac{a}{p} \right) \) be the quadratic residue symbol. We also define \( \left( \frac{a}{2} \right) = \left( \frac{a}{-1} \right) = 1 \) for any \( a \). Then we have (see \[6\] Th. 3)

\[
|\tilde{\sigma}_p| = \left( 1 + \log_{-1} \left( \frac{q^*}{p} \right) \right) |\sigma_p| \quad \text{and} \quad |\tilde{\sigma}_q| = \left( 1 + \log_{-1} \left( \frac{p^*}{q} \right) \right) |\sigma_q|,
\]

with the exception that \( |\tilde{\sigma}_2| = 2 |\sigma_2| \) when \((p, q) = (1, 2)\). If \( p = 2 \), we have furthermore \( |\tilde{\sigma}_{-1}| = |\sigma_{-1}| \). Thus we have completely determined the group \( \tilde{G} \) by generators and relations.
2. Abelian subgroup of index 2. In this section we construct a special abelian subgroup of $\tilde{G}$ of index 2 and determine its structure. We consider the following three cases separately:

Case A: $|\tilde{\sigma}_p| = |\sigma_p|$ and $|\tilde{\sigma}_q| = |\sigma_q|$;
Case B: $|\tilde{\sigma}_p| = 2|\sigma_p|$, $|\tilde{\sigma}_q| = |\sigma_q|$ or $|\tilde{\sigma}_p| = |\sigma_p|$, $|\tilde{\sigma}_q| = 2|\sigma_q|$;
Case C: $|\tilde{\sigma}_p| = 2|\sigma_p|$ and $|\tilde{\sigma}_q| = 2|\sigma_q|$.

All the three cases may happen: Case A if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q^*}{p}\right) = 1$;
Case B if and only if $\left(\frac{p^*}{q}\right) \neq \left(\frac{q^*}{p}\right)$ or $(p, q) = (-1, 2)$; Case C if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q^*}{p}\right) = -1$.

In Case A, we define the subgroup $N$ of $\tilde{G}$ to be

$$N = \begin{cases} \langle \tilde{\sigma}_-1, \tilde{\sigma}_2, \tilde{\sigma}_q^2, \varepsilon \rangle & \text{if } p = 2, \\ \langle \tilde{\sigma}_p, \tilde{\sigma}_q^2, \varepsilon \rangle & \text{if } p \neq 2. \end{cases}$$

It is easy to see that the subgroup $N$ is abelian of index 2 in $\tilde{G}$ and is a direct sum of the cyclic groups generated by the above elements. Thus we have

$$N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q - 1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = -1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q - 1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/(p - 1)\mathbb{Z} \oplus \mathbb{Z}/((q - 1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p > 2. \end{cases}$$

In Case B, we define the subgroup $N$ of $\tilde{G}$ to be

$$N = \begin{cases} \langle \tilde{\sigma}_-1, \tilde{\sigma}_2, \tilde{\sigma}_q^2 \rangle & \text{if } p = 2, \\ \langle \tilde{\sigma}_p, \tilde{\sigma}_q^2 \rangle & \text{if } p \neq 2 \text{ and } |\tilde{\sigma}_q| = 2|\sigma_q|, \\ \langle \tilde{\sigma}_p^2, \tilde{\sigma}_q \rangle & \text{if } |\tilde{\sigma}_p| = 2|\sigma_p|. \end{cases}$$

Again $N$ is abelian and has index 2 in $\tilde{G}$. In addition, we have

$$N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } (p, q) = (-1, 2), \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q - 1)\mathbb{Z} & \text{if } p = -1, q > 2, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q - 1)\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/(p - 1)\mathbb{Z} \oplus \mathbb{Z}/(q - 1)\mathbb{Z} & \text{if } p > 2. \end{cases}$$

In Case C, $p, q$ are both odd prime numbers. Let $v_2(p - 1)$ denote the power of 2 in $p - 1$. We define the subgroup $N$ of $\tilde{G}$ to be

$$N = \begin{cases} \langle \tilde{\sigma}_p^2, \tilde{\sigma}_q \rangle & \text{if } v_2(p - 1) \leq v_2(q - 1), \\ \langle \tilde{\sigma}_p, \tilde{\sigma}_q^2 \rangle & \text{if } v_2(p - 1) > v_2(q - 1). \end{cases}$$

Then $N$ is an abelian subgroup of $\tilde{G}$. When $v_2(p - 1) \leq v_2(q - 1)$, we have

$$|N| = \frac{|\tilde{\sigma}_p^2| \cdot |\tilde{\sigma}_q|}{|\langle \tilde{\sigma}_p^2 \rangle \cap \langle \tilde{\sigma}_q \rangle|} = \frac{(p - 1) \cdot 2(q - 1)}{2},$$
thus $[\widetilde{G} : N] = 2$ and $N$ is a normal subgroup of $\widetilde{G}$. We have the same result when $v_2(p-1) > v_2(q-1)$. Although the subgroup $\langle \tilde{\sigma}_p^2, \tilde{\sigma}_q \rangle$ is always an abelian subgroup of $\widetilde{G}$ of index 2, when $v_2(p-1) > v_2(q-1)$ we are not able to get all irreducible representations of $\widetilde{G}$ from this subgroup. So we define $N$ in two cases.

Next we determine the structure of the subgroup $N$ in Case C. We consider the case $v_2(p-1) \leq v_2(q-1)$ in detail. Let $d = \text{gcd}((p-1)/2, q-1)$, $s = (p-1)/2d$ and $t = (q-1)/d$. Choose $u, v \in \mathbb{Z}$ such that $us + vt = 1$. We have the relations

$$\tilde{\sigma}_p^{p-1} = 1, \quad (\tilde{\sigma}_p^{2})^{(p-1)/2} = \varepsilon = \tilde{\sigma}_q^{-1}.$$

Let $M$ be the free abelian group generated by two words $\alpha, \beta$. Let

$$\alpha_1 = (p-1)\alpha, \quad \beta_1 = \frac{p-1}{2} \alpha - (q-1)\beta,$$

and let $M_1$ be the subgroup of $M$ generated by $\alpha_1, \beta_1$. Then $M_1$ is the kernel of the homomorphism

$$M \to N, \quad \alpha \mapsto \tilde{\sigma}_p^2, \quad \beta \mapsto \tilde{\sigma}_q.$$

So we have $N \cong M/M_1$. Define the matrix

$$A = \begin{pmatrix} p-1 & (p-1)/2 \\ 0 & 1 - q \end{pmatrix}.$$

Then $(\alpha_1, \beta_1) = (\alpha, \beta) \cdot A$. We determine the structure of $M_1$ by considering the standard form of $A$. Define

$$P = \begin{pmatrix} u & v \\ -t & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \quad Q = \begin{pmatrix} 1 & 2tv - 1 \\ -1 & -2tv + 2 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$

Then

$$B = PAQ = \begin{pmatrix} d & 0 \\ 0 & -2s(q-1) \end{pmatrix}$$

is the standard form of $A$. Let

$$(\tau, \mu) = (\alpha, \beta)P^{-1} \quad \text{and} \quad (\tau_1, \mu_1) = (\alpha_1, \beta_1)Q.$$ 

Then $(\tau_1, \mu_1) = (\tau, \mu)B$, $M = \mathbb{Z}\tau \oplus \mathbb{Z}\mu$ and $M_1 = \mathbb{Z}d\tau \oplus \mathbb{Z}2s(q-1)\mu$. We thus have

$$N \cong M/M_1 \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z}.$$ 

By abuse of notation, we also write

$$(\tau, \mu) = (\tilde{\sigma}_p^2, \tilde{\sigma}_q)P^{-1} = (\tilde{\sigma}_p^{2s}, \tilde{\sigma}_q^{2t}, \tilde{\sigma}_q^{-2v}, \tilde{\sigma}_q^{-u}).$$

Then $\tau, \mu$ are of order $d, 2s(q-1)$ respectively, and $N$ is a direct sum of $\langle \tau \rangle$ and $\langle \mu \rangle$. We have $\tilde{\sigma}_p^2 = \tau^v \mu^{-t}$ and $\tilde{\sigma}_q = \tau^v \mu^s$. When $v_2(p-1) > v_2(q-1)$,
we get the structure of $N$ in the same way. So in Case C we have
\[(C2.2) \quad N \cong \begin{cases} \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z} & \text{if } v_2(p-1) \leq v_2(q-1), \\ \mathbb{Z}/d'\mathbb{Z} \oplus \mathbb{Z}/2s'(p-1)\mathbb{Z} & \text{if } v_2(p-1) > v_2(q-1), \end{cases}\]
where $d = \gcd((p-1)/2, q-1)$, $s = (p-1)/2d$ and $d' = \gcd(p-1, (q-1)/2)$, $s' = (q-1)/2d'$.

Now we summarize our results in the following

**Proposition 2.1.** The abelian subgroup $N$ of the group $\tilde{G}$ of index 2 defined in (A2.1), (B2.1) and (C2.1) has the structure described in (A2.2), (B2.2) and (C2.2) in Cases A, B and C, respectively. In particular, every irreducible representation of $\tilde{G}$ has dimension 1 or 2.

### 3. 2-dimensional representations.

We determine all irreducible representations of $\tilde{G}$ in this section. We will freely use some basic facts from representation theory. For the details, see [5].

It is well-known that the 1-dimensional representations of $\tilde{G}$ correspond bijectively to those of the maximal abelian quotient $G$ of $\tilde{G}$, which are Dirichlet characters. So we construct the 2-dimensional irreducible representations of $\tilde{G}$. From the dimension formula for all irreducible representations, we see that $\tilde{G}$ has $|G|/4$ irreducible representations of dimension 2, up to isomorphism. Let $N$ be the subgroup of $\tilde{G}$ defined in the previous section. Let $\tilde{G} = N \cup \sigma N$ be the decomposition into cosets. If $\rho : N \to \mathbb{C}^*$ is a representation of $N$, the induced representation $\tilde{\rho}$ of $\rho$ is a representation of $\tilde{G}$ of dimension 2. The space of the representation $\tilde{\rho}$ is $V = \text{Ind}_N^\tilde{G}(\mathbb{C}) = \mathbb{C}[\tilde{G}] \otimes_{\mathbb{C}[N]} \mathbb{C}$ with basis $e_1 = 1 \otimes 1$ and $e_2 = \sigma \otimes 1$. The group homomorphism
\[
\tilde{\rho} : \tilde{G} \to \text{GL}(V) \simeq \text{GL}_2(\mathbb{C})
\]
is given by
\[(3.1) \quad \tilde{\rho}(\tilde{\sigma}) = \begin{pmatrix} \rho(\tilde{\sigma}) & \rho(\tilde{\sigma}\sigma) \\ \rho(\sigma^{-1}\tilde{\sigma}) & \rho(\sigma^{-1}\tilde{\sigma}\sigma) \end{pmatrix}, \quad \forall \tilde{\sigma} \in \tilde{G},
\]
where $\rho(\tilde{\sigma}) = 0$ if $\tilde{\sigma} \notin N$. The representation $\tilde{\rho}$ is irreducible if and only if $\rho \not\cong \rho^\tau$ for every $\tau \in \tilde{G} \setminus N$, where $\rho^\tau$ is the conjugate representation of $\rho$ defined by
\[
\rho^\tau(x) = \rho(x^{-1}x\tau), \quad \forall x \in N.
\]
Since $N$ is abelian, we only need to check $\rho \not\cong \rho^\sigma$.

Now we begin to construct all 2-dimensional irreducible representations of $\tilde{G}$. As in the previous section, we consider the three cases separately. In addition, we consider the case when $p$ and $q$ are odd prime numbers in detail, and only state the results when $p = -1$ or 2.
3.1. Case A. Assume \( p > 2 \). In this case we have \( N = \langle \sigma_p, \sigma^2_q, \varepsilon \rangle \) and 
\[ N \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \]
Every irreducible representation of \( N \) can be written as \( \rho_{ijk} : N \to \mathbb{C}^* \) with 
\[ \rho_{ijk}(\sigma_p) = \zeta_{p-1}^i, \quad \rho_{ijk}(\sigma^2_q) = \zeta_{q-1}^{2j}, \quad \rho_{ijk}(\varepsilon) = (-1)^k, \]
where \( 0 \leq i < p-1, 0 \leq j < (q-1)/2 \) and \( k = 0, 1 \). Since \( \tilde{G} = N \cup \sigma_qN \) and \( \rho^q_{ijk}(\sigma_p) = \rho_{ijk}(\varepsilon)\rho_{ijk}(\sigma_p) = (-1)^k\rho_{ijk}(\sigma_p) \), we have
\[ \rho^q_{ijk} \neq \rho_{ijk} \iff k = 1. \]
Write \( \rho_{ij} = \rho_{ij1} \). The representation \( \tilde{\rho}_{ij} : \tilde{G} \to \text{GL}_2(\mathbb{C}) \) induced from \( \rho_{ij} \) is given by
\[(A3.1) \quad \tilde{\rho}_{ij}(\sigma_p) = \begin{pmatrix} \zeta_{p-1}^i & 0 \\ 0 & -\zeta_{p-1}^i \end{pmatrix}, \quad \tilde{\rho}_{ij}(\sigma_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}, \quad \tilde{\rho}_{ij}(\varepsilon) = -I, \]
where \( I \) is the identity matrix of degree 2. Since
\[ \tilde{\rho}_{ij}(\sigma^2_p) = \begin{pmatrix} \zeta_{p-1}^{2i} & 0 \\ 0 & \zeta_{p-1}^{2i} \end{pmatrix} \quad \text{and} \quad \tilde{\rho}_{ij}(\sigma^2_q) = \begin{pmatrix} \zeta_{q-1}^{2j} & 0 \\ 0 & \zeta_{q-1}^{2j} \end{pmatrix}, \]
we see that the representations \( \tilde{\rho}_{ij} \) with \( 0 \leq i < (p-1)/2, 0 \leq j < (q-1)/2 \) are irreducible and are not isomorphic to each other, by considering the values of the characters of these representations at \( \sigma^2_p \) and \( \sigma^2_q \). The number of these representations is \( \frac{p-1}{2} \cdot \frac{q-1}{2} = \frac{|G|}{4} \). So they are all the irreducible representations of \( \tilde{G} \) of dimension 2.

Similarly, when \( p = -1 \), all irreducible representations of \( \tilde{G} \) of dimension 2 are \( \tilde{\rho}_j \) with \( 0 \leq j < (q-1)/2 \), where
\[(A3.2) \quad \tilde{\rho}_j(\sigma_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_j(\sigma_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}, \quad \tilde{\rho}(\varepsilon) = -I, \]
and when \( p = 2 \), all irreducible representations of \( \tilde{G} \) of dimension 2 are \( \tilde{\rho}_{ij} \) with \( 0 \leq i \leq 1 \) and \( 0 \leq j < (q-1)/2 \), where \( \tilde{\rho}_{ij}(\varepsilon) = -I \) and
\[(A3.3) \quad \tilde{\rho}_{ij}(\sigma_{-1}) = (-1)^iI, \quad \tilde{\rho}_{ij}(\sigma_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_{ij}(\sigma_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}. \]

3.2. Case B. Assume \( p > 2 \) and \( |\sigma_q| = 2|\sigma_q| \). Then \( N = \langle \sigma_p, \sigma^2_q, \sigma_q \rangle \), and
\[ N \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z}. \]
Any irreducible representation of \( N \) has the form \( \rho_{ij} : N \to \mathbb{C}^* \), where
\[ \rho_{ij}(\sigma_p) = \zeta_{p-1}^i, \quad \rho_{ij}(\sigma^2_q) = \zeta_{q-1}^j, \quad \rho_{ij}(\varepsilon) = \rho_{ij}(\sigma_q^{(q-1)/2}) = (-1)^j, \]
and $0 \leq i < p - 1$, $0 \leq j < q - 1$. It is easy to check that
\[ \tilde{\rho}_{ij} \not\sim \rho_{ij} \iff j \equiv 1 \mod 2. \]

The representation $\tilde{\rho}_{ij} : \tilde{G} \to \text{GL}_2(\mathbb{C})$ induced from $\rho_{ij}$ with odd $j$ is given by
\[
(B3.1) \quad \tilde{\rho}_{ij}(\tilde{\sigma}_p) = \left( \begin{array}{cc} \zeta_p^{i-1} & 0 \\ 0 & -\zeta_p^{-i} \end{array} \right), \quad \tilde{\rho}_{ij}(\tilde{\sigma}_q) = \left( \begin{array}{cc} 0 & \zeta_q^j \\ 1 & 0 \end{array} \right).
\]

Since
\[
\tilde{\rho}_{ij}(\tilde{\sigma}_p^2) = \left( \begin{array}{cc} \zeta_p^{2i} & 0 \\ 0 & \zeta_p^{-2i} \end{array} \right) \quad \text{and} \quad \tilde{\rho}_{ij}(\tilde{\sigma}_q^2) = \left( \begin{array}{cc} \zeta_q^j & 0 \\ 0 & \zeta_q^{-j} \end{array} \right),
\]
we see that the representations $\tilde{\rho}_{ij}$ with $0 \leq i < (p - 1)/2$ and $0 \leq j < q - 1$, $2 \nmid j$ are irreducible and are not isomorphic to each other. The number of these representations is $|G|/4$. So they are all the irreducible representations of $\tilde{G}$ of dimension 2.

Similarly, when $(p, q) = (-1, 2)$, there is only one irreducible representation $\tilde{\rho}_0$ of dimension 2 defined by
\[
(B3.2) \quad \tilde{\rho}_0(\tilde{\sigma}_-1) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \tilde{\rho}_0(\tilde{\sigma}_2) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).
\]

When $p = -1$ and $q > 2$, all irreducible representations of dimension 2 are $\hat{\rho}_j$ with $0 \leq j < q - 1$, $2 \nmid j$, where $\hat{\rho}_j$ is defined by
\[
(B3.3) \quad \hat{\rho}_j(\hat{\sigma}_-1) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \hat{\rho}_j(\hat{\sigma}_q) = \left( \begin{array}{cc} 0 & \zeta_q^j \\ 1 & 0 \end{array} \right).
\]

When $p = 2$, all irreducible representations of dimension 2 are $\hat{\rho}_{ij}$ with $0 \leq i \leq 1$ and $0 \leq j < q - 1$, $2 \nmid j$, where $\hat{\rho}_{ij}$ is defined by
\[
(B3.4) \quad \hat{\rho}_{ij}(\hat{\sigma}_-1) = (-1)^i I, \quad \hat{\rho}_{ij}(\hat{\sigma}_2) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \hat{\rho}_{ij}(\hat{\sigma}_q) = \left( \begin{array}{cc} 0 & \zeta_q^j \\ 1 & 0 \end{array} \right).
\]

When $|\tilde{\sigma}_p| = 2|\sigma_p|$, all irreducible representations of dimension 2 are $\hat{\rho}_{ij}$ with $0 \leq i < p - 1$, $2 \nmid i$ and $0 \leq j < (q - 1)/2$, where $\hat{\rho}_{ij}$ is defined by
\[
(B3.5) \quad \hat{\rho}_{ij}(\hat{\sigma}_p) = \left( \begin{array}{cc} \zeta_p^i & 0 \\ 1 & 0 \end{array} \right), \quad \hat{\rho}_{ij}(\hat{\sigma}_q) = \left( \begin{array}{cc} \zeta_q^j & 0 \\ 0 & -\zeta_q^{-j} \end{array} \right).
\]

### 3.3. Case C.
Assume $v_2(p - 1) \leq v_2(q - 1)$. Let
\[
d = \gcd\left(\frac{p - 1}{2}, q - 1\right), \quad s = \frac{p - 1}{2d}, \quad t = \frac{q - 1}{d}, \quad us + vt = 1.
\]
as before. Here \( t \) must be even and \( u \) odd. Let \( \tau = \tilde{\sigma}^{2s} \cdot \tilde{\sigma}^t \) and \( \mu = \tilde{\sigma}^{-2v} \cdot \tilde{\sigma}^u \). Then \( N = \langle \tilde{\sigma}_p^2, \tilde{\sigma}_q \rangle = \langle \tau, \mu \rangle \) and

\[
N \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z}.
\]

Any irreducible representation \( \rho_{ij} : N \to \mathbb{C}^* \) is of the form

\[
\rho_{ij}(\tau) = \zeta_d^i = \zeta_{2s(q-1)}^{2s(q-1) i}, \quad \rho_{ij}(\mu) = \zeta_d^j = \zeta_{2s(q-1)}^{2s(q-1) j}.
\]

From \( \tilde{\sigma}_p^2 = \tau^u \mu^{-t} \) and \( \tilde{\sigma}_q = \tau^v \mu^s \), we have

\[
\rho_{ij}(\tilde{\sigma}_p^2) = \zeta_{2s(q-1)}^{2su-i-j}, \quad \rho_{ij}(\tilde{\sigma}_q) = \zeta_{2s(q-1)}^{2tv+i-j}, \quad \rho_{ij}(\varepsilon) = \rho_{ij}(\tilde{\sigma}_p^2)^{(p-1)/2} = (-1)^j.
\]

It is easy to show

\[
\tilde{\rho}_{ij} \neq \rho_{ij} \iff j \equiv 1 \mod 2.
\]

The representation \( \tilde{\rho}_{ij} : \tilde{G} \to \text{GL}_2(\mathbb{C}) \) induced from \( \rho_{ij} \) with odd \( j \) is given by

\[
\tilde{\rho}_{ij}(\tau) = \begin{pmatrix} \zeta_d^i & 0 \\ 0 & \zeta_d^i \end{pmatrix}, \quad \tilde{\rho}_{ij}(\mu) = \begin{pmatrix} \zeta_{2s(q-1)}^j & 0 \\ 0 & -\zeta_{2s(q-1)}^j \end{pmatrix}.
\]

Here in the first equality we used the fact that \( t \) is even, and in the second equality we used the fact that \( u \) is odd. Furthermore, we have

\[
(C3.1) \quad \tilde{\rho}_{ij}(\tilde{\sigma}_p) = \begin{pmatrix} 0 & \zeta_{2s(q-1)}^{2su-i-j} \\ 1 & 0 \end{pmatrix}, \quad \tilde{\rho}_{ij}(\tilde{\sigma}_q) = \begin{pmatrix} \zeta_{2s(q-1)}^{2tv+i-j} & 0 \\ 0 & -\zeta_{2s(q-1)}^{2tv+i-j} \end{pmatrix}.
\]

By considering the values of the character of \( \tilde{\rho}_{ij} \) at \( \tau \) and \( \mu^2 \), we see that all the representations \( \tilde{\rho}_{ij} \) with \( 0 \leq i < d \) and \( 0 \leq j < s(q-1) \), \( 2 \nmid j \) are irreducible and are not isomorphic to each other. The number of these representations is \( d \cdot s(q-1)/2 = |G|/4 \). So they are all the irreducible representations of \( \tilde{G} \) of dimension 2.

Similarly, if \( v_2(p-1) > v_2(q-1) \), we let

\[
d' = \gcd\left(p-1, \frac{q-1}{2}\right), \quad s' = p-1 \quad d', \quad t' = q-1 \quad 2d', \quad u's' + v't' = 1.
\]

Then all the irreducible representations of \( \tilde{G} \) of dimension 2 are \( \tilde{\rho}_{ij} \) with \( 0 \leq i < d' \) and \( 0 \leq j < t'(p-1) \), \( 2 \nmid j \), where \( \tilde{\rho}_{ij} \) is defined by

\[
(C3.2) \quad \tilde{\rho}_{ij}(\tilde{\sigma}_p) = \begin{pmatrix} \zeta_{2s'(q-1)}^{2su'i+j} & 0 \\ -\zeta_{2s'(q-1)}^{2s'u'i+j} & 1 \end{pmatrix}, \quad \tilde{\rho}_{ij}(\tilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{2s'(q-1)}^{2tv'i+j} \\ 1 & 0 \end{pmatrix}.
\]

Let \( R^2(\tilde{G}) \) be the set of all irreducible representations, up to isomorphism, of \( \tilde{G} \) of dimension 2. To summarize, we have proved the following

**Theorem 3.1.** All 2-dimensional irreducible representations of \( \tilde{G} \) are induced from representations of \( N \). In detail, we have:
In Case A

\[ R^2(\tilde{G}) = \begin{cases} 
\{\tilde{\rho}_j \mid 0 \leq j < (q - 1)/2\} & \text{if } p = -1, \\
\{\tilde{\rho}_{ij} \mid i = 0,1, 0 \leq j < (q - 1)/2\} & \text{if } p = 2, \\
\{\tilde{\rho}_{ij} \mid 0 \leq i < (p-1)/2, 0 \leq j < (q - 1)/2\} & \text{if } p > 2,
\end{cases} \]

where \(\tilde{\rho}_j\), \(\tilde{\rho}_{ij}\) and \(\tilde{\rho}_{ij}\) are defined in (A3.2), (A3.3) and (A3.1) respectively.

In Case B

\[ R^2(\tilde{G}) = \begin{cases} 
\{\bar{\rho}_0\} & \text{if } (p,q) = (-1,2), \\
\{\bar{\rho}_j \mid 0 \leq j < q-1, 2 \nmid j\} & \text{if } p = -1, q > 2, \\
\{\bar{\rho}_{ij} \mid i = 0,1, 0 \leq j < q-1, 2 \nmid j\} & \text{if } p = 2, \\
\{\bar{\rho}_{ij} \mid 0 \leq i < p-1, 2 \nmid i, 0 \leq j < (q-1)/2\} & \text{if } |\bar{\sigma}_p| = 2|\sigma_p|, \\
\{\bar{\rho}_{ij} \mid 0 \leq i < (p-1)/2, 0 \leq j < q-1, 2 \nmid j\} & \text{otherwise},
\end{cases} \]

where \(\bar{\rho}_0\), \(\bar{\rho}_j\), \(\bar{\rho}_{ij}\) and \(\bar{\rho}_{ij}\) are defined in (B3.2), (B3.3), (B3.4), (B3.5) and (B3.1) respectively.

In Case C

\[ R^2(\tilde{G}) = \begin{cases} 
\{\tilde{\rho}_{ij} \mid 0 \leq i < d, 0 \leq j < s(q-1), 2 \nmid j\} & \text{if } v_2(p-1) \leq v_2(q-1), \\
\{\tilde{\rho}_{ij} \mid 0 \leq i < d', 0 \leq j < t'(p-1), 2 \nmid j\} & \text{otherwise},
\end{cases} \]

where \(\tilde{\rho}_{ij}\) and \(\tilde{\rho}_{ij}\) are defined in (C3.1) and (C3.2) respectively.

4. The Frobenius maps. This section is a preparation for the next two sections where we will compute the Artin conductors of representations and the Artin L-functions of some quasi-cyclotomic fields \(\tilde{K}\). For a prime number \(\ell\), we say that \(\ell\) is ramified (resp. inert, splitting) in the relative quadratic extension \(\tilde{K}/K\) if the prime ideals of \(K\) over \(\ell\) are ramified (resp. inert, splitting) in \(\tilde{K}\). For a prime number \(\ell\) which is unramified in \(\tilde{K}/K\), let \(I_\ell\) (resp. \(\tilde{I}_\ell\)) be the inert group of \(\ell\) in the extension \(K/Q\) (resp. \(\tilde{K}/Q\)). Let \(\text{Fr}_\ell\) be the Frobenius automorphism of \(\ell\) in \(G/I_\ell\), and \(\tilde{\text{Fr}}_\ell\) the Frobenius automorphism of \(\ell\) in \(\tilde{G}/\tilde{I}_\ell\) associated to some prime ideal over \(\ell\).

To compute the Artin conductors of representations, we need to construct a uniformizer in the completion of \(\tilde{K}\) at a prime ideal, in particular at a prime ideal over 2. Generally we are not able to get such a uniformizer, but we can do it in the case \(p = -1\). In addition, to calculate the Artin L-functions of representations, we need to know \(\tilde{\text{Fr}}_\ell\), in particular for \(\ell = 2\), and so we need to know the decomposition of 2 in \(\tilde{K}\). For odd \(p < q \in S\), we calculated some examples by computer which suggest that 2 is always unramified in \(\tilde{K}\). But we are not able to show this. Furthermore, we do not know when 2 splits in \(\tilde{K}/K\) and when 2 is inert in \(\tilde{K}/K\). But when \(p = -1\), we can solve these problems (see below). So in this paper we only compute the Artin conductors and Artin L-functions of representations in the case \(p = -1\).
From now on, we always assume that \( p = -1 \), so \( K = \mathbb{Q}(\zeta_{4q}) \) and \( \tilde{K} = K(\sqrt[4]{q^*}) \). In this section we determine \( \tilde{F}_{\tau \ell} \) by \( F_{\tau \ell} \) for \( \ell = 2 \). In [6, Sect. 5] the decomposition of some odd prime numbers in \( \tilde{K}/K \) was determined. Now we determine the decomposition of 2 in \( \tilde{K}/K \). The result below is a more explicit reformulation of Theorem 2 in [7].

**Proposition 4.1.** If \( q = 2 \), then 2 is ramified in \( \tilde{K}/K \). If \( q \) is odd, then 2 is unramified in \( \tilde{K}/K \) if and only if \( (\frac{2}{q}) = 1 \), and in this case 2 splits in \( \tilde{K}/K \) if \( q^* \equiv 1 \mod 16 \), and is inert in \( \tilde{K}/K \) otherwise.

**Proof.** We first consider the case \( q = 2 \). The unique prime ideal of \( K \) over 2 is the principal ideal generated by \( \pi_2 = 1 - \zeta_8 \). Since the ramification degree of 2 in \( K/\mathbb{Q} \) is 4 and \( \sqrt{2} = \pi_2(\pi_2 + 2\zeta_8)\zeta_8 \), we deduce that 2 is ramified in \( \tilde{K}/K \) if and only if \( x^2 \equiv \sqrt{2} \mod \pi_2^3 \) is not solvable in the ring \( \mathcal{O}_K \) of integers of \( K \) by [7, Th. 2(1)], which is equivalent to \( (1 + \frac{2}{\pi_2} \zeta_8) \zeta_8 \) not being a square modulo \( \pi_2^6 \). Since \( 2 = u\pi_2^4 \) for some unit \( u \), we have

\[
\left(1 + \frac{2}{\pi_2} \zeta_8\right) \zeta_8 \equiv \zeta_8 \equiv (1 - \pi_2) \mod \pi_2^3,
\]

hence \( (1 + \frac{2}{\pi_2} \zeta_8) \zeta_8 \) is not a square modulo \( \pi_2^3 \). So 2 is ramified in \( \tilde{K}/K \).

Now we assume that \( q \) is odd. Let \( \pi_2 = 1 - \zeta_4 \). Since the ramification degree of 2 in \( K \) is 2, we see that 2 is unramified in \( \tilde{K}/K \) if and only if \( x^2 \equiv \sqrt{q^*} \mod \pi_2^4 \) is solvable in \( \mathcal{O}_K \) (see [7, Th. 2(1)]). Furthermore, 2 splits in \( \tilde{K}/K \) if and only if \( x^2 \equiv \sqrt{q^*} \mod \pi_2^5 \) is solvable in \( \mathcal{O}_K \). The explicit computation of the Gauss sum gives

\[
\sqrt{q^*} = \sum_{a=1}^{a-1} \left(\frac{a}{q}\right) \zeta_q^a = 1 + 2 \sum_{\left(\frac{a}{q}\right) = 1} \zeta_q^a.
\]

Let \( \alpha = \sum_{\left(\frac{a}{q}\right) = 1} \zeta_q^a \), \( \beta = \sum_{\left(\frac{a}{q}\right) = 1} \zeta_q^a \), and \( \gamma = \sum_{\left(\frac{a}{q}\right) = 1} \sum_{\left(\frac{b}{q}\right) = 1, a < b} \zeta_q^{a+b} \), where in the summations \( \alpha, \beta \) run over \( 1, \ldots, q-1 \). Then \( \alpha = \beta^2 - 2\gamma \), which together with the equality \( 2 = \pi_2^2 - \pi_2^3 \) gives

\[
\sqrt{q^*} = 1 + 2\beta^2 - 4\gamma = 1 + \pi_2^2\beta^2 - \pi_2^3\beta^2 - 4\gamma \\
\equiv (1 + \pi_2\beta)^2 - \pi_2^3(\beta + \beta^2) + \pi_2^4(\beta - \gamma) \\
\equiv (1 + \pi_2\beta)^2 - \pi_2^3(\alpha + \beta) + \pi_2^4(\beta + \gamma) \mod \pi_2^5.
\]

Since \( \zeta_{2q} = -\zeta_q^{-(q-1)/2} = -\zeta_q^t \), where \( t \) is the inverse of 2 in \( (\mathbb{Z}/q\mathbb{Z})^* \), we see that \( \beta = \sum_{\left(\frac{a}{q}\right) = 1} (-1)^a \zeta_q^{ta} = \sum_{\left(\frac{a}{q}\right) = 1} \zeta_q^a \mod 2 \). So if \( \left(\frac{q}{2}\right) = 1 \) we have \( \alpha \equiv \beta \mod 2 \) and thus 2 is unramified in \( \tilde{K}/K \), and if \( \left(\frac{q}{2}\right) = -1 \) we have \( \alpha + \beta \equiv \sum_{a=1}^{q-1} \zeta_q^a = -1 \mod 2 \) and thus 2 is ramified in \( \tilde{K}/K \).
Now we assume \((\frac{2}{q}) = 1\). Then \(\sqrt{q^*} \mod \pi_2^5\) is a square if and only if \(\pi_2 | \beta + \gamma\). We consider \(2(\beta + \gamma)\). Since \(\alpha \equiv \beta \mod 2\), we have
\[
2(\beta + \gamma) = 2\beta + \beta^2 - \alpha \equiv \alpha(\alpha + 1) \mod 4. 
\]
From \(\sqrt{q^*} = 1 + 2\alpha\), we see that \(\alpha(\alpha + 1) = (q^* - 1)/4\). Since \(8 | q^* - 1\) under the assumption \((\frac{2}{q}) = 1\), we have \(\beta + \gamma \equiv (q^* - 1)/8 \mod 2\). So \(\pi_2 | \beta + \gamma\) if and only if \(\pi_2 | (q^* - 1)/8\), that is, \(2 | (q^* - 1)/8\). The proof is complete. 

Now we assume that \(2\) is unramified in \(\tilde{K}/K\). Let \(\text{Fr}_2 \in G\) be such that \(\text{Fr}_2(\zeta_4) = 1\) and \(\text{Fr}_2(\zeta_q) = \zeta_q^2\). It is a Frobenius element of \(2\) in \(G\) modulo \(I_2\). We have \(\text{Fr}_2 = \sigma_2^{b_2}\) for some \(b_2 \in \mathbb{Z}\) with \(2 | b_2\) as \((\frac{2}{q}) = 1\). Thus \(\tilde{\text{Fr}}_2 = \tilde{\sigma}_2^{b_2}\) or \(\tilde{\text{Fr}}_2 = \tilde{\sigma}_2^{b_2} \varepsilon\). We need to determine \(\tilde{\text{Fr}}_2\) completely. Since \((\frac{2}{q}) = 1\), we have
\[
\sqrt{q^*} \equiv (1 + \pi_2^2\alpha)^2 + \pi_2^4(\beta + \gamma) \mod \pi_2^5. 
\]
Write \(u = 1 + \pi_2\alpha\) for simplicity. Since \(\sqrt{q^*} \equiv u^2 \mod \pi_2^4\), we see \((\sqrt{q^*} - u)/2 \in O_{\tilde{K}}\). Let \(\wp\) be the prime ideal of \(\tilde{K}\) over \(2\) associated to \(\text{Fr}_2\). By the definition, we have
\[
\tilde{\text{Fr}}_2 \left(\frac{\sqrt{q^*} - u}{2}\right) \equiv \left(\frac{\sqrt{q^*} - u}{2}\right)^2 \equiv (\beta + \gamma) + \frac{\sqrt{q^*} - u}{2} \mod \wp.
\]
On the other hand, since \(\tilde{\sigma}_2^{b_2}(\sqrt{q^*}) = (-1)^{b_2/2}\sqrt{q^*}\) and \(\tilde{\sigma}_2^{b_2}(u) = u\) as \(2 | b_2\), we have
\[
\tilde{\sigma}_2^{b_2} \left(\frac{\sqrt{q^*} - u}{2}\right) = \frac{(-1)^{b_2/2}\sqrt{q^*} - u}{2}
\]
and
\[
\tilde{\sigma}_2^{b_2} \varepsilon \left(\frac{\sqrt{q^*} - u}{2}\right) = \frac{(-1)^{b_2/2+1}\sqrt{q^*} - u}{2}.
\]
So if \(2 | b_2/2\) we have \(\tilde{\text{Fr}}_2 = \tilde{\sigma}_2^{b_2}\) if and only if \(\pi_2 | \beta + \gamma\) (that is, \(2\) splits in \(\tilde{K}/K\)), and if \(2 \nmid b_2/2\) we have \(\tilde{\text{Fr}}_2 = \tilde{\sigma}_2^{b_2}\) if and only if \(\pi_2 \nmid \beta + \gamma\) (that is, \(2\) is inert in \(\tilde{K}/K\)). In the case \(q \equiv 3 \mod 4\), we can always assume that \(2 \nmid b_2/2\), since if \(4 | b_2\), we may replace \(b_2\) by \(b_2 + (q - 1)\). In the case \(q \equiv 1 \mod 4\), we have \(2 | b_2/2\) iff \(2^{(q-1)/4} \equiv 1 \mod q\) iff \(q\) has the form \(A^2 + 64B^2\) for \(A, B \in \mathbb{Z}\), by Exercise 28 in Chap. 5 of [4]. So we get the following result:

**Proposition 4.2.** Assume that \(2\) is unramified in \(\tilde{K}/K\). Let \(\text{Fr}_2 = \sigma_2^{b_2}\). Then \(2 | b_2\). If \(q \equiv 3 \mod 4\), we always assume \(b_2 \equiv 2 \mod 4\). Let \(P_0\) be the set of prime numbers of the form \(A^2 + 64B^2\) with \(A, B \in \mathbb{Z}\). Then
\[
\tilde{\text{Fr}}_2 = \begin{cases} 
\tilde{\sigma}_2^{b_2} & \text{if } q \not\in P_0, \ 16 \nmid q^* - 1, \text{ or } q \in P_0, \ 16 | q^* - 1, \\
\tilde{\sigma}_2^{b_2} \varepsilon & \text{if } q \in P_0, \ 16 | q^* - 1, \text{ or } q \not\in P_0, \ 16 | q^* - 1.
\end{cases}
\]

The following lemma is useful in the computation of Artin L-functions.

**Lemma 4.3.** We have \(\varepsilon \in I_\ell\) if and only if \(\ell\) is ramified in \(\tilde{K}/K\).
Proof. The canonical projection $\tilde{G} \to G \simeq \hat{G}$ induces a surjective homomorphism $\tilde{I}_\ell \to I_\ell$ which implies the isomorphism $\tilde{I}_\ell/\langle \varepsilon \rangle \cap \tilde{I}_\ell \cong I_\ell$. Thus $\ell$ is ramified in $K/K$ iff $|\tilde{I}_\ell| = 2|I_\ell|$ iff $|\tilde{I}_\ell \cap \langle \varepsilon \rangle| = 2$ iff $\varepsilon \in \tilde{I}_\ell$.

5. The conductors of representations. In this section we compute the Artin conductors of all 2-dimensional irreducible representations of $\hat{G}$ in the case $p = -1$. First we recall the definition of the Artin conductor. For details, see [2, Chap. 6].

The notations are as before. Let $\ell$ be a prime number in $\mathbb{Q}$, and choose a prime ideal $\mathfrak{p}$ in $\tilde{K}$ over $\ell$. Let $\tilde{G}_\ell = \tilde{G}(\tilde{K}_\mathfrak{p}/\mathbb{Q}_\ell)$ be the corresponding decomposition subgroup. Let $v$ be the normalized valuation in $\tilde{K}_\mathfrak{p}$. For $i \geq 0$, define the ramification groups

$$\tilde{G}_{\ell,i} = \{ \sigma \in \tilde{G}_\ell \mid v(\sigma(x) - x) > i \text{ for all } x \in O_{\tilde{K}_\mathfrak{p}} \}.$$ 

The group $\tilde{G}_{\ell,0}$ is the inertia subgroup of $\tilde{G}_\ell$. Let $\pi$ be a uniformizer in $\tilde{K}_\mathfrak{p}$. Then for $i > 0$,

$$\tilde{G}_{\ell,i} = \{ \sigma \in \tilde{G}_\ell \mid v(\sigma(\pi) - \pi) > i \}.$$

For a representation $\rho$ of $\tilde{G}$ with character $\chi$ and representation space $V$, let

$$f(\chi, \ell) = f(\rho, \ell) = \sum_{i=0}^{\infty} \frac{|\tilde{G}_{\ell,i}|}{|\tilde{G}_{\ell,0}|} (\chi(1) - \chi(\tilde{G}_{\ell,i})),$$

where $\chi(\tilde{G}_{\ell,i}) = |\tilde{G}_{\ell,i}|^{-1} \sum_{s \in \tilde{G}_{\ell,i}} \chi(s)$. We have $f(\chi, \ell) = 0$ if $\rho$ is unramified over $\ell$, i.e. $V = V^{\tilde{G}_{\ell,0}}$. The Artin conductor of the representation $\rho$ is defined as

$$f(\chi) = f(\rho) = \prod_{\ell} \ell^{f(\chi, \ell)}.$$

From the result in the previous section, we know that $\ell$ is unramified in $\tilde{K}/\mathbb{Q}$ if $\ell \neq 2, q$. Thus to compute the conductor $f(\chi)$, we only need to calculate $f(\chi, 2)$ and $f(\chi, q)$. We consider the cases $q = 2$ and $q$ odd separately.

5.1. Case $q = 2$. In the case $(p, q) = (-1, 2)$, there is only one 2-dimensional irreducible representation $\tilde{\rho}_0$ of $\hat{G}$. Let $\tilde{\chi}_0$ be the character of $\tilde{\rho}_0$. Since only 2 is ramified in $\tilde{K}$, we only need to calculate $f(\tilde{\chi}_0, 2)$.

As in the previous section, let $\pi_2 = 1 - \zeta_8$. Let $\mathfrak{p}$ be a prime ideal in $\tilde{K}$ over 2 and let $v$ be the normalized valuation in $\tilde{K}_\mathfrak{p}$. From the proof of Proposition 4.1, we see that $\sqrt{2}/\pi_2^2 \equiv 1 - \pi_2 \pmod{\pi_2^3}$. Thus

$$v\left(\frac{\sqrt{2}}{\pi_2^2} - 1\right) = v\left(\frac{\sqrt{2}}{\pi_2} - 1\right) + v\left(\frac{\sqrt{2}}{\pi_2} + 1\right) = v(\pi_2) = 2.$$
We have $v(\sqrt{2}/\pi_2 - 1) = v(\sqrt{2}/\pi_2 + 1) = 1$. So $\pi = \sqrt{2}/\pi_2 - 1$ is a uniformizer of $\tilde{K}_{\varphi}$. The group $\tilde{G}$ is generated by $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, and $\tilde{\sigma}_1(\sqrt{2}) = \sqrt{2}$ and $\tilde{\sigma}_2(\sqrt{2}) = \sqrt{2}/\sqrt{-1}$. Clearly $G_{2,0} = \tilde{G}$. Furthermore,

\[
v(\tilde{\sigma}_1(\pi) - \pi) = v\left(\frac{\sqrt{2}}{1 - \zeta_8^{-1}} - \frac{\sqrt{2}}{1 - \zeta_8}\right) = v\left(-\frac{1 + \zeta_8}{1 - \zeta_8}\right) = 2,
v(\tilde{\sigma}_2(\pi) - \pi) = v\left(\frac{\sqrt{2}/\sqrt{-1}}{1 + \zeta_8} - \frac{\sqrt{2}}{1 - \zeta_8}\right) = v\left(-\frac{1 - \zeta_8^3}{1 - \zeta_8}\right) = 2,
v(\varepsilon(\pi) - \pi) = v(-2(\pi + 1)) = 8.
\]

Thus $G_{2,1} = G_{2,0} = \tilde{G}$ and $G_{2,2} = \cdots = G_{2,7} = \langle \varepsilon \rangle$. By an easy computation we get $\tilde{\chi}_0(G_{2,0}) = \tilde{\chi}_0(G_{2,1}) = \cdots = \tilde{\chi}_0(G_{2,7}) = 0$, and $\tilde{\chi}_0(G_{2,n}) = 2$ for $n \geq 8$. So we obtain

\[(5.1) \quad f(\tilde{\chi}_0, 2) = 2 + 2 + \frac{1}{4} \cdot 2 \cdot 6 = 7.\]

**5.2. Case of $q$ odd.** To compute $f(\chi, q)$, we consider the cases $(\frac{-1}{q}) = 1$ and $(\frac{-1}{q}) = -1$ separately. Let $\varphi$ be a prime ideal in $\tilde{K}$ over $q$. Let $v$ be the normalized valuation in $\tilde{K}_{\varphi}$.

**5.2.1. Assume $(\frac{-1}{q}) = 1$.** Then $q$ is unramified in $\tilde{K}/K$ but ramified in $K/Q$. We see $\pi = 1 - \zeta_q$ is a uniformizer of $\tilde{K}_{\varphi}$. Now all 2-dimensional irreducible representations of $\tilde{G}$ are as in Case A. Let $\tilde{\chi}_j$ be the character of $\tilde{\rho}_j$. It is easy to see that $\tilde{G}_{q,0} = \langle \tilde{\sigma}_q \rangle$. Notice that $\varepsilon \notin \tilde{G}_{q,0}$.

Let $1 \neq \tilde{\sigma} \in \tilde{G}_{q,0}$ and $\tilde{\sigma}(\zeta_q) = \zeta_q^a$, $1 < a \leq q - 1$. We have

\[v(\tilde{\sigma}\pi - \pi) = v(\zeta_q - \zeta_q^a) = v(1 - \zeta_q^{a-1}) = 1.\]

Thus $\tilde{G}_{q,n} = \{1\}$ for $n \geq 1$. By an easy computation we get $\tilde{\chi}_j(\tilde{G}_{q,0}) = 0$ and $\tilde{\chi}_j(\tilde{G}_{q,n}) = 2$ for $n \geq 1$. We obtain

\[(5.2) \quad f(\tilde{\chi}_j, q) = 2.\]

**5.2.2. Assume $(\frac{-1}{q}) = -1$.** Then $q$ is ramified both in $\tilde{K}/K$ and in $K/Q$, and all 2-dimensional irreducible representations are as in Case B. Let $\tilde{\chi}_j$ be the character of $\tilde{\rho}_j$. Since $v(1 - \zeta_q) = 2$ and $v(\sqrt{-q}) = \frac{1}{4}(2(q - 1)) = (q - 1)/2$, we see that $\pi = \sqrt{-q}/(1 - \zeta_q(q-3)/4$ is a uniformizer of $q$ in $\tilde{K}$. It is obvious that $\tilde{G}_{q,0} = \langle \tilde{\sigma}_q \rangle$. Notice that in this case $\varepsilon \in \tilde{G}_{q,0}$. 
Let \( 1 \neq \bar{\sigma} \in \tilde{G}_{q,0} \) and \( \bar{\sigma}(\zeta_q) = \zeta_q^a, 1 < a \leq q - 1 \). We have

\[
v(\bar{\sigma}\pi - \pi) + v(\bar{\sigma}\varepsilon\pi - \pi) = v(\bar{\sigma}\pi - \pi) + v(-\bar{\sigma}\pi - \pi) = v(\bar{\sigma}\pi^2 - \pi^2)
\]

\[
= v\left( \frac{\left(\frac{a}{q}\right)\sqrt{-q}}{(1 - \zeta_q^a)(q-3)/2} - \frac{\sqrt{-q}}{(1 - \zeta_q)(q-3)/2} \right)
\]

\[
= v(\pi^2) + v\left( 1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1} \zeta_q^i \right) \right) \frac{(q-3)/2}{(q-3)/2}
\]

\[
= 2 + v\left( 1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1} \zeta_q^i \right) \right) \frac{(q-3)/2}{(q-3)/2}.
\]

Let \( t = v\left( 1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1} \zeta_q^i \right) \right) \frac{(q-3)/2}{(q-3)/2} \). We claim that \( t = 0 \). Otherwise \( t > 0 \). Since

\[
\left(\sum_{i=0}^{a-1} \zeta_q^i \right) \frac{(q-3)/2}{(q-3)/2} \equiv 1 \mod (1 - \zeta_q) \quad \text{if} \quad \left(\frac{a}{q}\right) = 1,
\]

\[
\left(\sum_{i=0}^{a-1} \zeta_q^i \right) \frac{(q-3)/2}{(q-3)/2} \equiv -1 \mod (1 - \zeta_q) \quad \text{if} \quad \left(\frac{a}{q}\right) = -1,
\]

we always have \( a \equiv 1 \mod q \) and thus \( a = 1 \), which contradicts the assumption that \( a > 1 \). This shows the claim. Thus \( v(\bar{\sigma}\pi - \pi) = v(\bar{\sigma}\varepsilon\pi - \pi) = 1 \), as \( v(\bar{\sigma}\pi - \pi + \bar{\sigma}\varepsilon\pi - \pi) = v(2\pi) = 1 \). So we get \( \tilde{G}_{q,n} = \{1\} \) for \( n \geq 1 \). By an easy computation, \( \tilde{\chi}_j(\tilde{G}_{q,0}) = 0 \) and \( \tilde{\chi}_j(\tilde{G}_{q,n}) = 2 \) for \( n \geq 1 \). We obtain

\[
f(\tilde{\chi}_j, q) = 2.
\]

Next we compute \( f(\tilde{\chi}_j, 2) \). We consider the cases \( \left(\frac{2}{q}\right) = 1 \) and \( \left(\frac{2}{q}\right) = -1 \) separately. Let \( \wp \) be a prime ideal in \( \tilde{K} \) over 2. Let \( v \) be the normalized valuation in \( \tilde{K}_\wp \).

**5.2.3. Assume \( \left(\frac{2}{q}\right) = 1 \).** Then 2 is unramified in \( \tilde{K}/K \) but ramified in \( K'/Q \), and \( \pi = 1 - \zeta_4 \) is a uniformizer in \( \tilde{K}_\wp \). It is easy to see that \( \tilde{G}_{2,0} = \langle \bar{\sigma}_1 \rangle \). Notice that in this case \( \varepsilon \notin \tilde{G}_{2,0} \). We have

\[
v(\bar{\sigma}_1 - \pi) = v(\zeta_4 - \zeta_4^{-1}) = v(2) = 2.
\]

Thus \( \tilde{G}_{2,0} = \tilde{G}_{2,1} = \langle \bar{\sigma}_1 \rangle \) and \( \tilde{G}_{2,n} = \{1\} \) for \( n > 1 \). By an easy computation, \( \tilde{\chi}_j(\tilde{G}_{2,0}) = \tilde{\chi}_j(\tilde{G}_{2,1}) = 1 \) and \( \tilde{\chi}_j(\tilde{G}_{2,n}) = 2 \) for \( n > 1 \). We obtain

\[
f(\tilde{\chi}_j, 2) = 1 + 1 = 2.
\]

**5.2.4. Assume \( \left(\frac{2}{q}\right) = -1 \).** Now 2 is ramified both in \( \tilde{K}/K \) and in \( K'/Q \). As in the previous section, let \( \pi_2 = 1 - \zeta_4, \alpha = \sum_{\left(\frac{a}{q}\right)=1} \zeta_q^a \) and \( \beta = \sum_{\left(\frac{a}{q}\right)=1} \zeta_2^a \), where the summations are over \( 1 \leq a \leq q - 1 \). From the previous section we have

\[
\sqrt{q^\pi} \equiv (1 + \pi_2\beta)^2 + \pi_2^3 \mod \pi_2^4.
\]
Let $\mu = 1 + \pi_2 \beta$. We claim that $\pi = (\sqrt[q]{q^*} + \mu)/\pi_2$ is a uniformizer in $\tilde{K}_0$. In fact, since

$$v(\sqrt[q]{q^*} + \mu) + v(\sqrt[q]{q^*} - \mu) = v(\sqrt[q]{q^*} - \mu^2) = v(\pi_2^3) = 6$$

and $v((\sqrt[q]{q^*} + \mu) + (\sqrt[q]{q^*} - \mu)) = v(2\sqrt[q]{q^*}) = 4$, we must have

$$v(\sqrt[q]{q^*} + \mu) = v(\sqrt[q]{q^*} - \mu) = 3,$$

and thus $v((\sqrt[q]{q^*} + \mu)/\pi_2) = 1$.

It is obvious that $\tilde{G}_{2,0} = \{1, \varepsilon, \tilde{\sigma}_1, \tilde{\sigma}_1 \varepsilon\}$. Since $\tilde{\sigma}_1(\sqrt[q]{q^*}) = \sqrt[q]{q^*}$ and $\tilde{\sigma}_1 \varepsilon(\sqrt[q]{q^*}) = -\sqrt[q]{q^*}$, we have

$$v(\tilde{\sigma}_1 - \pi - \pi) = v\left(\tilde{\sigma}_1 - \frac{\sqrt[q]{q^*} + 1 + \pi_2 \beta}{\pi_2} - \frac{\sqrt[q]{q^*} + 1 + \pi_2 \beta}{\pi_2}\right)$$

$$= v\left(\tilde{\sigma}_1 - \frac{\sqrt[q]{q^*} + 1}{\pi_2} - \frac{\sqrt[q]{q^*} + 1}{\pi_2}\right)$$

(since $\tilde{\sigma}_1 \beta = \beta$)

$$= v\left(\frac{\sqrt[q]{q^*} + 1}{1 - \zeta_4} - \frac{\sqrt[q]{q^*} + 1}{1 - \zeta_4}\right) = v(\sqrt[q]{q^*} + 1).$$

To compute it, we first claim that $\pi_2 \nmid \beta$. Otherwise, $2 | \beta$ as $\beta \in \mathbb{Q}(\zeta_q)$. From the previous section, we have $\sqrt[q]{q^*} = 1 + 2\alpha$ and $\alpha + \beta \equiv 1 \mod 2$, thus $\sqrt[q]{q^*} \equiv -1 + 2\beta \equiv -1 \mod 4$ and so $q^* \equiv 1 \mod 8$. This contradicts the assumption $(\frac{2}{q}) = -1$. We have shown the claim. Thus $v(\beta) = 0$. Since $v(\sqrt[q]{q^*} + 1 + \pi_2 \beta) = 3$, we have $v(\sqrt[q]{q^*} + 1) = 2$, so $v(\tilde{\sigma}_1 - \pi - \pi) = 2$.

We now compute $v(\tilde{\sigma}_1 \varepsilon \pi - \pi)$. We have

$$v(\tilde{\sigma}_1 \varepsilon \pi - \pi) = v\left(\tilde{\sigma}_1 \varepsilon - \frac{\sqrt[q]{q^*} + 1 + \pi_2 \beta}{\pi_2} - \frac{\sqrt[q]{q^*} + 1 + \pi_2 \beta}{\pi_2}\right)$$

$$= v\left(-\sqrt[q]{q^*} + 1 - \sqrt[q]{q^*} + 1\right) = v(\sqrt[q]{q^*} + \zeta_4).$$

Observe that

$$v(\sqrt[q]{q^*} + \zeta_4) + v(\sqrt[q]{q^*} - \zeta_4) = v(\sqrt[q]{q^*} + 1) = v\left(\frac{2 \sqrt[q]{q^*} + 1}{2}\right) = 4,$$

since $\pi_2 \nmid (\sqrt[q]{q^*} + 1)/2$. Furthermore, since

$$v((\sqrt[q]{q^*} + \zeta_4) + (\sqrt[q]{q^*} - \zeta_4)) = v(2 \sqrt[q]{q^*}) = 4,$$

we must have $v(\sqrt[q]{q^*} + \zeta_4) = v(\sqrt[q]{q^*} - \zeta_4) = 2$, so $v(\tilde{\sigma}_1 \varepsilon \pi - \pi) = 2$. In addition, we have

$$v(\varepsilon \pi - \pi) = v\left(-\sqrt[q]{q^*} + 1 + \pi_2 \beta - \sqrt[q]{q^*} + 1 + \pi_2 \beta\right) = 2.$$
By the discussion above we have \( \tilde{G}_{2,0} = \tilde{G}_{2,1} \) and \( \tilde{G}_{2,n} = \{1\} \) for \( n > 1 \). By an easy computation, \( \tilde{\chi}_j(\tilde{G}_{2,0}) = \tilde{\chi}_j(\tilde{G}_{2,1}) = 0 \) and \( \tilde{\chi}_j(\tilde{G}_{2,n}) = 2 \) for \( n > 2 \). We obtain

(5.6) \quad f(\tilde{\chi}_j, 2) = 2 + 2 = 4.

5.3. Global conductors. By the equalities (5.1)–(5.6) above, we get the following

**Theorem 5.1.** In the case \( q = 2 \), the conductor of the unique 2-dimensional irreducible representation \( \tilde{\rho}_0 \) of \( \tilde{G} \) is equal to \( f(\tilde{\rho}_0) = 2^7 \). In the case that \( q \) is odd, all the 2-dimensional irreducible representations \( \tilde{\rho}_j \) of \( \tilde{G} \) have the conductor \( f(\tilde{\rho}_j) = 2^{2(1+\log_{-1}(\frac{2}{q}))}q^2 \).

6. The Artin \( L \)-functions. In this section we compute the Artin \( L \)-functions of the quasi-cyclotomic fields \( \tilde{K} = \mathbb{Q}(\zeta_{4q}, \sqrt[4]{q^*}) \).

The \( L \)-functions associated to the 1-dimensional representations of \( \tilde{G} \) are the well-known Dirichlet \( L \)-functions. So we compute the \( L \)-functions associated to the 2-dimensional irreducible representations of \( \tilde{G} \). Let \( \varphi : \tilde{G} \to \text{GL}(V) \) be a 2-dimensional irreducible representation. The Artin \( L \)-function \( L(\varphi, s) \) associated to \( \varphi \) is defined as the product

\[
L(\varphi, s) = \prod_{\ell \text{ prime}} L_{\ell}(\varphi, s),
\]

where the local factors are defined as \( L_{\ell}(\varphi, s) = \det(1 - \varphi(\tilde{\text{Fr}}_{\ell})\ell^{-s}|V^{\tilde{\ell}})^{-1} \). Now we begin to compute them. First we notice that if \( \ell \) is ramified in \( \tilde{K}/K \), then \( V^{\tilde{\ell}} = 0 \) and \( L_{\ell}(\varphi, s) = 1 \), which is due to the facts that \( \varepsilon \in \tilde{I}_{\ell} \) by Lemma 4.3 and \( \varphi(\varepsilon) = -I \) for any irreducible representation \( \varphi \) of \( \tilde{G} \) by Theorem 3.1.

6.1. Case \( q = 2 \). By Section 3, there is only one 2-dimensional representation \( \tilde{\rho}_0 \) in this case, which is defined by

\[
\tilde{\rho}_0(\tilde{\sigma}_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_0(\tilde{\sigma}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Since 2 is ramified in \( \tilde{K}/K \), we have \( L_2(\tilde{\rho}_0, s) = 1 \). Assume that \( \ell \) is an odd prime number.

If \( \ell \equiv 7 \mod 8 \), then \( \text{Fr}_\ell = \sigma_{-1} \) and thus \( \tilde{\text{Fr}}_{\ell} = \tilde{\sigma}_{-1} \) or \( \tilde{\sigma}_{-1} \varepsilon \). In any case we have

\[
L_{\ell}(\tilde{\rho}_0, s) = \det\left( I \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ell^{-s} \right)^{-1} = (1 - \ell^{-2s})^{-1}.
\]
We first determine the local factors \( \ell \equiv 5 \mod 8 \), then \( \text{Fr}_\ell = \sigma_2 \) and thus \( \tilde{\text{Fr}}_\ell = \tilde{\sigma}_2 \) or \( \tilde{\sigma}_2 \varepsilon \). We have

\[
L_\ell(\tilde{\rho}_0, s) = \det \left( I \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s} \right)^{-1} = (1 + \ell^{-2s})^{-1}.
\]

If \( \ell \equiv 3 \mod 8 \), then \( \text{Fr}_\ell = \sigma_2 \) and thus \( \tilde{\text{Fr}}_\ell = \tilde{\sigma}_2 \) or \( \tilde{\sigma}_2 \varepsilon \). We have

\[
L_\ell(\tilde{\rho}_0, s) = \det \left( I \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s} \right)^{-1} = (1 - \ell^{-2s})^{-1}.
\]

If \( \ell \equiv 1 \mod 8 \), then \( \text{Fr}_\ell = 1 \) and thus \( \tilde{\text{Fr}}_\ell = 1 \) or \( \varepsilon \). In this case we must determine \( \tilde{\text{Fr}}_\ell \) completely. Since \( \tilde{\text{Fr}}_\ell(\sqrt{2}) \equiv (\sqrt{2})^\ell \mod \varphi \) for the prime \( \varphi \) of \( K \) over \( \ell \) associated to \( \tilde{\text{Fr}}_\ell \), we have \( \tilde{\text{Fr}}_\ell = 1 \) if \( 2^{(\ell-1)/4} \equiv 1 \mod \ell \), and \( \tilde{\text{Fr}}_\ell = \varepsilon \) if \( 2^{(\ell-1)/4} \equiv -1 \mod \ell \). As in the previous section, we find that for \( \ell \equiv 1 \mod 8 \), \( 2^{(\ell-1)/4} \equiv 1 \mod \ell \) if and only if \( \ell \in P_0 \). So we have

\[
L_\ell(\tilde{\rho}_0, s) = \begin{cases} (1 - \ell^{-s})^{-2} & \text{if } \ell \in P_0, \\ (1 + \ell^{-s})^{-2} & \text{otherwise.} \end{cases}
\]

We get the Artin L-function in the case \( (p, q) = (-1, 2) \) as follows:

\[
L(\tilde{\rho}_0, s) = \prod_{\ell \equiv 3 \text{ or } 7 \mod 8} (1 - \ell^{-2s})^{-1} \cdot \prod_{\ell \equiv 5 \mod 8} (1 + \ell^{-2s})^{-1} \\
\times \prod_{\ell \in P_0} (1 - \ell^{-s})^{-2} \cdot \prod_{\ell \equiv 1 \mod 8, \ell \not\in P_0} (1 + \ell^{-s})^{-2}.
\]

6.2. Case of \( q \) odd. In this case, all 2-dimensional irreducible representations of \( \tilde{G} \) are \( \tilde{\rho}_j \) with \( 0 \leq j < q - 1, 2 | j \) if \( q \equiv 1 \mod 4 \), and \( 0 \leq j < q - 1, 2 \nmid j \) if \( q \equiv 3 \mod 4 \), where \( \tilde{\rho}_j \) is defined by

\[
\tilde{\rho}_j(\tilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_j(\tilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1} \zeta_j \\ \zeta_{2(q-1)} & 0 \end{pmatrix}, \quad \tilde{\rho}_j(\varepsilon) = -I.
\]

We first determine the local factors \( L_\ell(\tilde{\rho}_j, s) \) for \( \ell \neq 2, q \). For such \( \ell \) we have \( V^{\text{Fr}_\ell} = V \). Let \( \text{Fr}_\ell = \sigma_{-1}^{-1} \sigma_q^{b_\ell} \), which is equivalent to \( \ell \equiv (-1)^a \ell \mod 4 \) and \( \ell \equiv g^{b_\ell} \mod q \), where \( g \) is the primitive root mod \( q \) associated to \( \sigma_q \). It is easy to compute that

\[
\tilde{\rho}_j(\tilde{\sigma}_q^{b_\ell}) = \begin{pmatrix} 0 & \zeta_j^{q-1} \\ 1 & 0 \end{pmatrix}^{b_\ell} = \begin{cases} \zeta_{2(q-1)}^{b_\ell} I & \text{if } 2 | b_\ell, \\ \begin{pmatrix} 0 & \zeta_j^{(b_\ell+1)} \\ \zeta_{2(q-1)} & 0 \end{pmatrix} & \text{if } 2 \nmid b_\ell. \end{cases}
\]
Furthermore,

$$\det(I - \tilde{\rho}_j(\tilde{\sigma}_{-1}^{a\ell} \tilde{\sigma}_{q}^{b\ell})^s) = \begin{cases} 
(1 - \zeta_{2(q-1)}^{jb\ell} \ell^{-s})^2 & \text{if } a\ell = 0, 2 \mid b\ell, \\
1 - \zeta_{q-1}^{jb\ell} \ell^{-2s} & \text{if } a\ell = 0, 2 \not\mid b\ell, \\
1 + \zeta_{q-1}^{jb\ell} \ell^{-2s} & \text{if } a\ell = 1, 2 \mid b\ell, 
\end{cases}$$

and

$$\det(I + \tilde{\rho}_j(\tilde{\sigma}_{-1}^{a\ell} \tilde{\sigma}_{q}^{b\ell})^s) = \begin{cases} 
(1 + \zeta_{2(q-1)}^{jb\ell} \ell^{-s})^2 & \text{if } a\ell = 0, 2 \mid b\ell, \\
1 - \zeta_{q-1}^{jb\ell} \ell^{-2s} & \text{if } a\ell = 0, 2 \not\mid b\ell, \\
1 + \zeta_{q-1}^{jb\ell} \ell^{-2s} & \text{if } a\ell = 1, 2 \mid b\ell. 
\end{cases}$$

So we get

$$L_\ell(\tilde{\rho}_j, s) = (1 - \zeta_{q-1}^{jb\ell} \ell^{-2s})^{-1}$$

if \(\ell \equiv 1 \mod 4\) and \(\ell \equiv g^{b\ell} \mod q\) with \(2 \not\mid b\ell\), and also if \(\ell \equiv 3 \mod 4\) and \(\ell \equiv g^{b\ell} \mod q\) with \(2 \mid b\ell\), while

$$L_\ell(\tilde{\rho}_j, s) = (1 + \zeta_{q-1}^{jb\ell} \ell^{-2s})^{-1}$$

if \(\ell \equiv 3 \mod 4\) and \(\ell \equiv g^{b\ell} \mod q\) with \(2 \not\mid b\ell\).

To compute the local factors when \(\ell \equiv 1 \mod 4\) and \(\ell \equiv g^{b\ell} \mod q\) with \(2 \mid b\ell\) we must determine \(\tilde{\mathrm{Fr}}_\ell\) completely. Since \((\frac{\ell}{q}) = 1\), we have \((\frac{q}{\ell}) = 1\) and \((\frac{q}{\ell^2}) = 1\). Let \(\alpha\ell \in \mathbb{Z}\) be such that \(\alpha\ell^2 \equiv q^* \mod \ell\). From \(\tilde{\sigma}_{q}^{b\ell} (\sqrt[4]{q^*}^z) = (-1)^{b\ell/2} \sqrt[4]{q^*}\), we see that \(\tilde{\mathrm{Fr}}_\ell = \tilde{\sigma}_{q}^{b\ell} \) if \((\frac{\alpha\ell}{\ell}) = (-1)^{b\ell/2}\), and \(\tilde{\mathrm{Fr}}_\ell = \tilde{\sigma}_{q}^{b\ell} \) if \((\frac{\alpha\ell}{\ell^2}) = (-1)^{b\ell/2+1}\). So when \(\ell \equiv 1 \mod 4\) and \(\ell \equiv g^{b\ell} \mod q\) with \(2 \mid b\ell\), we have

$$L_\ell(\tilde{\rho}_j, s) = \begin{cases} 
(1 - \zeta_{2(q-1)}^{jb\ell} \ell^{-s})^{-2} & \text{if } (\frac{\alpha\ell}{\ell}) = (-1)^{b\ell/2}, \\
(1 + \zeta_{2(q-1)}^{jb\ell} \ell^{-s})^{-2} & \text{if } (\frac{\alpha\ell}{\ell^2}) = (-1)^{b\ell/2+1}. 
\end{cases}$$

Next we compute the local factors \(L_2(\tilde{\rho}_j, s)\) and \(L_q(\tilde{\rho}_j, s)\). When \((\frac{2}{q}) = -1\), we know from the previous section that 2 is ramified in \(\tilde{K}/K\). So \(L_2(\tilde{\rho}_j, s) = 1\) in this case. Now we assume \((\frac{2}{q}) = 1\). Since \(I_2 = \langle \sigma_{-1} \rangle\) and 2 is unramified in \(\tilde{K}/K\), we have \(\tilde{I}_2 = \langle \tilde{\sigma}_{-1} \rangle\) or \(\tilde{I}_2 = \langle \tilde{\sigma}_{-1} \rangle\). The matrices \(I + \tilde{\rho}_j(\tilde{\sigma}_{-1})\) and \(I + \tilde{\rho}_j(\tilde{\sigma}_{-1})\) have rank 1, thus \(V^{I_2}\) has dimension 1. Write \(\mathrm{Fr}_2 = \sigma_{2}^{b_2}\) with \(2 \mid b_2\). As in the previous section, we always assume \(b_2 \equiv 2 \mod 4\) if \(q \equiv 3 \mod 4\). Recall that \(P_0\) is the set of all prime numbers of the form \(A^2 + 64B^2\) with \(A, B \in \mathbb{Z}\). Since \(\tilde{\rho}_j(\tilde{\sigma}_{2}^{b_2}) = \zeta_{2(q-1)}^{jb_2} \), by Lemma 4.3
we have

\[ L_2(\tilde{\rho}_j, s) = \begin{cases} 
1 - \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \notin P_0, \ 16 \mid q^* - 1, \text{ or } q \in P_0, \ 16 \mid q^* - 1, \\
1 + \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \in P_0, \ 16 \mid q^* - 1, \text{ or } q \notin P_0, \ 16 \mid q^* - 1.
\end{cases} \]

When \( q \equiv 3 \pmod{4} \), we know that \( q \) is ramified in \( \widetilde{K}/K \). So \( L_q(\tilde{\rho}_j, s) = 1 \) for odd \( j \) in this case. Assume \( q \equiv 1 \pmod{4} \). Since \( I_q = \langle \sigma_q \rangle \) and \( q \) is unramified in \( \widetilde{K}/K \), we have \( \tilde{I}_q = \langle \tilde{\sigma}_q \rangle \) or \( \tilde{I}_2 = \langle \tilde{\sigma}_q \varepsilon \rangle \). Thus \( V\tilde{I}_q = 0 \) if \( j \neq 0 \), and \( V\tilde{I}_q \) has dimension 1 if \( j = 0 \).

The Frobenius map \( \text{Fr}_q \) of \( q \) in \( G \) modulo \( I_q \) is the identity map. So \( \text{Fr}_q = 1 \) or \( \varepsilon \). In [7 Sect. 5] we have shown that \( q \) splits in \( \widetilde{K}/K \) if \( q \equiv 1 \pmod{8} \), and is inert if \( q \equiv 5 \pmod{8} \). So \( \text{Fr}_2 = 1 \) if \( q \equiv 1 \pmod{8} \), and \( \text{Fr}_2 = \varepsilon \) if \( q \equiv 5 \pmod{8} \). Thus we get

\[ L_q(\tilde{\rho}_j, s) = \begin{cases} 
1 & \text{if } j \neq 0, \\
1 - q^{-s} & \text{if } j = 0, \ q \equiv 1 \pmod{8}, \\
1 + q^{-s} & \text{if } j = 0, \ q \equiv 5 \pmod{8}.
\end{cases} \]

We have computed all the local factors, obtaining

\begin{equation}
(6.2) \quad L(\tilde{\rho}_j, s) = (1 - u_q\zeta_{2(q-1)}^{jb_2} 2^{-s})^{-1} (1 - (-1)^{(q-1)/4} q^{-s})^{-\delta_{0j}} \\
\times \prod_{\ell \equiv 1, 2 \mid b_\ell \text{ or } \ell \equiv 3, 2 \mid b_\ell} (1 - \zeta_{q-1}^{jb_\ell} \ell^{-2s})^{-1} \\
\times \prod_{\ell \equiv 3, 2 \mid b_\ell} (1 + \zeta_{q-1}^{jb_\ell} \ell^{-2s})^{-1} \prod_{\ell \equiv 1, 2 \mid b_\ell} (1 - u_\ell \zeta_{2(q-1)}^{jb_\ell} \ell^{-s})^{-2},
\end{equation}

where \( u_q = 1 \) if \( q \notin P_0 \), \( 16 \mid q^* - 1 \) or \( q \in P_0 \), \( 16 \mid q^* - 1 \), and \( u_q = -1 \) otherwise; \( \delta_{0j} = 0 \) if \( j \neq 0 \) and \( \delta_{00} = 1 \); and \( u_\ell = (\frac{\alpha_\ell}{\ell}) (-1)^{b_\ell/2} \). In the above products, “\( \equiv \)” denotes congruence modulo 4.

**Theorem 6.1.** Except for the Dirichlet \( L \)-functions, all Artin \( L \)-functions of the Galois extension \( \widetilde{K}/Q \) are explicitly given by (6.1) in the case \( q = 2 \) and by (6.2) in the case of \( q \) odd, where in (6.2) \( 0 \leq j < q - 1, \ 2 \mid j \) if \( q \equiv 1 \pmod{4} \) and \( 0 \leq j < q - 1, \ 2 \nmid j \) if \( q \equiv 3 \pmod{4} \).

**6.3. A formula.** Let \( \zeta_{\widetilde{K}}(s) \) and \( \zeta_K(s) \) be the Dedekind zeta functions of \( \widetilde{K} \) and \( K \) respectively. By Artin’s formula for the decomposition of Dedekind zeta functions, we have

\[ \frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} = \prod_{\tilde{\rho}_j} \prod_{\ell \text{ prime}} L_\ell(\tilde{\rho}_j, s)^2, \]

where \( \tilde{\rho}_j \) runs over all 2-dimensional irreducible representations of \( \widetilde{G} \). When \( q = 2 \), there is only one 2-dimensional irreducible representation of \( \widetilde{G} \). So the
square of (6.1) gives the formula. When $q$ is odd, by computing $\prod L_\ell(\tilde{\rho}_j, s)$, we get the following

**Corollary 6.2.** For a prime number $\ell \neq q$, let

$$f_\ell = \frac{q - 1}{\gcd(b_\ell, q - 1)}$$

be the order of $\ell$ mod $q$ and let

$$g_\ell = \gcd(b_\ell, q - 1) = \frac{q - 1}{f_\ell}.$$

If $q \equiv 1 \pmod{4}$, then

$$\frac{\zeta_K(s)}{\zeta_K(s)} = (1 - u_q^2 2^{-f_2 s})^{g_2} (1 - (-1)(q-1)/4 q^{-s})^{-2} \prod_{\ell \equiv 1, 2|b_\ell \text{ or } \ell \equiv 3} (1 - \ell^{-f_\ell s})^{-2g_\ell} \times \prod_{\ell \equiv 1, 2|\ell} (1 - u_\ell^{f_\ell \ell^{-f_\ell s}})^{-2g_\ell},$$

and if $q \equiv 3 \pmod{4}$, then

$$\frac{\zeta_K(s)}{\zeta_K(s)} = (1 + u_q^2 2^{-f_2 s})^{g_2} \prod_{\ell \equiv 1, 2|b_\ell} (1 + \ell^{-f_\ell s})^{-2g_\ell} \prod_{\ell \equiv 3} (1 - \ell^{-2f_\ell s})^{-g_\ell} \times \prod_{\ell \equiv 1, 2|\ell} (1 + u_\ell^{f_\ell \ell^{-f_\ell s}})^{-2g_\ell},$$

where $u_q$ and $u_\ell$ are as above.

### 6.4. The corresponding modular forms

All the 2-dimensional irreducible representations of $\tilde{G}$ in the case $p = -1$ are monomial. It is easy to see that they are odd. By Deligne–Serre’s theorem [6, Th. 2], these Artin $L$-functions above are equal to the $L$-functions of some normalized newforms of weight one, which allows one to determine a newform of weight one from a 2-dimensional irreducible odd representation of $\tilde{G}$. More precisely, the irreducible representation $\tilde{\rho}_j$ of conductor $N$ corresponds to a normalized newform $f_j(z)$ of weight one on $\Gamma_0(N)$ with nebentype $\phi_j = \det(\tilde{\rho}_j)$, which has a Fourier expansion at infinity

$$f_j(z) = \sum_{n=1}^{\infty} a_n^{(j)} q^n, \quad q = e^{2\pi i z},$$

where $a_1^{(j)} = 1$ and the other coefficients $a_n$ are equal to those of the $L$-function $L(\phi_j, s) = \sum_{n=1}^{\infty} a_n n^{-s}$. In this subsection we describe these modular forms explicitly. Since these newforms are eigenfunctions of Hecke operators, to determine all $a_n^{(j)}$ it is enough to determine $a_{\ell}^{(j)}$ for all primes $\ell$.

When $q = 2$, we get one normalized newform $f_0(z)$ of weight 1 on $\Gamma_0(2^7)$ with nebentype $\phi_0 : (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{C}^*$, where $\phi_0(\sigma_{-1}) = -1$ and $\phi_0(\sigma_2) = 1$. 
By (6.1), we directly see that for primes \( \ell \) the coefficients \( a^{(0)}_\ell \) of the newform are given by

\[
a^{(0)}_\ell = \begin{cases} 
0 & \text{if } \ell = 2 \text{ or } \ell \equiv 3, 5, 7 \mod 8, \\
2 & \text{if } \ell \in P_0, \\
-2 & \text{if } \ell \equiv 1 \mod 8 \text{ but } \ell \not\in P_0.
\end{cases}
\]

When \( q \) is odd, we get \( (q - 1)/2 \) normalized newforms \( f_j(z) \) of weight 1 on \( \Gamma_0(4^{1+\log_2(\ell/3)}q^2) \) with nebentype \( \phi_j : (\mathbb{Z}/4q\mathbb{Z})^* \rightarrow \mathbb{C}^* \), where \( \phi_j(\sigma_{-1}) = -1 \) and \( \phi_j(\sigma_q) = -\zeta_{q-1}^j \). By (6.2) we directly see that for primes \( \ell \neq q \) the coefficients of the newforms are given by

\[
a^{(j)}_\ell = \begin{cases} 
u_q \zeta_{2(q-1)}^{jb_\ell} & \text{if } \ell = 2, \\
2u_\ell \zeta_{2(q-1)}^{jb_\ell} & \text{if } \ell \equiv 1 \mod 4 \text{ and } 2 \mid b_\ell, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
a^{(j)}_q = \begin{cases} 
0 & \text{if } j \neq 0, \\
(-1)^{(q-1)/4} & \text{if } j = 0,
\end{cases}
\]

where \( 0 \leq j < q - 1, 2 \mid j \) if \( q \equiv 1 \mod 4 \), and \( 0 \leq j < q - 1, 2 \nmid j \) if \( q \equiv 3 \mod 4 \); \( b_\ell \) is defined by \( \ell \equiv g^{b_\ell} \mod q \) for a primitive root \( g \) modulo \( q \); \( u_\ell = (\alpha_\ell/\ell) (-1)^{b_\ell/2} \) for an integer \( \alpha_\ell \) such that \( \alpha_\ell^2 \equiv q^* \mod \ell \); and

\[
u_q = \begin{cases} 
1 & \text{if } q \not\in P_0, 16 \nmid q^* - 1 \text{ or } q \in P_0, 16 \mid q^* - 1, \\
-1 & \text{if } q \not\in P_0, 16 \mid q^* - 1 \text{ or } q \in P_0, 16 \nmid q^* - 1.
\end{cases}
\]

Here \( P_0 \) is the set of all primes of the form \( A^2 + 64B^2 \) with \( A, B \in \mathbb{Z} \).

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